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ON APPROXIMATION PROPERTIES OF STANCU VARIANT λ-SZÁSZ-MIRAKJAN-DURRMEYER OPERATORS

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Abstract. In the present paper, we aim to obtain several approximation properties of Stancu form Szász-Mirakjan-Durrmeyer operators based on Bézier basis functions with shape parameter $\lambda \in [-1,1]$. We estimate some auxiliary results such as moments and central moments. Then, we obtain the order of convergence in terms of the Lipschitz-type class functions and Peetre's K-functional. Further, we prove weighted approximation theorem and also Voronovskaya-type asymptotic theorem. Finally, to see the accuracy and effectiveness of discussed operators, we present comparison of the convergence of constructed operators to certain functions with some graphical illustrations under certain parameters.

1. Introduction

In [26] Mirakjan and Szász [44] proposed and studied the following sequence of linear positive operators, for any $m \in \mathbb{N}$ and for the bounded functions $\mu(z)$ in $C[0,\infty)$

$$S_m(\mu;z) = \sum_{j=0}^{\infty} s_{m,j}(z)\mu\left(\frac{j}{m}\right), \tag{1}$$

where $z \geq 0$ and $s_{m,j}(z) = e^{-mz}\frac{(mz)^j}{j!}$.

A Durrmeyer type integral modification of operators (1) is established by Mazhar and Totik [25] as below:

$$D_m(\mu;z) = m\sum_{j=0}^{\infty} s_{m,j}(z)\int_0^{\infty} s_{m,j}(z)\mu(t)\,dt,\ \ z \in [0,\infty). \tag{2}$$

Recently, several approximation properties such as uniform approximation, weighted approximations, simultaneous approximation and Voronovskaja type result of operators (1) and (2) and their modifications are considered. We refer for the readers to [7, 8, 13, 19–21, 27–29, 31].

Bézier curves with shape parameters are one of the prominent research areas for modeling in computer graphics (CG) and computer-aided geometric design (CAGD).

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Due to their computational simplicity and stability, the Bézier curves have various applications such as airframe design, numerical solution of partial differential equations, font design, networks, animation, robotics and so on. A choice of shape parameter is significant, wherefore Bézier curves and surfaces are characterized with their control meshes. One can has some applications in (CAGD) (see: [17, 22, 32, 40]).

Very recently, Ye et al. [45] presented the Bézier basis with shape parameter $\lambda \in [-1,1]$. In 2018, Cai et al. [14] introduced $\lambda-$Bernstein operators and studied various approximation theorems, uniform convergence, local approximation and Voronovskaya-type asymptotic. Acu et al. [1] discussed some approximation properties such as order of convergence by Ditzian-Totik modulus of smoothness and Voronovskaya and Grüss-Voronovskaya-type results. On the other hand, Özger [35] proposed a new type of Schurer operators with Bézier-Schurer basis and investigated several weighted A-statistical convergence results of these operators.

In 2019, Qi et al. [38] introduced the following Szász-Mirakjan operators with shape parameter $\lambda \in [-1,1]$

$$S_{m,\lambda}(\mu;z)=\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;z)\mu\left(\frac{j}{m}\right), \tag{3}$$

where Szász-Mirakjan bases functions $\widetilde{s}_{m,j}(\lambda;z)$ with shape parameter $\lambda \in [-1,1]$ given by

$$\begin{aligned}
\widetilde{s}_{m,0}(\lambda;z) &= s_{m,0}(z)-\frac{\lambda}{m+1}s_{m+1,1}(z);\\
\widetilde{s}_{m,i}(\lambda;z) &= s_{m,i}(z)+\lambda\left(\frac{m-2i+1}{m^2-1}s_{m+1,i}(z)\right.\\
&\left.-\frac{m-2i-1}{m^2-1}s_{m+1,i+1}(z)\right)\ \ (i=1,2,...,\infty,\ z\in[0,\infty)).
\end{aligned} \tag{4}$$

Motivated by operators (3), Aslan [6] constructed following Szász-Mirakjan-Durrmeyer operators with shape parameter $\lambda \in [-1,1]$:

$$D_{m,\lambda}(\mu;z)=m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;z)\int_0^{\infty}s_{m,j}(t)\mu(t)\,dt, \quad z\in[0,\infty), \tag{5}$$

where $\widetilde{s}_{m,j}(\lambda;y)$ $(j=0,1,..\infty)$ defined in (4) and $\lambda \in [-1,1]$.

He estimated the rate of convergence in terms of the usual modulus of continuity and Peetre's K-functional and proved a uniform convergence theorem on weighted spaces and derived a Voronovskaya type asymptotic theorem for these operators.

One may see the recent works that include linear positive operators which have the shape parameter λ: [2, 4, 5, 9–12, 15, 24, 30, 33, 34, 36, 37, 39, 41, 42].

In the present work, we construct a Stancu [43] type Szász-Mirakjan-Durrmeyer operators based on shape parameter $\lambda \in [-1,1]$:

$$D_{m,\lambda}^{\alpha,\beta}(\mu;z)=m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;z)\int_0^{\infty}s_{m,j}(t)\mu\left(\frac{mt+\alpha}{m+\beta}\right)dt, \tag{6}$$

where $z \in [0,\infty)$, $\mu(z) \in C[0,\infty)$, α and β are non-negative parameters verifing the conditions $0 \leq \alpha \leq \beta$. Note that for $\alpha = \beta = 0$, the operators (6) reduce to (5) and with $\lambda = \alpha = \beta = 0$, they reduce to operators (2).

