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q-COEFFICIENT TABLE OF NEGATIVE EXPONENT POLYNOMIAL WITH q-COMMUTING VARIABLES

Eunmi Choi

Abstract. Let $N^{(q)}$ be an arithmetic table of a negative exponent polynomial with q-commuting variables. We study sequential properties of diagonal sums of $N^{(q)}$. We first device a q-coefficient table $\hat{N}$ of $N^{(q)}$, find sequences of diagonal sums over $\hat{N}$, and then retrieve the findings of $\hat{N}$ to $N^{(q)}$. We also explore recurrence rules of s-slope diagonal sums of $N^{(q)}$ with various s and q.

1. Introduction

Assume x and y are q-commuting variables satisfying $yx = qxy$ $(q \in \mathbb{Z})$. Let $(x+y)^i = \sum_{j=0}^{i} c_{i,j}^{(q)} x^{i-j} y^j$ be a polynomial and $C^{(q)} = [c_{i,j}^{(q)}]$ be its arithmetic table. By expanding $(x+y)^4$, for instance, we observe $c_{4,2}^{(q)} = 1 + q + 2q^2 + q^3 + q^4$ and write $c_{4,2}^{(q)} = \hat{c}_{4,2} \circ (1, q, q^2, q^3, q^4)$ with q-coefficients $\hat{c}_{4,2} = (1, 1, 2, 1, 1)$ (often 11211). In this way, let $c_{i,j}^{(q)} = \hat{c}_{i,j} \circ (1, q, q^2, \cdots)$ for all $i, j \geq 0$, and call $\hat{C} = [\hat{c}_{i,j}]$ the q-coefficient table of $C^{(q)}$.

Table 1. arithmetic table $C^{(q)} = [c_{i,j}^{(q)}]$ — q-coefficient table $\hat{C} = [\hat{c}_{i,j}]$

$i\backslash j$	0	1	2	3	0	1	2	3	4
1	1	1			1	1_1			
2	1	$1+q$	1		1	1_2	1		
3	1	$1+q+q^2$	$1+q+q^2$	1	1	1_3	1_3	1	
4	1	$1+q+q^2+q^3$	$1+q+2q^2+q^3+q^4$	$1+q+q^2+q^3$	1	1_4	$1_2 2 1_2$	1_4	1
5	1		$\cdots$		1	1_5	$1_2 2_3 1_2$	$1_2 2_3 1_2$	1_5

Here, $i_k = (\underbrace{i, \cdots, i}_{k-\text{tuple}})$. The tables $C^{(q)}$ and $\hat{C}$ satisfy recurrence rules

$$c_{i+1,j+1}^{(q)} = c_{i,j}^{(q)} + q^{j+1} c_{i,j+1}^{(q)} = q^{i-j} c_{i,j}^{(q)} + c_{i,j+1}^{(q)},$$
$$\hat{c}_{i+1,j+1} = \hat{c}_{i,j} + 0_{j+1}\hat{c}_{i,j+1} = 0_{i-j}\hat{c}_{i,j} + \hat{c}_{i,j+1}, \quad (i \geq j \geq 0) \tag{1}$$

where $0_k\hat{c}_{i,j}$ is a k-tuple of 0's followed by $\hat{c}_{i,j}$ ([2], [3]). Indeed, $\hat{c}_{6,3} = \hat{c}_{5,2} + 0_3\hat{c}_{5,3}$

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$= \left(\begin{array}{r} 1122211 \\ \underline{+0001122211} \\ 1123333211 \end{array}\right) = 1_2 23_4 21_2$. Clearly $C^{(1)} = [c^{(1)}_{i,j}]$ and $C^{(-1)} = [c^{(-1)}_{i,j}]$ are the Pascal and Pauli Pascal tables respectively [4], so the diagonal sums of $C^{(1)}$ and $C^{(-1)}$ give Fibonacci numbers and interlocked Fibonacci numbers [2].

Now for q-commuting variables satisfying $yx = qxy$, we consider a polynomial $(x+y)^{-i} = x^{-i}\sum_{j=0}^{\infty} n^{(q)}_{i,j} x^{-j} y^j$ $(i \geq 0)$ having negative exponent, and let $N^{(q)} = [n^{(q)}_{i,j}]$ be its arithmetic table. A purpose of the work is to study diagonal sums over $N^{(q)}$. When $q = \pm 1$, the diagonal sums of $N^{(q)}$ yield either Padovan or interlocked Padovan sequence [1]. But with any $q \in \mathbb{Z}$, the expansion of $(x+y)^{-i}$ has long expressions, for instance $(x+y)^{-1} = \frac{1}{x} - \frac{y}{2x^2} + \frac{y^2}{2^3x^3} - \frac{y^3}{2^6x^4} + \cdots$ with $q = 2$. So we first device a coefficient table $\hat{N}$ of $N^{(q)}$ which is independent of q, find sequential properties of diagonal sums over $\hat{N}$ and then retrieve the findings to $N^{(q)}$. In fact, we prove the table $\hat{N}$ is a type of skew symmetricity (Theorem 3) and the sequence of diagonal sums over $\hat{N}$ is a sort of generalized opposite fibonacci sequence (Theorem 6). After then, considering various s and q's, we explore general s-slope diagonal sums and their sequential properties over $N^{(q)}$. In this work, $r_i(M)$ and $c_j(M)$ denote the i^{th}row and the j^{th}column of a matrix M for $i, j \geq 0$.

2. Construction of coefficient tables of $N^{(q)}$

The table $C^{(q)}$ of $(x+y)^i$ for $i \geq 0$ can be extended to all integers $i \in \mathbb{Z}$ following the recurrence (1). In fact, the 0^{th} row $r_0(C^{(q)}) = (1, 0, 0, 0, \cdots)$ yields

the $(-1)^{\text{th}}$row: $(1, -\frac{1}{q}, \frac{1}{q^3}, -\frac{1}{q^6}, \frac{1}{q^{10}}, -\frac{1}{q^{15}}, \frac{1}{q^{21}}, \cdots)$,

the $(-2)^{\text{th}}$row: $(1, -\frac{1}{q}(1+\frac{1}{q}), \frac{1}{q^3}(1+\frac{1}{q}+\frac{1}{q^2}), \cdots)$,

and so on. So the upper part of the extended table of $C^{(q)}$ corresponds to the negative arithmetic table $N^{(q)} = [n^{(q)}_{i,j}]$ of $(x+y)^{-i}$ with $yx = qxy$ $(q \in \mathbb{Z})$. If let $\frac{1}{q} = p$ then a recurrence of $N^{(q)}$ comes from the rule (1) of $C^{(q)}$ that

$$n^{(q)}_{i+1,j+1} = p^{j+1}(n^{(q)}_{i,j+1} - n^{(q)}_{i+1,j}). \tag{2}$$

Indeed the 1^{th}row $r_1(N^{(q)}) = (1, -p, p^3, -p^6, p^{10}, -p^{15}, p^{21}, \cdots)$, and $n^{(q)}_{2,1} = p(n^{(q)}_{1,1} - n^{(q)}_{2,0}) = -p(1+p)$, $n^{(q)}_{2,2} = p^2(n^{(q)}_{1,2} - n^{(q)}_{2,1}) = p^2(p^3 + p(1+p)) = p^3(1+p+p^2)$, etc. Hence $N^{(q)}$ forms as in Table 2.