The focus of this paper is organized as follows: In Sect. 2, we calculate some preliminaries results such as moments and central moments. In Sect. 3, we estimate the order of convergence in terms of the functions belong to Lipschitz-type class and Peetre's K-functional. In Sect. 4, we obtain a result concerning the weighted approximation. In Sect. 5, we investigate a Voronovskaya-type asymptotic theorem. Finally, to see the accuracy and effectiveness of proposed operators, we show the comparison of the convergence of operators (6) to the certain functions with some illustrations for different values of m, α, β and λ.

2. Preliminaries results

LEMMA 2.1. [38]. *Let the operators $S_{m,\lambda}(\mu;z)$ be defined by (3). Then, we get the following expressions:*

$$\begin{aligned}
S_{m,\lambda}(1;z) &= 1,\\
S_{m,\lambda}(t;z) &= z + \left[\frac{1-e^{-(m+1)z}-2z}{m(m-1)}\right]\lambda,\\
S_{m,\lambda}(t^2;z) &= z^2 + \frac{z}{m} + \left[\frac{2z+e^{-(m+1)z}-1-4(m+1)z^2}{m^2(m-1)}\right]\lambda,\\
S_{m,\lambda}(t^3;z) &= z^3 + \frac{3z^2}{m} + \frac{z}{m^2} + \left[\frac{1-e^{-(m+1)z}-2z}{m^3(m-1)}\right.\\
&\quad \left. + \frac{3(m-3)(m+1)z^2-6(m+1)z^3}{m^3(m-1)}\right]\lambda,\\
S_{m,\lambda}(t^4;z) &= z^4 + \frac{6z^3}{m} + \frac{7z^2}{m^2} + \frac{z}{m^3} + \left[\frac{e^{-(m+1)z}-1+2mz}{m^4(m-1)}\right.\\
&\quad \left. + \frac{2(3m-11)(m+1)z^2+4(m-8)(m+1)^2z^3-8(m+1)^3z^4}{m^4(m-1)}\right]\lambda.
\end{aligned}$$

LEMMA 2.2. [6]. *For the operators defined by (5), we get the following moments*

$$\begin{aligned}
D_{m,\lambda}(1;z) &= 1,\\
D_{m,\lambda}(t;z) &= z + \frac{1}{m} + \left[\frac{1-e^{-(m+1)z}-2z}{m(m-1)}\right]\lambda,\\
D_{m,\lambda}(t^2;z) &= z^2 + \frac{4z}{m} + \frac{2}{m^2} + \left[\frac{1-e^{-(m+1)z}-2z-2(m+1)z^2}{m^2(m-1)}\right]2\lambda,\\
D_{m,\lambda}(t^3;z) &= z^3 + \frac{9z^2}{m} + \frac{18z}{m^2} + \frac{6}{m^3}\\
&\quad + \left[\frac{2-2e^{-(m+1)z}-4z+(m-11)(m+1)z^2-2(m+1)z^3}{m^3(m-1)}\right]3\lambda,
\end{aligned}$$

$$D_{m,\lambda}(t^4;z) = z^4 + \frac{16z^3}{m} + \frac{72z^2}{m^2} + \frac{96z}{m^3} + \frac{24}{m^4} + \left[\frac{24 - 24e^{-(m+1)z} + z(m-25) + 18(m-7)(m+1)z^2}{m^4(m-1)} - \frac{2(m^2-7m-23)(m+1)z^3 + 4(m+1)^3z^4}{m^4(m-1)}\right]2\lambda.$$

Lemma 2.3. *For $n = 1, 2, ...,$ we have following relation:*

$$D^{\alpha,\beta}_{m,\lambda}(t^n;z) = \sum_{k=0}^{n}\binom{n}{k}\frac{m^k\alpha^{n-k}}{(m+\beta)^n}D_{m,\lambda}(t^k;z),$$

where $D_{m,\lambda}(\mu;z)$ and $D^{\alpha,\beta}_{m,\lambda}(\mu;z)$ are defined by (5) and (6), respectively.

Proof. In view of (5) and (6), it follows

$$\begin{aligned} D^{\alpha,\beta}_{m,\lambda}(t^n;z) &= m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;z)\int_0^{\infty}s_{m,j}(t)\mu\left(\frac{mt+\alpha}{m+\beta}\right)^n dt \\ &= m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;z)\int_0^{\infty}s_{m,j}(t)\left(\sum_{k=0}^{n}\binom{n}{k}\frac{m^k\alpha^{n-k}}{(m+\beta)^n}t^k\right)dt \\ &= \sum_{k=0}^{n}\binom{n}{k}\frac{m^k\alpha^{n-k}}{(m+\beta)^n}\left(m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;z)\int_0^{\infty}s_{m,j}(t)t^k dt\right) \\ &= \sum_{k=0}^{n}\binom{n}{k}\frac{m^k\alpha^{n-k}}{(m+\beta)^n}D_{m,\lambda}(t^k;z). \end{aligned}$$