Table 2. $N^{(q)} = [n^{(q)}_{i,j}]$ $(i \geq 1, j \geq 0)$ with $p = \frac{1}{q}$

$i\backslash j$	0	1	2	3	4
1	1	$-p$	p^3	$-p^6$	p^{10}
2	1	$-p(1+p)$	$p^3(1+p+p^2)$	$-p^6(1+p+p^2+p^3)$	$p^{10}(1+\cdots+p^4)$
3	1	$-p(1+p+p^2)$	$p^3(1+p+2p^2+p^3+p^4)$	$\cdots$	

We observe $n^{(q)}_{3,1} = -p(1+p+p^2) = \hat{n}_{3,1} \circ (1, p, p^2, p^3)$ with p-coefficient $\hat{n}_{3,1} = -(0, 1, 1, 1) = -01_3$, and $n^{(q)}_{2,2} = \hat{n}_{2,2} \circ (1, p, \cdots, p^5)$ with $\hat{n}_{2,2} = (0, 0, 0, 1, 1, 1) = 0_3 1_3$. Thus for all $i \geq 1$, $j \geq 0$, by letting

$$n^{(q)}_{i,j} = \hat{n}_{i,j} \circ (1, p, p^2, \cdots) \text{ with } p = \frac{1}{q}, \tag{3}$$

we have the p-coefficient table $\hat{N} = [\hat{n}_{i,j}]$ of $N^{(q)}$ as in Table 3.

Table 3. p-coefficient table $\hat{N} = [\hat{n}_{i,j}]$ $(i \geq 1, j \geq 0)$

$i\backslash j$	0	1	2	3	4	5
1	1	-01	$0_3 1$	$-0_6 1$	$0_{10} 1$	$-0_{15} 1$
2	1	-01_2	$0_3 1_3$	$-0_6 1_4$	$0_{10} 1_5$	$-0_{15} 1_6$
3	1	-01_3	$0_3 1_2 21_2$	$-0_6 1_2 2_3 1_2$	$0_{10} 1_2 2_2 32_2 1_2$	$-0_{15} 1_2 2_2 3_3 2_2 1_2$
4	1	-01_4	$0_3 1_2 2_3 1_2$	$-0_6 1_2 23_4 21_2$	$0_{10} 1_2 234_2 54_2 321_2$	$-0_{15} 1_2 23456_4 54321_2$
5	1	-01_5	0_3 $1_2 2_2 3$ $2_2 1_2$	-0_6 $1_2 234_2 5$ $4_2 321_2$	0_{10} $1_2 235_2 7_2 8$ $7_2 5_2 321_2$	-0_{15} $1_2 235689(11)_2(12)$ $(11)_2 9865321_2$

Here i_k means the k-tuple of i's, as in Table 1. Clearly the recurrence (2) of $N^{(q)}$ can be transformed over $\hat{N}$ to

$$\hat{n}_{i+1,j+1} = 0_{j+1}(\hat{n}_{i,j+1} - \hat{n}_{i+1,j}). \tag{4}$$

LEMMA 1. [2] *Let the length len($\hat{c}_{i,j}$) be the number of digits in the q-coefficient $\hat{c}_{i,j}$ of $\hat{C}$. Then* $\text{len}(\hat{c}_{i,j}) = 1 + (i-j)j$ *if $i \geq j$. Otherwise, $\hat{c}_{i,j} = 0$.*

For instance, $\text{len}(\hat{c}_{7,3}) = 13$ so $\hat{c}_{7,3} \circ (1, q, \cdots, q^{12}) = c^{(q)}_{7,3}$. Similar to this, the next theorem finds the length $\text{len}(\hat{n}_{i,j})$ of the p-coefficient $\hat{n}_{i,j}$ in $\hat{N}$.

THEOREM 2. *Let len($\hat{n}_{i,j}$) be the number of digits in $\hat{n}_{i,j}$ for $i \geq 1$, $j \geq 0$. Then $\hat{n}_{i,j} = (-1)^j 0_{\lambda_j} \hat{c}_{i+j-1,j}$ with $\lambda_j = \frac{j(j+1)}{2}$, and* $\text{len}(\hat{n}_{i,j}) = \lambda_j + 1 + (i-1)j$.

Proof. Tables 1 and 3 show $\hat{n}_{2,2} = 0_3 111 = 0_3\hat{c}_{3,1} = 0_3\hat{c}_{3,2}$, $\hat{n}_{2,3} = -0_6 1_4 = -0_6\hat{c}_{4,3}$ and $\hat{n}_{2,4} = 0_{10}1_5 = 0_{10}\hat{c}_{5,4}$. And the recurrence (4) implies, for instance,

$$\hat{n}_{4,7} = 0_7(-0_{28}1_2 2\cdots 1_2 - 0_{21}1_2 2\cdots 1_2) = -0_{28}(1_2 234\cdots 1_2) = -0_{28}\hat{c}_{10,7},$$
$$\hat{n}_{5,6} = 0_6(0_{21}1_2\cdots 1_2 + 0_{15}1_2\cdots 1_2) = 0_{21}(1_2 235\cdots 1_2) = 0_{21}\hat{c}_{10,6}.$$

Now for some i, j, we assume $\hat{n}_{i,j} = (-1)^j 0_{\lambda_j}\hat{c}_{i+j-1,j}$ with $\lambda_j = \sum_{k=1}^{j} k$. Then

$$\begin{aligned}\hat{n}_{i+1,j} &= 0_j(\hat{n}_{i,j} - n_{i+1,j-1})\\ &= 0_j((-1)^j 0_{\lambda_j}\hat{c}_{i+j-1,j} - (-1)^{j-1}0_{\lambda_{j-1}}\hat{c}_{i+j-1,j-1})\\ &= (-1)^j 0_j 0_{\lambda_{j-1}}(0_j\hat{c}_{i+j-1,j} + \hat{c}_{i+j-1,j-1}) = (-1)^j 0_{\lambda_j}\hat{c}_{i+j,j},\end{aligned}$$

because $\lambda_j = \lambda_{j-1} + j$. Similarly

$$\begin{aligned}\hat{n}_{i,j+1} &= 0_{j+1}((-1)^{j+1}0_{\lambda_{j+1}}\hat{c}_{i+j-1,j+1} - (-1)^j 0_{\lambda_j}\hat{c}_{i+j-1,j})\\ &= (-1)^{j+1}0_{j+1}0_{\lambda_j}(0_{j+1}\hat{c}_{i+j-1,j+1} + \hat{c}_{i+j-1,j}) = (-1)^{j+1}0_{\lambda_{j+1}}\ \hat{c}_{i+j,j+1}.\end{aligned}$$

Moreover due to Lemma 1, the length of $\hat{n}_{i,j}$ is

$$\text{len}(\hat{n}_{i,j}) = \text{len}((-1)^j 0_{\lambda_j}\hat{c}_{i+j-1,j}) = \lambda_j + \text{len}(\hat{c}_{i+j-1,j}) = \lambda_j + 1 + (i-1)j. \qquad \square$$

The Pascal table $C^{(1)}$ provide binomial expansions $(x+y)^i$, say $r_4(C^{(1)})$ yields $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$. As a generalization, $C^{(q)}$ as well as $\hat{C}$ in Table 1 give expansions $(x+y)^i$ with q-commuting variables, for instance

$$\begin{aligned}(x+y)^4 &= x^4 + (1+q+q^2+q^3)x^3y + (1+q+2q^2+q^3+q^4)x^2y^2 + (1+q+q^2+q^3)xy^3 + y^4\\ &= x^4 + 1_4\circ(1,\cdots,q^3)x^3y + 1_2 21_2\circ(1,\cdots,q^4)x^2y^2 + 1_4\circ(1,\cdots,q^3)xy^3 + y^4.\end{aligned}$$

Similarly, the $N^{(q)}$ and $\hat{N}$ in Table 3 provide expansions $(x+y)^{-i}$. For example with $p = 1/q$, $r_4(\hat{N})$ yields

$$\begin{aligned}(x+y)^{-4} &= x^{-4} - 01_4\circ(1,p,\cdots,p^4)x^{-5}y + 0_3 1_2 2_3 1_2\circ(1,\cdots,p^9)x^{-6}y^2\\ &\quad -0_6 1_2 23_4 21_2\circ(1,\cdots,p^{15})x^{-7}y^3 + 0_{10}1_2 234_2 54_2 321_2\circ(1,\cdots,p^{22})x^{-8}y^4 + \cdots,\end{aligned}$$

where $\text{len}(\hat{n}_{4,2}) = 10$, $\text{len}(\hat{n}_{4,3}) = 16$ and $\text{len}(\hat{n}_{4,4}) = 23$ by Theorem 2. So if $q = 1$ then $(x+y)^{-4} = x^{-4} - 4x^{-5}y + 10x^{-6}y^2 - 20x^{-7}y^3 + 35x^{-8}y^4 + \cdots$. Theorem 2 shows

$n^{(q)}_{i,j}$ in $N^{(q)}$ equals $(-1)^j \hat{c}_{i+j-1,j}$ inner product with $(p^{\lambda_j}, p^{\lambda_j+1}, \cdots, p^{\mathrm{len}(\hat{n}_{i,j})-1})$, so the table $\hat{N}$ in Table 3 can be represented in terms of $\hat{c}_{i,j}$ as in Table 4.

Table 4. $\hat{N} = [\hat{n}_{i,j}]$

$i\backslash j$	0	1	2	3	4	5	6	7
1	$\hat{c}_{0,0}$	$-0\hat{c}_{1,1}$	$0_3\hat{c}_{2,2}$	$-0_6\hat{c}_{3,3}$	$-0_{10}\hat{c}_{4,4}$	$-0_{15}\hat{c}_{5,5}$	$0_{21}\hat{c}_{6,6}$	$-0_{28}\hat{c}_{7,7}$
2	$\hat{c}_{1,0}$	$-0\hat{c}_{2,1}$	$0_3\hat{c}_{3,2}$	$-0_6\hat{c}_{4,3}$	$-0_{10}\hat{c}_{5,4}$	$-0_{15}\hat{c}_{6,5}$	$0_{21}\hat{c}_{7,6}$	$-0_{28}\hat{c}_{8,7}$
3	$\hat{c}_{2,0}$	$-0\hat{c}_{3,1}$	$0_3\hat{c}_{4,2}$	$-0_6\hat{c}_{5,3}$	$-0_{10}\hat{c}_{6,4}$	$-0_{15}\hat{c}_{7,5}$	$0_{21}\hat{c}_{8,6}$	$-0_{28}\hat{c}_{9,7}$
4	$\hat{c}_{3,0}$	$-0\hat{c}_{4,1}$	$0_3\hat{c}_{5,2}$	$-0_6\hat{c}_{6,3}$	$-0_{10}\hat{c}_{7,4}$	$-0_{15}\hat{c}_{8,5}$	$0_{21}\hat{c}_{9,6}$	$-0_{28}\hat{c}_{10,7}$
5	$\hat{c}_{4,0}$	$-0\hat{c}_{5,1}$	$0_3\hat{c}_{6,2}$	$-0_6\hat{c}_{7,3}$	$-0_{10}\hat{c}_{8,4}$	$-0_{15}\hat{c}_{9,5}$	$0_{21}\hat{c}_{10,6}$	$-0_{28}\hat{c}_{11,7}$

Table 4 gives more relations between $\hat{N}$ and $\hat{C}$ that if we place λ_j zeros in front of nonzero entries in $c_j(\hat{C})$ then we get $(-1)^j c_j(\hat{N})$, in fact,

$$0_{\lambda_j} c_j(\hat{C}) = 0_{\lambda_j} \begin{bmatrix} \hat{c}_{j,j} \\ \hat{c}_{j+1,j} \\ \hat{c}_{j+2,j} \\ \cdots \end{bmatrix} = (-1)^j \begin{bmatrix} \hat{n}_{1,j} \\ \hat{n}_{2,j} \\ \hat{n}_{3,j} \\ \cdots \end{bmatrix} = (-1)^j c_j(\hat{N}).$$

Moreover $r_i(\hat{N})$ is obtained from $c_{i-1}(\hat{C})$, for example, $r_4(\hat{N}) = \{1, -01_4, 0_31_22_31_2, -0_61_223_421_2, \cdots\}$ in Table 3 and $c_3(\hat{C}) = \{1, 1_4, 1_22_31_2, 1_223_421_2, \cdots\}$ in Table 1. Theorem 3 shows $\hat{N}$ satisfies a type of skew symmetricity (See Table 5).

THEOREM 3. $\hat{n}_{i,j} = (-1)^{j-i-1} 0_{\mu_j} \hat{n}_{j+1,i-1}$ *with* $\mu_j = \sum_{k=i}^{j} k$ *for* $j \geq i \geq 1$.

Proof. Note $\hat{n}_{3,3} = -0_3\hat{n}_{4,2}$, $\hat{n}_{3,4} = 0_7\hat{n}_{5,2}$ and $\hat{n}_{3,5} = -0_{12}\hat{n}_{6,2}$ from Table 4. The symmetricity $\hat{c}_{n,k} = \hat{c}_{n,n-k}$ of $\hat{C}$ provides $\hat{n}_{2,5} = -0_{15}\hat{c}_{6,5} = 0_{14}(-0\hat{c}_{6,1}) = 0_{14}\hat{n}_{6,1}$, $\hat{n}_{3,5} = -0_{15}\hat{c}_{7,5} = -0_{12}(0_3\hat{c}_{7,2}) = -0_{12}\hat{n}_{6,2}$, and $\hat{n}_{4,5} = -0_{15}\hat{c}_{8,5} = -0_{15}\hat{c}_{8,3} = 0_9(-0_6\hat{c}_{8,3}) = 0_9\hat{n}_{6,3}$, etc. Hence Theorem 2 shows

$$\begin{aligned}\hat{n}_{i,j} &= (-1)^j 0_{\lambda_j}\hat{c}_{i+j-1,j} = (-1)^j 0_{\lambda_j}\hat{c}_{i+j-1,i-1} \\ &= (-1)^{j-i+1}(-1)^{i-1}0_{\mu_j}0_{\lambda_{i-1}}\hat{c}_{i+j-1,i-1} \\ &= (-1)^{j-i+1}0_{\mu_j}((-1)^{i-1}0_{\lambda_{i-1}}\hat{c}_{i+j-1,i-1}) = (-1)^{j-i-1}0_{\mu_j}\hat{n}_{j+1,i-1},\end{aligned}$$

because $\lambda_j = \sum_{k=1}^{j} k = \lambda_{i-1} + \mu_j$ for $j \geq i$. □

Table 5. $\hat{N} = [\hat{n}_{i,j}]$

$i\backslash j$	0	1	2	3	4	5	6
1	$\mathbf{\hat{n}_{1,0}}$	$-0\hat{n}_{2,0}$	$0_3\hat{n}_{3,0}$	$-0_6\hat{n}_{4,0}$	$0_{10}\hat{n}_{5,0}$	$-0_{15}\hat{n}_{6,0}$	$0_{21}\hat{n}_{7,0}$
2	$\hat{n}_{2,0}$	$\mathbf{\hat{n}_{2,1}}$	$-0_2\hat{n}_{3,1}$	$0_5\hat{n}_{4,1}$	$-0_9\hat{n}_{5,1}$	$0_{14}\hat{n}_{6,1}$	$-0_{20}\hat{n}_{7,1}$
3	$\hat{n}_{3,0}$	$\hat{n}_{3,1}$	$\mathbf{\hat{n}_{3,2}}$	$-0_3\hat{n}_{4,2}$	$0_7\hat{n}_{5,2}$	$-0_{12}\hat{n}_{6,2}$	$0_{18}\hat{n}_{7,2}$
4	$\hat{n}_{4,0}$	$\hat{n}_{4,1}$	$\hat{n}_{4,2}$	$\mathbf{\hat{n}_{4,3}}$	$-0_4\hat{n}_{5,3}$	$0_9\hat{n}_{6,3}$	$-0_{15}\hat{n}_{7,3}$
5	$\hat{n}_{5,0}$	$\hat{n}_{5,1}$	$\hat{n}_{5,2}$	$\hat{n}_{5,3}$	$\mathbf{\hat{n}_{5,4}}$	$-0_5\hat{n}_{6,4}$	$0_{11}\hat{n}_{7,4}$

THEOREM 4. $\hat{n}_{i+1,j+1} = 0_{j+1}(\hat{n}_{i,j+1} - \hat{n}_{i+1,j})$ *and* $\hat{n}_{i,j} - \hat{n}_{i+1,j} = 0_{i+j}\hat{n}_{i+1,j-1}$.