□

Lemma 2.4. *Let $e_u(t) = t^u, \quad t = 0, 1, 2, 3, 4.$ Then, we obtain*

(7) $$D^{\alpha,\beta}_{m,\lambda}(e_0;z) = 1,$$

(8) $$D^{\alpha,\beta}_{m,\lambda}(e_1;z) = \frac{m}{m+\beta}z + \frac{\alpha+1}{m+\beta} + \left[\frac{1 - e^{-(m+1)z} - 2z}{(m+\beta)(m-1)}\right]\lambda,$$

(9) $$D^{\alpha,\beta}_{m,\lambda}(e_2;z) = \frac{m^2}{(m+\beta)^2}z^2 + \frac{2m(\alpha+2)}{(m+\beta)^2}z + \frac{(\alpha+1)^2+1}{(m+\beta)^2} + \left[\frac{(\alpha+1)(1-e^{-(m+1)z}-2z) - 2(m+1)z^2}{(m+\beta)^2(m-1)}\right]2\lambda,$$

(10) $$\begin{aligned} D^{\alpha,\beta}_{m,\lambda}(e_3;z) &= \frac{m^3}{(m+\beta)^3}z^3 + \frac{3(\alpha+3)m^2}{(m+\beta)^3}z^2 + \frac{3((\alpha+2)^2+2)m}{(m+\beta)^3}z \\ &+ \frac{3(\alpha+1)^2}{(m+\beta)^3} + \left[\frac{(\alpha^2+2\alpha-2)(1-e^{-(m+1)z}-2z)}{(m+\beta)^3(m-1)} + \frac{(m-11-4\alpha)(m+1)z^2 - 2(m+1)z^3}{(m+\beta)^3(m-1)}\right]3\lambda, \end{aligned}$$

$$
\begin{aligned}
D_{m,\lambda}^{\alpha,\beta}(e_4;z) &= \frac{m^4}{(m+\beta)^4}z^4 + \frac{4(\alpha+4)m^3}{(m+\beta)^4}z^3 + \frac{6(\alpha^2+6\alpha+12)m^2}{(m+\beta)^4}z^2 \\
&+ \frac{4(\alpha^3+6\alpha^2+18\alpha+24)m}{(m+\beta)^4}z + \frac{(\alpha+1)^4+6\alpha^2+20\alpha+23}{(m+\beta)^4} \\
&+ \left[\frac{((\alpha^2-6)(2\alpha+6)+12)(1-e^{-(m+1)z})+}{(m+\beta)^4(m-1)}\right. \\
&+ \frac{(18(m-7)+6\alpha(m-11)-12\alpha^2)(m+1)z^2}{(m+\beta)^4(m-1)} \\
&- \left.\frac{2(m^2-7m-23+6\alpha)(m+1)z^3+4(m+1)^3z^4}{(m+\beta)^4(m-1)}\right]2\lambda.
\end{aligned}
\tag{11}
$$

Proof. For $n=2$ and using Lemma 2.2, we obtain

$$
\begin{aligned}
D_{m,\lambda}^{\alpha,\beta}(e_2;z) &= \sum_{k=0}^{2}\binom{2}{k}\frac{m^k\alpha^{2-k}}{(m+\beta)^2}D_{m,\lambda}(t^k;z) \\
&= \frac{1}{(m+\beta)^2}\left[\alpha^2 D_{m,\lambda}(1;z) + 2m\alpha D_{m,\lambda}(t;z) + m^2 D_{m,\lambda}(t^2;z)\right] \\
&= \frac{1}{(m+\beta)^2}\left[\alpha^2 + 2m\alpha z + 2\alpha + \left\{\frac{1-e^{-(m+1)z}-2z}{m-1}\right\}2\alpha\lambda\right. \\
&+ \left. m^2z^2 + 4mz + 2 + \left\{\frac{1-e^{-(m+1)z}-2z-2(m+1)z^2}{m-1}\right\}2\lambda\right] \\
&= \frac{m^2}{(m+\beta)^2}z^2 + \frac{2m(\alpha+2)}{(m+\beta)^2}z + \frac{(\alpha+1)^2+1}{(m+\beta)^2} \\
&+ \left[\frac{(\alpha+1)(1-e^{-(m+1)z}-2z)-2(m+1)z^2}{(m+\beta)^2(m-1)}\right]2\lambda.
\end{aligned}
$$

Other expressions can be calculated by similar methods, thus we omitted details. □

LEMMA 2.5. *Let $z\in[0,\infty)$, $\lambda\in[-1,1]$ and $m>1$. Then, we get the following central moments:*

$$
\begin{aligned}
(i)\ D_{m,\lambda}^{\alpha,\beta}(t-z;z) &= \frac{\alpha+1}{m+\beta} - \frac{\beta}{m+\beta}z + \left[\frac{1-e^{-(m+1)z}-2z}{(m+\beta)(m-1)}\right]\lambda \\
&\le \frac{(\alpha+1)m+e^{-(m+1)z}+2z}{(m+\beta)(m-1)} := \Phi_m^{\alpha,\beta}(z), \\
(ii)\ D_{m,\lambda}^{\alpha,\beta}((t-z)^2;z) &\le \frac{2(m(\alpha+2)+\beta(m+\beta))}{(m+\beta)^2}z + \frac{(\alpha+1)^2+1}{(m+\beta)^2} \\
&+ \frac{2((\alpha+1)(1+e^{-(m+1)z}+2z)+2(m+1)z^2)}{(m+\beta)^2(m-1)} := \Omega_m^{\alpha,\beta}(z),
\end{aligned}
$$