Proof. Theorem 2 together with the identity (2) shows

$$\begin{aligned}\hat{n}_{i,j+1} - \hat{n}_{i+1,j} &= (-1)^{j+1}0_{\lambda_{j+1}}\hat{c}_{i+j,j+1} - (-1)^j 0_{\lambda_j}\hat{c}_{i+j,j} \\ &= (-1)^{j+1}0_{\lambda_j}(0_{j+1}\hat{c}_{i+j,j+1} + \hat{c}_{i+j,j}) = (-1)^{j+1}0_{\lambda_j}\hat{c}_{i+j+1,j+1}\end{aligned}$$

where $\lambda_j = \frac{j(j+1)}{2}$. Thus

$$0_{j+1}(\hat{n}_{i,j+1} - \hat{n}_{i+1,j}) = (-1)^{j+1}0_{j+1}0_{\lambda_j}\hat{c}_{i+j+1,j+1} = \hat{n}_{i+1,j+1},$$

because $j + 1 + \lambda_j = \lambda_{j+1}$. Moreover we also have

$$\begin{aligned}\hat{n}_{i,j} - \hat{n}_{i+1,j} &= (-1)^{j-1}0_{\lambda_j}(\hat{c}_{i+j,j} - \hat{c}_{i+j-1,j}) = (-1)^{j-1}0_{\lambda_j}(0_i\hat{c}_{i+j-1,j-1}) \\ &= 0_{i+j}(-1)^{j-1}0_{\lambda_{j-1}}\hat{c}_{i+j-1,j-1} = 0_{i+j}\hat{n}_{i+1,j-1}.\end{aligned}$$

□

3. Diagonal sums of $\hat{N}$ and $N^{(q)}$

A s/t-slope diagonal over a table means a diagonal that moves t steps along x axis direction and s steps along y axis direction for $s \geq 0$, $t \geq 1$. We simply call it s-slope if $t = 1$. Over the table $\hat{C}$, let $\hat{d}_{\langle s/t\rangle,i}$ denote the set of entries on s/t-slope diagonal starting from $\hat{c}_{i,0}$ $(i \geq 0)$ toward northeast direction, and $\hat{D}_{\langle s/t\rangle,i}$ be the s/t-slope i^{th}diagonal sum, i.e., the sum of all entries in $\hat{d}_{\langle s/t\rangle,i}$. Similarly, over $\hat{N}$, $\hat{g}_{\langle s/t\rangle,j}$ denotes the set of entries on s/t-slope diagonal starting from $\hat{n}_{1,j}$ $(j \geq 0)$ toward southwest direction, and $\hat{G}_{\langle s/t\rangle,j}$ is the s/t-slope j^{th}diagonal sum. We simply write $\hat{D}_{\langle s\rangle,i}$ and $\hat{G}_{\langle s\rangle,j}$ if $t = 1$. Analogous to $\hat{D}_{\langle s/t\rangle,i}$ over $\hat{C}$ [resp. $\hat{G}_{\langle s/t\rangle,j}$ over $\hat{N}$], let $D^{(q)}_{\langle s/t\rangle,i}$ [resp. $G^{(q)}_{\langle s/t\rangle,j}$] be the corresponding notion over $C^{(q)}$ [resp. $N^{(q)}$]. Theorem 5 shows some recurrence rules of $\hat{G}_{\langle 1/s\rangle,i}$.

Theorem 5. *For any $i > 1$, $\hat{G}_{\langle 1\rangle,i+1} = \hat{G}_{\langle 1\rangle,i} - 0_{i+1}\hat{G}_{\langle 1\rangle,i}$, where $0_{i+1}\hat{G}_{\langle 1\rangle,i}$ is the $(i+1)$ tuple of 0's followed by $\hat{G}_{\langle 1\rangle,i}$, and moreover $G^{(q)}_{\langle 1\rangle,i+1} = G^{(q)}_{\langle 1\rangle,i}(1 - p^{i+1})$ with $p = \frac{1}{q}$. In particular $G^{(q)}_{\langle 1\rangle,2} = (1 + G^{(q)}_{\langle 1\rangle,1})(1 - p^2)$.*

Proof. By means of the next table, we observe the followings.

i	$\hat{g}_{\langle 1\rangle,i}$	$\hat{G}_{\langle 1\rangle,i}$
1	$\{-01\}$	-01
2	$\{0_31, -01_2, 1\}$	$1(-1)_21$
3	$\{-0_61, 0_31_3, -01_3, 1\}$	$1(-1)_201_2(-1)$
4	$\{0_{10}1, -0_61_4, 0_31_221_2, -01_4, 1\}$	$1(-1)_20_220_2(-1)_21$
5	$\{-0_{15}1, 0_{10}1_5, -0_61_22_31_2, 0_31_22_31_2, -01_5, 1\}$	$1(-1)_20_21_3(-1)_30_21_2(-1)$

$\hat{G}_{\langle 1\rangle,2} = 1001 + 0(-1)(-1) = 1(-1)_21$,
$\hat{G}_{\langle 1\rangle,3} = (0_31_3 + 1) - (0_61 + 01_3) = (10_21_3) - (01_30_21) = 1(-1)_201_2(-1)$,
$\hat{G}_{\langle 1\rangle,4} = (0_{10}1 + 0_31_221_2 + 1) - (0_61_4 + 01_4) = 1(-1)_20_220_2(-1)_21$, etc. So we have
$\hat{G}_{\langle 1\rangle,2} - 0_3\hat{G}_{\langle 1\rangle,2} = 1(-1)_21 - 0_31(-1)_21 = 1(-1)_201_2(-1) = \hat{G}_{\langle 1\rangle,3}$,
$\hat{G}_{\langle 1\rangle,3} - 0_4\hat{G}_{\langle 1\rangle,3} = 1(-1)_201_2(-1) - 0_41(-1)_201_2(-1) = \hat{G}_{\langle 1\rangle,4}$.
Therefore for any $i > 1$, by Theorem 4 we have

$$\hat{G}_{\langle 1\rangle,i} - 0_{i+1}\hat{G}_{\langle 1\rangle,i} = \begin{pmatrix} \hat{n}_{1,i} \\ \cdots \\ \hat{n}_{k,i-k+1} \\ \cdots \\ \hat{n}_{i,1} \\ +\hat{n}_{i+1,0} \end{pmatrix} - 0_{i+1}\begin{pmatrix} \hat{n}_{1,i} \\ \cdots \\ \hat{n}_{k,i-k+1} \\ \cdots \\ \hat{n}_{i,1} \\ +\hat{n}_{i+1,0} \end{pmatrix} = \begin{pmatrix} -0_{i+1}\hat{n}_{1,i} \\ \hat{n}_{1,i} - 0_{i+1}\hat{n}_{2,i-1} \\ \hat{n}_{2,i-1} - 0_{i+1}\hat{n}_{3,i-2} \\ \cdots \\ \hat{n}_{i,1} - 0_{i+1}\hat{n}_{i+1,0} \\ +\hat{n}_{i+1,0} \end{pmatrix} = \begin{pmatrix} \hat{n}_{1,i+1} \\ \hat{n}_{2,i} \\ \hat{n}_{3,i-1} \\ \cdots \\ \hat{n}_{i+1,1} \\ +\hat{n}_{i+2,0} \end{pmatrix} = \hat{G}_{\langle 1\rangle,i+1}.$$

Now over $N^{(q)}$, since $\hat{G}_{\langle 1\rangle,i} \circ (1, p, p^2, \cdots) = G^{(q)}_{\langle 1\rangle,i}$ and $0_{i+1}\hat{G}_{\langle 1\rangle,i} \circ (1, p, p^2, \cdots) = p^{i+1}G^{(q)}_{\langle 1\rangle,i}$, we have

$$G^{(q)}_{\langle 1\rangle,i+1} = (\hat{G}_{\langle 1\rangle,i} - 0_{i+1}\hat{G}_{\langle 1\rangle,i}) \circ (1, p, p^2, \cdots) = G^{(q)}_{\langle 1\rangle,i} - p^{i+1}G^{(q)}_{\langle 1\rangle,i}.$$

In particular, $G^{(q)}_{\langle 1\rangle,2} = 1 - p - p^2 + p^3 = (1 + G^{(q)}_{\langle 1\rangle,1})(1 - p^2)$. □

Clearly all diagonal sums $G^{(1)}_{\langle 1\rangle,i}$ $(i > 1)$ over $N^{(1)}$ are zeros. A sequence $\{a_i\}$ satisfying a recurrence $a_{i+1} = a_i + a_{i-k}$ $(k > 0)$ is called a k-fibonacci sequence([5], [6]), so it is fibonacci if $k = 1$. And $\{a_i\}$ is called an opposite k-fibonacci sequence if

$a_i = a_{i+k} + a_{i+(k-1)}$ $(k > 1)$, so it is opposite fibonacci if $k = 2$. Consider a certain modified 1/2-slope diagonal sum $\hat{\mathfrak{G}}_{\langle 1/2\rangle,i} = \begin{pmatrix} 0_{i+1}\ \hat{n}_{1,i} \\ 0_i\ \ \hat{n}_{2,i-2} \\ \cdots \\ 0_{i-k+2}\hat{n}_{k,i-2(k-1)} \\ +\cdots \end{pmatrix}$, in which $(i-k+1)$ zeros are placed in front of each diagonal entries $\hat{n}_{k,i-2k}$ in $\hat{N}$.

THEOREM 6. *The 1/2-slope diagonal sums $\hat{G}_{\langle 1/2\rangle,i}$ satisfy an opposite fibonacci rule $\hat{G}_{\langle 1/2\rangle,i+2} + \hat{\mathfrak{G}}_{\langle 1/2\rangle,i+1} = \hat{G}_{\langle 1/2\rangle,i}$ by modified diagonal sum $\hat{\mathfrak{G}}_{\langle 1/2\rangle,i}$.*

Proof. Observe the diagonal sets $\hat{g}_{\langle 1/2\rangle,i}$ and diagonal sums $\hat{G}_{\langle 1/2\rangle,i}$ $(i = 1, 2, 3)$.

i	$\hat{g}_{\langle 1/2\rangle,i}$	$\hat{G}_{\langle 1/2\rangle,i}$	i	$\hat{g}_{\langle 1/2\rangle,i}$	$\hat{G}_{\langle 1/2\rangle,i}$
1	$\{-01\}$	-01	4	$\{0_{10}1, 0_31_3, 1\}$	$10_21_30_41$
2	$\{0_31, 1\}$	10_21	5	$\{-0_{15}1, -0_61_4, -01_3\}$	$-01_30_21_40_51$
3	$\{-0_61, -01_2\}$	-01_20_31	6	$\{0_{21}1, 0_{10}1_5, 0_31_221_2, 1\}$	$10_21_221_20_21_50_61$

Indeed, $\hat{G}_{\langle 1/2\rangle,3} = -\begin{pmatrix} 0000001 \\ +011 \end{pmatrix} = -01_20_31$, $\hat{G}_{\langle 1/2\rangle,4} = \begin{pmatrix} 00000000001 \\ 000111 \\ +1 \end{pmatrix} = 10_21_30_41$, and so on. So we have the identities

$$\hat{G}_{\langle 1/2\rangle,2} - \hat{\mathfrak{G}}_{\langle 1/2\rangle,3} = \begin{pmatrix} 0_31 \\ +1 \end{pmatrix} + \begin{pmatrix} 0_40_61 \\ +0_301_2 \end{pmatrix} = \begin{pmatrix} 0_{10}1 \\ 0_31_3 \\ +1 \end{pmatrix} = \hat{G}_{\langle 1/2\rangle,4},$$

$$\hat{G}_{\langle 1/2\rangle,3} - \hat{\mathfrak{G}}_{\langle 1/2\rangle,4} = -\begin{pmatrix} 0_61 \\ +01_2 \end{pmatrix} - \begin{pmatrix} 0_50_{10}1 \\ 0_40_31_3 \\ +0_31 \end{pmatrix} = -\begin{pmatrix} 0_{15}1 \\ 0_61_4 \\ +01_3 \end{pmatrix} = \hat{G}_{\langle 1/2\rangle,5},$$

etc. Therefore for any $i \geq 1$, we have

$$\hat{G}_{\langle 1/2\rangle,i} - \hat{\mathfrak{G}}_{\langle 1/2\rangle,i+1} = \begin{pmatrix} \hat{n}_{1,i} \\ \hat{n}_{2,i-2} \\ \hat{n}_{3,i-2(2)} \\ \cdots \\ \hat{n}_{k,i-2(k-1)} \\ +\cdots \end{pmatrix} - \begin{pmatrix} 0_{i+2}\hat{n}_{1,i+1} \\ 0_{i+1}\hat{n}_{2,i-1} \\ 0_i\hat{n}_{3,i-3} \\ \cdots \\ \hat{0}_{i-k+2}n_{k+1,(i+1)-2k} \\ +\cdots \end{pmatrix}$$

$$= \begin{pmatrix} -0_{i+2}\hat{n}_{1,i+1} \\ \hat{n}_{1,i} - 0_{i+1}\hat{n}_{2,i-1} \\ \hat{n}_{2,i-2} - 0_i\hat{n}_{3,i-3} \\ \cdots \\ \hat{n}_{k,i-2(k-1)} - 0_{i-k+2}\hat{n}_{k+1,(i+1)-2k} \\ +\cdots \end{pmatrix} = \begin{pmatrix} \hat{n}_{1,i+2} \\ \hat{n}_{2,i} \\ \hat{n}_{3,i-2} \\ \cdots \\ \hat{n}_{k+1,i-2(k-1)} \\ +\cdots \end{pmatrix} = \hat{G}_{\langle 1/2\rangle,i+2},$$

since $-0_{i+2}\hat{n}_{1,i+1} = \hat{n}_{1,i+2}$, $\hat{n}_{1,i} - 0_{i+1}\hat{n}_{2,i-1} = \hat{n}_{2,i}$, $\hat{n}_{2,i-2} - 0_i\hat{n}_{3,i-3} = \hat{n}_{3,i-2}$, and $\hat{n}_{k,i-2(k-1)} - 0_{i-k+2}\hat{n}_{k+1,(i+1)-2k} = \hat{n}_{k+1,i-2(k-1)}$, etc, by Theorem 4. □