$$(iii)\ D_{m,\lambda}^{\alpha,\beta}((t-z)^4;z) = \frac{\beta^4}{(m+\beta)^4}z^4 + \frac{4(6m-\alpha-\beta)\beta^2}{(m+\beta)^4}z^3$$
$$+\frac{6(\beta^2(\alpha+1)^2+2m(m-2\alpha\beta-4\beta)}{(m+\beta)^4}z^2$$
$$+\frac{4(\alpha^3+3\alpha^2+12\alpha+21)m-12\beta(\alpha+1)^2}{(m+\beta)^4}z$$
$$+\frac{(\alpha+1)^4+(2\alpha+5)^2+2(\alpha^2-1)}{(m+\beta)^4}$$
$$+\left[\frac{(2(\alpha^2-6)(2\alpha+6)+12)-12(\alpha^2+2\alpha-2)(m+\beta)z}{(m+\beta)^4(m-1)}\right.$$
$$+\frac{12(\alpha+1)(m+\beta)^2z^2-4(m+\beta)^3z^3)(1-e^{-(m+1)z})}{(m+\beta)^4(m-1)}$$
$$+\frac{(18(m-7)+6\alpha(m-11-12\alpha^2))(m+1)z^2}{(m+\beta)^4(m-1)}$$
$$-\frac{4((m^2-7m-23)+3\alpha)-3(m-11-4\alpha)(m+\beta)(m+1)z^3}{(m+\beta)^4(m-1)}$$
$$\left.+\frac{8((m+1)^2+3(m+\beta))\,(m+1)\,z^4}{(m+\beta)^4(m-1)}\right]\lambda.$$

3. Local approximation

Let the space $C[0,\infty)$ denote the all continuous and bounded functions μ on $[0,\infty)$ and it is equipped with the norm $\|\mu\|_{[0,\infty)} = \sup_{z\in[0,\infty)} |\mu(z)|$. Firstly we define some notations, which will be fundamental of our following theorems. Let the Peetre's K-functional is given by

$$K_2(\mu,\eta) = \inf_{\nu\in C^2[0,\infty)} \{\|\mu-\nu\| + \eta\,\|\nu''\|\},$$

where $\eta>0$ and $C^2[0,\infty) = \{\nu \in C[0,\infty) : \nu', \nu'' \in C[0,\infty)\}$.

There exists an absolute constant $C>0$ such that (see:([16]))

$$K_2(\mu;\eta) \le C\omega_2(\mu;\sqrt{\eta}), \qquad \eta>0, \tag{12}$$

where

$$\omega_2(\mu;\eta) = \sup_{0<u\le\eta}\ \sup_{z\in[0,\infty)} |\mu(z+2u)-2\mu(z+u)+\mu(z)|,$$

is the second order modulus of smoothness of $\mu \in C[0,\infty)$. Further, we denote the ordinary modulus of continuity of $\mu\in C[0,\infty)$ by

$$\omega(\mu;\eta) := \sup_{0<u\le\eta}\ \sup_{z\in[0,\infty)} |\mu(z+u)-\mu(z)|,$$

(see details [3]).

With $Lip_L(\zeta)$, we denote an element of Lipschitz type continuous function, where $D>0$ and $0<\zeta\le 1$. Since the following expression

$$|\mu(t)-\mu(z)| \le D\,|t-z|^{\zeta}, \qquad (t,z\in\mathbb{R}),$$

verifies, then a function μ is belong to $Lip_L(\zeta)$.

THEOREM 3.1. *Let $z \in [0,\infty)$, $\mu \in Lip_L(\zeta)$ and $\lambda \in [-1,1]$. Then,*

$$\left| D_{m,\lambda}^{\alpha,\beta}(\mu;z) - \mu(z) \right| \leq D(\Omega_m^{\alpha,\beta}(z))^{\frac{\zeta}{2}},$$

where $\Omega_m^{\alpha,\beta}(z)$ is defined in Lemma 2.5.

Proof. By using the linearity and monotonicity properties of the operators (6), it deduce following

$$\begin{aligned}\left| D_{m,\lambda}^{\alpha,\beta}(\mu;z) - \mu(z) \right| &\leq D_{m,\lambda}^{\alpha,\beta}(\left|\mu(t)-\mu(z)\right|;z) \\ &\leq m\sum_{j=0}^{\infty} \widetilde{s}_{m,j}(\lambda;z)\int_0^{\infty} s_{m,j}(t)\left|\mu(t)-\mu(z)\right| dt \\ &\leq Dm\sum_{j=0}^{\infty} \widetilde{s}_{m,j}(\lambda;z)\int_0^{\infty} s_{m,j}(t)\left|t-z\right|^{\zeta} dt.\end{aligned}$$