THEOREM 7. *$\{\hat{G}_{\langle 1/s\rangle,i}\}$ satisfies a type of opposite s-fibonacci rule $\hat{G}_{\langle 1/s\rangle,i+s} + \hat{\mathfrak{G}}_{\langle 1/s\rangle,i+(s-1)} = \hat{G}_{\langle 1/s\rangle,i}$ where $\hat{\mathfrak{G}}_{\langle 1/s\rangle,i} = \begin{pmatrix} 0_{i+1}\hat{n}_{1,i} \\ 0_{i+1-(s-1)}\hat{n}_{2,i-s} \\ \cdots \\ 0_{i+1-(s-1)(k-1)}\hat{n}_{k,i-s(k-1)} \\ +\cdots \end{pmatrix}$ is a modified 1/s-slope diagonal sum in $\hat{N}$.*

Proof. If $s = 2$, see Theorem 6. For $s = 3, 4$, we have $1/s$-slope diagonal sums

i	$\hat{g}_{\langle 1/3\rangle,i}$	$\hat{G}_{\langle 1/3\rangle,i}$
1	$\{-01\}$	-01
2	$\{0_31\}$	0_31
3	$\{-0_61, 1\}$	$10_5(-1)$
4	$\{0_{10}1, -01_2\}$	$0(-1)_20_71$
5	$\{-0_{15}1, 0_31_3\}$	$0_31_30_9(-1)$

i	$\hat{g}_{\langle 1/4\rangle,i}$	$\hat{G}_{\langle 1/4\rangle,i}$
1	$\{-01\}$	-01
2	$\{0_31\}$	0_31
3	$\{-0_61\}$	-0_61
4	$\{0_{10}1, 1\}$	10_91
5	$\{-0_{15}1, -01_2\}$	$-01_20_{12}1$

and observe $\hat{G}_{\langle 1/3\rangle,1} - \hat{\mathfrak{G}}_{\langle 1/3\rangle,3} = 0(-1) - \begin{pmatrix} 0_40_6(-1) \\ +0_21 \end{pmatrix} = \begin{pmatrix} 0_{10}1 \\ +0(-1)_2 \end{pmatrix} = \hat{G}_{\langle 1/3\rangle,4}$ and

$\hat{G}_{\langle 1/3\rangle,2} - \hat{\mathfrak{G}}_{\langle 1/3\rangle,4} = 0_3 1 - \begin{pmatrix} 0_5 0_{10} 1 \\ +0_3 0(-1)_2 \end{pmatrix} = \begin{pmatrix} 0_{15}(-1) \\ +0_3 1_3 \end{pmatrix} = \hat{G}_{\langle 1/3\rangle,5}$. Moreover for all $i \geq 1$, we have

$$\hat{G}_{\langle 1/3\rangle,i} - \hat{\mathfrak{G}}_{\langle 1/3\rangle,i+2} = \begin{pmatrix} \hat{n}_{1,i} \\ \hat{n}_{2,i-3} \\ \hat{n}_{3,i-3(2)} \\ \cdots \\ \hat{n}_{k,i-3(k-1)} \\ +\cdots \end{pmatrix} - \begin{pmatrix} 0_{i+3}\hat{n}_{1,i+2} \\ 0_{i+1}\hat{n}_{2,i-1} \\ 0_{i-1}\hat{n}_{3,i-4} \\ \cdots \\ \hat{0}_{(i+3)-2k} n_{k+1,(i+2)-3k} \\ +\cdots \end{pmatrix}$$

$$= \begin{pmatrix} -0_{i+3}\hat{n}_{1,i+2} \\ \hat{n}_{1,i} - 0_{i+1}\hat{n}_{2,i-1} \\ \hat{n}_{2,i-3} - 0_{i-1}\hat{n}_{3,i-4} \\ \cdots \\ \hat{n}_{k,i-3(k-1)} - 0_{(i+3)-2k}\hat{n}_{k+1,(i+2)-3k} \\ +\cdots \end{pmatrix} = \begin{pmatrix} \hat{n}_{1,i+3} \\ \hat{n}_{2,i} \\ \hat{n}_{3,i-3} \\ \cdots \\ \hat{n}_{k+1,i-3(k-1)} \\ +\cdots \end{pmatrix} = \hat{G}_{\langle 1/3\rangle,i+3},$$

because $-0_{i+3}\hat{n}_{1,i+2} = \hat{n}_{1,i+3}$ and $\hat{n}_{1,i} - 0_{i+1}\hat{n}_{2,i-1} = \hat{n}_{2,i}$, $\hat{n}_{2,i-3} - 0_{i-1}\hat{n}_{3,i-4} = \hat{n}_{3,i-3}$ and $\hat{n}_{k,i-3(k-1)} - 0_{(i+3)-2k}\hat{n}_{k+1,(i+2)-3k} = \hat{n}_{k+1,i-3(k-1)}$, and so on.

Analogously we also see

$$\hat{G}_{\langle 1/4\rangle,1} - \hat{\mathfrak{G}}_{\langle 1/4\rangle,4} = 0(-1) - \begin{pmatrix} 0_5 0_{10} 1 \\ +0_2 1 \end{pmatrix} = -\begin{pmatrix} 0_{15} 1 \\ +0 1_2 \end{pmatrix} = \hat{G}_{\langle 1/4\rangle,5},$$

$$\hat{G}_{\langle 1/4\rangle,2} - \hat{\mathfrak{G}}_{\langle 1/4\rangle,5} = 0_3 1 + \begin{pmatrix} 0_6 0_{15} 1 \\ +0_3 0 1_2 \end{pmatrix} = \begin{pmatrix} 0_{21} 1 \\ +0_3 1_3 \end{pmatrix} = \hat{G}_{\langle 1/4\rangle,6},$$

and the identity $\hat{G}_{\langle 1/4\rangle,i} - \hat{\mathfrak{G}}_{\langle 1/4\rangle,i+3} = \hat{G}_{\langle 1/4\rangle,i+4}$ can be proved analogously. Therefore for any $s > 1$, it follows that

$$\hat{G}_{\langle 1/s\rangle,i} - \hat{\mathfrak{G}}_{\langle 1/s\rangle,i+(s-1)} = \begin{pmatrix} \hat{n}_{1,i} \\ \hat{n}_{2,i-s} \\ \hat{n}_{3,i-s(2)} \\ \cdots \\ \hat{n}_{k,i-s(k-1)} \\ +\cdots \end{pmatrix} - \begin{pmatrix} 0_{i+s}\hat{n}_{1,i+s-1} \\ 0_{i+s-(s-1)}\hat{n}_{2,(i+s-1)-s} \\ 0_{i+s-(s-1)2}\hat{n}_{3,(i+s-1)-2s} \\ \cdots \\ 0_{i+s-(s-1)k}\hat{n}_{k+1,(i+s-1)-ks} \\ +\cdots \end{pmatrix}$$

$$= \begin{pmatrix} -0_{i+s}\hat{n}_{1,i+s-1} \\ \hat{n}_{1,i} - 0_{1+i}\hat{n}_{2,i-1} \\ \hat{n}_{2,i-s} - 0_{2+i-s}\hat{n}_{3,i-s-1} \\ \cdots \\ \hat{n}_{k,i-s(k-1)} - 0_{k+i-s(k-1)}\hat{n}_{k+1,i-s(k-1)-1} \\ +\cdots \end{pmatrix} = \begin{pmatrix} \hat{n}_{1,i+s} \\ \hat{n}_{2,i} \\ \hat{n}_{3,i-2s} \\ \cdots \\ \hat{n}_{k+1,i-(k-1)s} \\ +\cdots \end{pmatrix}$$

$$= \hat{G}_{\langle 1/s\rangle,i+s}. \qquad \square$$

Similar to the lengthes of $\hat{c}_{i,j}$ and $\hat{n}_{i,j}$ in Theorem 2, let the length of diagonal sums of $\hat{D}_{\langle s/t\rangle,i}$ and $\hat{G}_{\langle s/t\rangle,j}$ be the numbers of digits in each diagonal sum.

THEOREM 8. *For all* $s, i \geq 1$, *the lengthes* $\mathrm{len}(\hat{D}_{\langle s\rangle,i})$ *and* $\mathrm{len}(\hat{G}_{\langle 1/s\rangle,i})$ *satisfy* $\mathrm{len}(\hat{D}_{\langle s\rangle,i}) = 1 + \left(i - (s+1)\lfloor\frac{i+s+1}{2(s+1)}\rfloor\right)\lfloor\frac{i+s+1}{2(s+1)}\rfloor = \mathrm{len}(\hat{c}_{i,(s+1)\lfloor\frac{i+s+1}{2(s+1)}\rfloor})$ *and* $\mathrm{len}(\hat{G}_{\langle 1/s\rangle,i}) = 1 + \frac{i(i+1)}{2} = \mathrm{len}(\hat{n}_{1,i})$.

Proof. Refer to [2] for $\mathrm{len}(\hat{D}_{\langle s\rangle,i})$ in $\hat{C}$. Now over the table $\hat{N}$, consider the $1/s$-slope i^{th}diagonal set $\hat{g}_{\langle 1/s\rangle,i} = \{\hat{n}_{1,i}, \hat{n}_{2,i-s}, \cdots, \hat{n}_{k,i-(k-1)s}, \cdots\}$. Since the subscripts of $\hat{n}_{k,i-(k-1)s}$ must satisfy $i - (k-1)s \geq 0$ and $k \leq \frac{i}{s} + 1$, the set $\hat{g}_{\langle 1/s\rangle,i}$ contains $\lfloor\frac{i}{s}\rfloor + 1$ elements in which $\hat{n}_{\lfloor\frac{i}{s}\rfloor+1,i-(\lfloor\frac{i}{s}\rfloor)s}$ is the last element. By Theorem 2, the length of $\hat{n}_{k,i-(k-1)s}$ in $\hat{g}_{\langle 1/s\rangle,i}$ satisfies

$$\mathrm{len}(\hat{n}_{k,i-(k-1)s}) = 1 + \tfrac{1}{2}(i-(k-1)s)(i-(k-1)s+1) + (k-1)(i-(k-1)s)$$

for $1 \leq k \leq \lfloor\frac{i}{s}\rfloor$. In order to compare the lengthes of two consecutive elements in $\hat{g}_{\langle 1/s\rangle,i}$, let $\Delta = \mathrm{len}(\hat{n}_{k,i-(k-1)s}) - \mathrm{len}(\hat{n}_{k+1,i-ks})$ be the difference. Then

$$\Delta = (1 + \tfrac{1}{2}(i-(k-1)s)(i-(k-1)s+1) + (k-1)(i-(k-1)s))$$

$$- (1 + \tfrac{1}{2}(i-ks)(i-ks+1) + k(i-ks))$$
$$= \tfrac{1}{2}(-2k+1)s^2 + \tfrac{1}{2}(2i+4k-1)s - i.$$

But since $i \geq ks$ and $s \geq 1$, we have $\Delta \geq \frac{1}{2}s(s+2k-1) \geq 0$, which shows $\operatorname{len}(\hat{n}_{k,i-(k-1)s}) \geq \operatorname{len}(\hat{n}_{k+1,i-ks})$ for all k.

So in the set $\hat{g}_{\langle 1/s\rangle,i}$, the first element $\hat{n}_{1,i}$ has the longest length, hence $\operatorname{len}(\hat{G}_{\langle 1/s\rangle,i}) = \operatorname{len}(\hat{n}_{1,i}) = \operatorname{len}(0_{\lambda_i}1) = 1 + \lambda_i$, with $\lambda_i = \frac{i(i+1)}{2}$. □

For instance, $\operatorname{len}(\hat{G}_{\langle 1\rangle,4}) = \operatorname{len}(\hat{n}_{1,4}) = \operatorname{len}(0_{10}1) = 11$ and $\operatorname{len}(\hat{G}_{\langle 1\rangle,5}) = \operatorname{len}(\hat{n}_{1,5}) = \operatorname{len}(-0_{15}1) = 16$, etc, so $G^{(q)}_{\langle 1\rangle,5} = \hat{G}_{\langle 1\rangle,5} \circ (1, p, \cdots, p^{15})$.

4. Interrelationship of $G^{(q)}_{\langle 1/s\rangle,n}$ with various q's

In this section, we study interrelationships of $1/s$-slope diagonal sums $\{G^{(q+t)}_{\langle 1/s\rangle,i}\}$ over $N^{(q+t)}$ with $t > 0$. Let $Y_k = \begin{bmatrix} 1 & \frac{1}{q} & \frac{1}{q^2} & \cdots & \frac{1}{q^k} \\ 1 & \frac{1}{q+1} & \frac{1}{(q+1)^2} & \cdots & \frac{1}{(q+1)^k} \\ & & \cdots & & \\ 1 & \frac{1}{q+k} & \frac{1}{(q+k)^2} & \cdots & \frac{1}{(q+k)^k} \end{bmatrix}$ be a $(k+1)$ square Vandermonde matrix and P_k be the $(k+1)$ Pascal matrix. Let $r_i(P_k^{-1})$ and $c_j(Y_k)$ be the i^{th}row of P_k^{-1} and j^{th}column of Y_k for $0 \leq i, j \leq k$.

THEOREM 9. *The $1/2$-slope diagonal sums satisfy* $G^{(q)}_{\langle 1/2\rangle,1} = G^{(q+1)}_{\langle 1/2\rangle,1} + r_1(P_1^{-1})c_1(Y_1)$ *and* $G^{(q)}_{\langle 1/2\rangle,2} = 3G^{(q+1)}_{\langle 1/2\rangle,2} - 3G^{(q+2)}_{\langle 1/2\rangle,2} + G^{(q+3)}_{\langle 1/2\rangle,2} - r_3(P_3^{-1})c_3(Y_3)$.

Proof. Write $P_k^{-1}Y_k = [r_i(P_k^{-1})c_j(Y_k)]$ for $i, j \geq 0$. Clearly $r_i(P_k^{-1})c_0(Y_k) = 0$ for all i and $r_i(P_k^{-1})Y_k = (0, r_i(P_k^{-1})c_1(Y_k), \cdots, r_i(P_k^{-1})c_k(Y_k))$.

We note $\operatorname{len}(\hat{G}_{\langle 1/2\rangle,1}) = 2$ by Theorem 8. Then with $r_1(P_1^{-1}) = (-1, 1)$, we have

$$\begin{aligned} r_1(P_1^{-1}) \circ (G^{(q)}_{\langle 1/2\rangle,1}, G^{(q+1)}_{\langle 1/2\rangle,1}) &= G^{(q)}_{\langle 1/2\rangle,1}(-1) + G^{(q+1)}_{\langle 1/2\rangle,1}(1) \\ &= \hat{G}_{\langle 1/2\rangle,1} \circ (-1)(1, \tfrac{1}{q}) + \hat{G}_{\langle 1/2\rangle,1} \circ (1)(1, \tfrac{1}{q+1}) \\ &= \hat{G}_{\langle 1/2\rangle,1} \circ (-1+1, -\tfrac{1}{q} + \tfrac{1}{q+1}) = \hat{G}_{\langle 1/2\rangle,1} \circ (-1, 1) \begin{bmatrix} 1 & \frac{1}{q} \\ 1 & \frac{1}{q+1} \end{bmatrix} \\ &= \hat{G}_{\langle 1/2\rangle,1} \circ r_1(P_1^{-1})Y_1 = -(0,1) \circ (0, r_1(P_1^{-1})c_1(Y_1)) \\ &= -r_1(P_1^{-1})c_1(Y_1), \end{aligned}$$

for $\hat{G}_{\langle 1/2\rangle,1} = -(0,1)$. So we have $G^{(q)}_{\langle 1/2\rangle,1} = G^{(q+1)}_{\langle 1/2\rangle,1} + r_1(P_1^{-1})c_1(Y_1)$.