Using the Hölder's inequality with $p_1 = \frac{2}{\zeta}$ and $p_2 = \frac{2}{2-\zeta}$ and taking Lemma 2.4-2.5 into account, therefore

$$\begin{aligned}\left| D_{m,\lambda}^{\alpha,\beta}(\mu;z) - \mu(z) \right| &\leq D\left\{ m\sum_{j=0}^{\infty} \widetilde{s}_{m,j}(\lambda;z)\int_0^{\infty} s_{m,j}(t)\,(t-z)^2\, dt \right\}^{\frac{\zeta}{2}} \\ &\quad . \left\{ m\sum_{j=0}^{\infty} \widetilde{s}_{m,j}(\lambda;z)\int_0^{\infty} s_{m,j}(t) \right\}^{\frac{2-\zeta}{2}} \\ &= D\left\{ D_{m,\lambda}^{\alpha,\beta}((e_1-z)^2;z) \right\}^{\frac{\zeta}{2}} \left\{ D_{m,\lambda}^{\alpha,\beta}(e_0;z) \right\}^{\frac{2-\zeta}{2}} \\ &\leq D(\Omega_m^{\alpha,\beta}(z))^{\frac{\zeta}{2}}.\end{aligned}$$

Hence, we get the desired proof. □

THEOREM 3.2. *Let $z \in [0,\infty)$, $\mu \in Lip_L(\zeta)$ and $\lambda \in [-1,1]$. Then for a constant $C > 0$ the following relation holds true*

$$\left| D_{m,\lambda}^{\alpha,\beta}(\mu;z) - \mu(z) \right| \leq C\omega_2(\mu; \frac{1}{2}\sqrt{\Omega_m^{\alpha,\beta}(z) + (\Phi_m^{\alpha,\beta}(z))^2} + \omega(\mu; \Phi_m^{\alpha,\beta}(z)),$$

where $\Phi_m^{\alpha,\beta}(z)$, $\Omega_m^{\alpha,\beta}(z)$ are same as in Lemma 2.5.

Proof. Let $\mu \in C[0,\infty)$ and we denote with $\phi_{m,\lambda}^{\alpha,\beta}(z) := \frac{m}{m+\beta}z + \frac{\alpha+1}{m+\beta} + \left[\frac{1-e^{-(m+1)z}-2z}{m+\beta(m-1)}\right]\lambda$. It is clear that $\phi_{m,\lambda}^{\alpha,\beta}(z) \in [0,\infty)$ for sufficently large m. Now, we give the following auxiliary operators:

$$\widetilde{D}_{m,\lambda}^{\alpha,\beta}(\mu;z) = D_{m,\lambda}^{\alpha,\beta}(\mu;z) - \mu(\phi_{m,\lambda}^{\alpha,\beta}(z)) + \mu(z). \tag{13}$$

Note that, by (7) and (8), it follows

$$\widetilde{D}_{m,\lambda}^{\alpha,\beta}(t-z;z) = 0.$$

In view of Taylor's expansion formula, hence

$$\sigma(t)=\sigma(z)+(t-z)\sigma'(z)+\int_z^t(t-u)\sigma''(u)du,\quad (\sigma\in C^2[0,\infty)). \tag{14}$$

Operating $\widetilde{D}_{m,\lambda}^{\alpha,\beta}(.;z)$ on (14), we get

$$\begin{aligned}\widetilde{D}_{m,\lambda}^{\alpha,\beta}(\sigma;z)-\sigma(z) &= \widetilde{D}_{m,\lambda}^{\alpha,\beta}((t-z)\sigma'(z);z)+\widetilde{D}_{m,\lambda}^{\alpha,\beta}(\int_z^t(t-u)\sigma''(u)du;z)\\ &= \sigma'(z)\widetilde{D}_{m,\lambda}^{\alpha,\beta}(t-z;z)+D_{m,\lambda}^{\alpha,\beta}(\int_z^t(t-u)\sigma''(u)du;z)\\ &- \int_z^{\phi_{m,\lambda}^{\alpha,\beta}(z)}(\phi_{m,\lambda}^{\alpha,\beta}(z)-u)\sigma''(u)du\\ &= D_{m,\lambda}^{\alpha,\beta}(\int_z^t(t-u)\sigma''(u)du;z)-\int_z^{\phi_{m,\lambda}^{\alpha,\beta}(z)}(\phi_{m,\lambda}^{\alpha,\beta}(z)-u)\sigma''(u)du.\end{aligned}$$

From Lemma 2.4 and (13), we obtain

$$\begin{aligned}\left|\widetilde{D}_{m,\lambda}^{\alpha,\beta}(\sigma;z)-\sigma(z)\right| &\le \left|D_{m,\lambda}^{\alpha,\beta}(\int_z^t(t-u)\sigma''(u)du;z)\right|+\left|\int_z^{\phi_{m,\lambda}^{\alpha,\beta}(z)}(\phi_{m,\lambda}^{\alpha,\beta}(z)-u)\sigma''(u)du\right|\\ &\le D_{m,\lambda}^{\alpha,\beta}(\int_z^t(t-u)\left|\sigma''(u)\right|du;z)+\int_z^{\phi_{m,\lambda}^{\alpha,\beta}(z))}(\phi_{m,\lambda}^{\alpha,\beta}(z)-u)\left|\sigma''(u)\right|du\\ &\le \left\|\sigma''\right\|\left\{D_{m,\lambda}^{\alpha,\beta}((t-z)^2;z)+\left(\phi_{m,\lambda}^{\alpha,\beta}(z)-z\right)^2\right\}\\ &\le \left\{\Omega_m^{\alpha,\beta}(z)+(\Phi_m^{\alpha,\beta}(z))^2\right\}\left\|\sigma''\right\|.\end{aligned}$$

On the other hand, taking (7), (8) and (13) into account, one has

$$\begin{aligned}\left|\widetilde{D}_{m,\lambda}^{\alpha,\beta}(\mu;z)\right| &\le \left|D_{m,\lambda}^{\alpha,\beta}(\mu;z)\right|+2\left\|\mu\right\|\\ &\le \left\|\mu\right\|D_{m,\lambda}^{\alpha,\beta}(1;z)+2\left\|\mu\right\|\le 3\left\|\mu\right\|.\end{aligned} \tag{15}$$

Further, with (14) and (15), we get

$$\begin{aligned}\left|D_{m,\lambda}^{\alpha,\beta}(\mu;z)-\mu(z)\right| &\le \left|\widetilde{D}_{m,\lambda}^{\alpha,\beta}(\mu-\sigma;z)-(\mu-\sigma)(z)\right|+\left|\widetilde{D}_{m,\lambda}^{\alpha,\beta}(\sigma;z)-\sigma(z)\right|\\ &+\left|\mu(z)-\mu(\phi_{m,\lambda}^{\alpha,\beta}(z))\right|\\ &\le 4\left\|\mu-\sigma\right\|+\left\{\Omega_m^{\alpha,\beta}(z)+(\Phi_m^{\alpha,\beta}(z))^2\right\}\left\|\sigma''\right\|+\omega\left(\mu;\Phi_m^{\alpha,\beta}(z)\right).\end{aligned}$$

If we take infimum on the right hand side over all $\sigma \in C^2[0,\infty)$ and by (12), so one can get following easily

$$\left|D_{m,\lambda}^{\alpha,\beta}(\mu;z)-\mu(z)\right| \leq 4K_2(\mu;\frac{\left\{\Omega_m^{\alpha,\beta}(z)+(\Phi_m^{\alpha,\beta}(z))^2\right\}}{4})+\omega(\mu;\Phi_m^{\alpha,\beta}(z))$$
$$\leq C\omega_2(\mu;\frac{1}{2}\sqrt{\Omega_m^{\alpha,\beta}(z)+(\Phi_m^{\alpha,\beta}(z))^2})+\omega(\mu;\Phi_m^{\alpha,\beta}(z)).$$

Thus, we arrive at the desired result. □

4. Weighted approximation

In this section, we prove a result concerning the weighted approximation for the operators $D_{m,\lambda}^{\alpha,\beta}$. Let $B_{z^2}[0,\infty)$ be the set of all functions h satisfying the condition $|h(z)| \leq M_h(1+z^2)$, $z \in [0,\infty)$ with constant M_h, which depend only on h. We denote by $C_{z^2}[0,\infty)$ the set of all continuous functions belonging to $B_{z^2}[0,\infty)$ endowed with the norm $\|h\|_{z^2} = \sup\limits_{z\in[0,\infty)} \frac{|h(z)|}{1+z^2}$ and $C_{z^2}^*[0,\infty) := \{h: \ h \in C_{z^2}[0,\infty), \ \lim\limits_{y\to\infty} \frac{|h(z)|}{1+z^2} < \infty\}$.

THEOREM 4.1. *For all $\mu \in C_{z^2}^*[0,\infty)$, we obtain*

$$\lim_{m\to\infty} \sup_{z\in[0,\infty)} \frac{\left|D_{m,\lambda}^{\alpha,\beta}(\mu;z)-\mu(z)\right|}{1+z^2} = 0.$$

Proof. Using the Korovkin type theorem given by Gadzhiev [18], we have to show that operators (6) verify the following condition:

$$\lim_{m\to\infty} \sup_{z\in[0,\infty)} \frac{\left|D_{m,\lambda}^{\alpha,\beta}(t^s;z)-z^s\right|}{1+z^2} = 0, \qquad s=0,1,2. \tag{16}$$

From (7), the first condition in (16) is clear for $s=0$.

For $s=1$, by (8), we find

$$\sup_{z\in[0,\infty)} \frac{\left|D_{m,\lambda}^{\alpha,\beta}(e_1;z)-z\right|}{1+z^2} \leq \left|\frac{(m-1)(\alpha+1)+\lambda}{(m+\beta)\,(m-1)}\right| \sup_{z\in[0,\infty)} \frac{1}{1+z^2}$$
$$+\left|\frac{(m-1)\beta+3\lambda}{(m+\beta)\,(m-1)}\right| \sup_{z\in[0,\infty)} \frac{z}{1+z^2},$$

which gives

$$\lim_{m\to\infty} \sup_{z\in[0,\infty)} \frac{\left|D_{m,\lambda}^{\alpha,\beta}(e_1;z)-z\right|}{1+z^2} = 0.$$

Also for $s = 2$, from (9), we get

$$\sup_{z\in[0,\infty)} \frac{\left|D^{\alpha,\beta}_{m,\lambda}(e_2;z) - z^2\right|}{1+z^2} \leq \left|\frac{((\alpha+1)^2+1)(m-1)+2(\alpha+1)\lambda}{(m+\beta)^2(m-1)}\right| \sup_{z\in[0,\infty)} \frac{1}{1+z^2}$$
$$+ \left|\frac{2m(\alpha+2)(m-1)-6(\alpha+1)\lambda)}{(m+\beta)^2(m-1)}\right| \sup_{z\in[0,\infty)} \frac{z}{1+z^2}$$
$$+ \left|\frac{\beta(2m+\beta)(m-1)+4(m+1)\lambda}{(m+\beta)^2(m-1)}\right| \sup_{z\in[0,\infty)} \frac{z^2}{1+z^2}.$$

Hence, we get

$$\lim_{m\to\infty} \sup_{z\in[0,\infty)} \frac{\left|D^{\alpha,\beta}_{m,\lambda}(e_2;z) - z^2\right|}{1+z^2} = 0.$$

This completes the proof. □

5. Voronovskaya type asymptotic theorem

In this section, we will consider Voronovskaja type asymptotic theorem. Before presenting our main theorem, let us give the following lemma, the results of which we will use.

Lemma 5.1. *Let $z \in [0,\infty)$ and $\lambda \in [-1,1]$. Then, we arrive at the following expressions:*

$$(i) \lim_{m\to\infty} mD^{\alpha,\beta}_{m,\lambda}(t-z;z) = \alpha+1,$$
$$(ii) \lim_{m\to\infty} mD^{\alpha,\beta}_{m,\lambda}((t-z)^2;z) = 2(\alpha+2+\beta)z,$$
$$(iii) \lim_{m\to\infty} m^2 D^{\alpha,\beta}_{m,\lambda}((t-z)^4;z) = 2z^2.$$

Theorem 5.2. *Let $\mu \in C^*_{z^2}[0,\infty)$ such that $\mu', \mu'' \in C^*_{z^2}[0,\infty)$ and $\lambda \in [-1,1]$, then we obtain for any $z \in [0,\infty)$ the following identity*

$$\lim_{m\to\infty} m\left[D^{\alpha,\beta}_{m,\lambda}(\mu;z) - \mu(z)\right] = (\alpha+1)\mu'(z) + (\alpha+2+\beta)z\mu''(z).$$

Proof. In view of Taylor's expansion formula, thus

$$\mu(t) = \mu(z) + (t-z)\mu'(z) + \frac{1}{2}(t-z)^2\mu''(z) + (t-z)^2\chi(t;z), \tag{17}$$

where $\chi(t;z)$ is a Peano of the rest term and for $\chi(.;z) \in C[0,\infty)$, we get $\lim_{t\to z}\chi(t;z) = 0$. Operating $D^{\alpha,\beta}_{m,\lambda}(.;z)$ on (17), we have

$$D^{\alpha,\beta}_{m,\lambda}(\mu;z) - \mu(z) = D^{\alpha,\beta}_{m,\lambda}((t-z);z)\mu'(z) + \frac{1}{2}D^{\alpha,\beta}_{m,\lambda}((t-z)^2;z)\mu''(z)$$
$$+ D^{\alpha,\beta}_{m,\lambda}((t-z)^2\chi(t;z);z).$$

Taking the limit of the both sides of above identity as $m \to \infty$, it follows

$$\lim_{m\to\infty} m(D^{\alpha,\beta}_{m,\lambda}(\mu;z) - \mu(z) = \lim_{m\to\infty} m \left(D^{\alpha,\beta}_{m,\lambda}((t-z);z)\mu'(z) + \frac{1}{2} D^{\alpha,\beta}_{m,\lambda}((t-z)^2;z)\mu''(z) \right.$$

(18)

$$\left. + D^{\alpha,\beta}_{m,\lambda}((t-z)^2\chi(t;z);z) \right).$$

Applying the Cauchy-Bunyakovsky-Schwarz inequality to the last term on the right hand side of the equation (18), thus

$$\lim_{m\to\infty} mD^{\alpha,\beta}_{m,\lambda}((t-z)^2\chi(t;z);z) \le \sqrt{\lim_{m\to\infty} D^{\alpha,\beta}_{m,\lambda}(\chi^2(t;z);z)} \sqrt{\lim_{m\to\infty} m^2 D^{\alpha,\beta}_{m,\lambda}((t-z)^4;z)}.$$

We observe that $\chi^2(z;z) = 0$ and $\chi^2(t;z) \in C_{z^2}[0,\infty)$. Hence

(19) $$\lim_{m\to\infty} D^{\alpha,\beta}_{m,\lambda}(\chi^2(t;z);z) = \chi^2(z;z) = 0,$$

uniformly with respect to $z \in [0,A]$, where $A > 0$. If we combine (18)-(19) and by Lemma 5.1 (iii), one has

$$\lim_{m\to\infty} mD^{\alpha,\beta}_{m,\lambda}((t-z)^2\chi(t;z);z) = 0.$$

Hence, we arrive at the following result

$$\lim_{m\to\infty} m \left[D^{\alpha,\beta}_{m,\lambda}(\mu;z) - \mu(z) \right] = (\alpha+1)\mu'(z) + (\alpha+2+\beta)z\mu''(z).$$

□

6. Graphical representation

In this section, we present the convergence of operators (6) to the certain functions with the different values of m, α, β and λ.

In Figure 1, we consider the function $\mu(z) = (z - 1/5)e^{-2z}$ to observe the convergence of the proposed operators for the values $\alpha = 0$, $\beta = 0.5$, $\lambda = 1$, $m = 10(red), 20(green), 50(blue)$, respectively.

In Figure 2, we consider the function $\mu(z) = (z - 1/5)e^{-2z}$ and we denote the error functions with $E^{\alpha,\beta}_{m,\lambda}(\mu;z) = \left| D^{\alpha,\beta}_{m,\lambda}(\mu;z) - \mu(z) \right|$. We choose $\alpha = 0.5$, $\beta = 2$ and $\lambda = 1$ and illustrate the error of approximation process of operators (6) to $\mu(z)$ for $m = 25(red), 50(green), 125(blue)$, respectively.

In Figure 3, we consider the trigonometric function $\mu(z) = \sin(\pi z)$. We compare the convergence of operators (6) to the function $\mu(z)$ (*black*) for $m = 10$, $\lambda = 1(red), 0(green), -1(blue)$, respectively.

One can check from Figure 1 that, as the values of m increases than the convergence of operators (6) to a function $\mu(z)$ is getting better. Moreover, in Figure 2 we show its error of approximation process for $m = 25, 50, 125$. We can see by Figure 3 that, in case $\lambda > 0$ operators (6) provides better approximation than in cases $\lambda = 0$ and $\lambda = -1$. Note that, the parameters α, β and λ give us more flexibility in modeling.

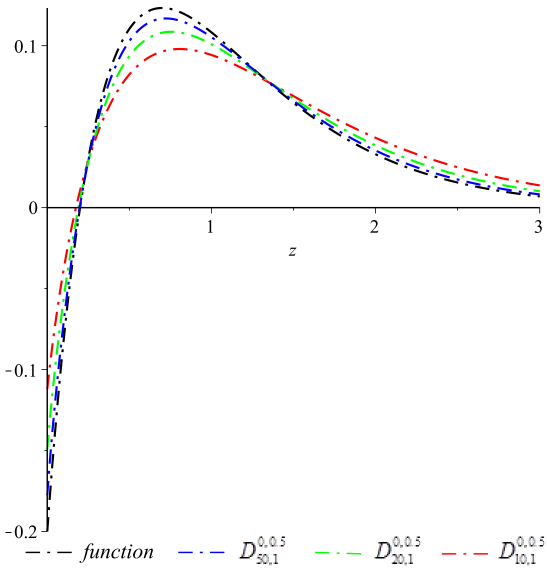

FIGURE 1. The convergence of operators $D^{0,0.5}_{m,1}(\mu; z)$ to $\mu(z) = (z - 1/5)e^{-2z}$

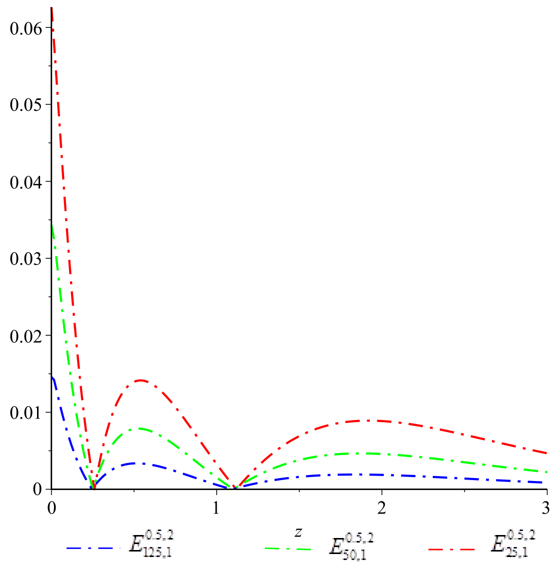

FIGURE 2. The error of approximation process of $D^{0.5,2}_{m,1}(\mu; z)$ to $\mu(z) = (z - 1/5)e^{-2z}$

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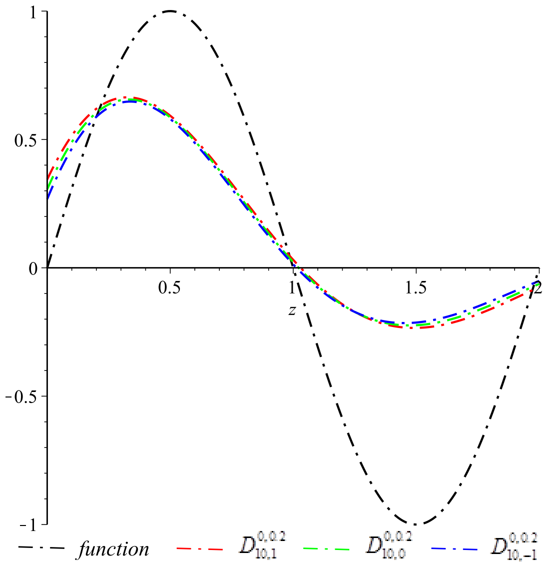

FIGURE 3. The convergence of operators $D^{0,0.2}_{10,\lambda}(\mu; z)$ to $\mu(z) = \sin(\pi z)$

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