On the other hand since $\operatorname{len}(\hat{G}_{\langle 1/2\rangle,2}) = 4$ and $r_3(P_3^{-1}) = (-1, 3, -3, 1)$, we have

$$\begin{aligned} & r_3(P_3^{-1}) \circ (G^{(q)}_{\langle 1/2\rangle,2}, G^{(q+1)}_{\langle 1/2\rangle,2}, G^{(q+2)}_{\langle 1/2\rangle,2}, G^{(q+3)}_{\langle 1/2\rangle,2}) \\ &= \hat{G}_{\langle 1/2\rangle,2} \circ (-1)(1, \tfrac{1}{q}, \tfrac{1}{q^2}, \tfrac{1}{q^3}) + \hat{G}_{\langle 1/2\rangle,2} \circ (3)(1, \tfrac{1}{q+1}, \tfrac{1}{(q+1)^2}, \tfrac{1}{(q+1)^3}) \\ &\quad + \hat{G}_{\langle 1/2\rangle,2} \circ (-3)(1, \tfrac{1}{q+2}, \tfrac{1}{(q+2)^2}, \tfrac{1}{(q+2)^3}) + \hat{G}_{\langle 1/2\rangle,2} \circ (1, \tfrac{1}{q+3}, \tfrac{1}{(q+3)^2}, \tfrac{1}{(q+3)^3}) \\ &= \hat{G}_{\langle 1/2\rangle,2} \circ (-1, 3, -3, 1) \begin{bmatrix} 1 & \frac{1}{q} & \frac{1}{q^2} & \frac{1}{q^3} \\ 1 & \frac{1}{q+1} & \frac{1}{(q+1)^2} & \frac{1}{(q+1)^3} \\ 1 & \frac{1}{q+2} & \frac{1}{(q+2)^2} & \frac{1}{(q+2)^3} \\ 1 & \frac{1}{q+3} & \frac{1}{(q+3)^2} & \frac{1}{(q+3)^3} \end{bmatrix} = \hat{G}_{\langle 1/2\rangle,2} \circ r_3(P_3^{-1})Y_3 \\ &= (1, 0, 0, 1) \circ (0, r_3(P_3^{-1})c_1(Y_3), r_3(P_3^{-1})c_2(Y_3), r_3(P_3^{-1})c_3(Y_3)) \end{aligned}$$

$= r_3(P_3^{-1})c_3(Y_3)$,
for $\hat{G}_{\langle 1/2\rangle,2} = 0_3 1 + 1 = 1001$. So we immediately have
$G^{(q)}_{\langle 1/2\rangle,2} = 3G^{(q+1)}_{\langle 1/2\rangle,2} - 3G^{(q+2)}_{\langle 1/2\rangle,2} + G^{(q+3)}_{\langle 1/2\rangle,2} - r_3(P_3^{-1})c_3(Y_3)$. □

Theorem 9 can be more sharpened by using $r_3(P_3^{-1})c_1(Y_3) = (-1,1) \circ (\frac{1}{q}, \frac{1}{q+1}) = \frac{-1}{q(q+1)}$ and $r_3(P_3^{-1})c_3(Y_3) = (-1,3,-3,1) \circ (\frac{1}{q^3}, \frac{1}{(q+1)^3}, \frac{1}{(q+2)^3}, \frac{1}{(q+3)^3})$. A generalization of Theorem 9 to $1/s$-slope diagonal is as follows.

Theorem 10. *Let $t = \mathrm{len}(\hat{G}_{\langle 1/s\rangle,i}) - 1$ for $s, i > 0$. Then $1/s$-slope diagonal sums satisfy* $G^{(q)}_{\langle 1/s\rangle,i} = \sum_{j=1}^{t} (-1)^{j-1}\binom{t}{j} G^{(q+j)}_{\langle 1/s\rangle,i} + (-1)^t \hat{G}_{\langle 1/s\rangle,i} \circ r_t(P_t^{-1})Y_t$.

Proof. Clearly $t = \mathrm{len}(\hat{G}_{\langle 1/s\rangle,i}) - 1 = \frac{i(i+1)}{2}$ by Theorem 8. When $s = 2$, see Theorem 9 with $i = 1, 2$. Now with the t^{th}row $r_t(P_t^{-1}) = ((-1)^t, (-1)^{t-1}\binom{t}{1}, \cdots, (-1)^{t-j}\binom{t}{j}, \cdots, (-1)\binom{t}{t-1}, 1)$, we have

$$\begin{aligned}
&r_t(P_t^{-1}) \circ (G^{(q)}_{\langle 1/s\rangle,i}, G^{(q+1)}_{\langle 1/s\rangle,i}, \cdots, G^{(q+t)}_{\langle 1/s\rangle,i})\\
&= \hat{G}_{\langle 1/s\rangle,i} \circ (-1)^t(1, \tfrac{1}{q}, \cdots, \tfrac{1}{q^t}) + \hat{G}_{\langle 1/s\rangle,1} \circ (-1)^{t-1}\tbinom{t}{1}(1, \tfrac{1}{q+1}, \cdots, \tfrac{1}{(q+1)^t})\\
&\quad + \cdots + \hat{G}_{\langle 1/s\rangle,1} \circ (1, \tfrac{1}{q+t}, \cdots, \tfrac{1}{(q+t)^t})\\
&= \hat{G}_{\langle 1/s\rangle,1} \circ ((-1)^t, (-1)^{t-1}\tbinom{t}{1}, \cdots, 1) \begin{bmatrix} 1 & \frac{1}{q} & \frac{1}{q^2} & \cdots & \frac{1}{q^k} \\ 1 & \frac{1}{q+1} & \frac{1}{(q+1)^2} & \cdots & \frac{1}{(q+1)^k} \\ & & \cdots & & \\ 1 & \frac{1}{q+k} & \frac{1}{(q+k)^2} & \cdots & \frac{1}{(q+k)^k} \end{bmatrix}\\
&= \hat{G}_{\langle 1/s\rangle,i} \circ r_t(P_t^{-1})Y_t,
\end{aligned}$$

thus

$$\begin{aligned}
&(-1)^t G^{(q)}_{\langle 1/s\rangle,i} + (-1)^{t-1}\tbinom{t}{1} G^{(q+1)}_{\langle 1/s\rangle,i} + \cdots + (-1)\tbinom{t}{t-1} G^{(q+t-1)}_{\langle 1/s\rangle,i} + G^{(q+t)}_{\langle 1/s\rangle,i}\\
&\quad = \hat{G}_{\langle 1/s\rangle,i} \circ r_t(P_t^{-1})Y_t.
\end{aligned}$$

Hence it follows immediately that

$$\begin{aligned}
G^{(q)}_{\langle 1/s\rangle,i} &= \tbinom{t}{1} G^{(q+1)}_{\langle 1/s\rangle,i} + (-1)\tbinom{t}{2} G^{(q+2)}_{\langle 1/s\rangle,i} + \cdots + (-1)^t \hat{G}_{\langle 1/s\rangle,i} \circ r_t(P_t^{-1})Y_t\\
&= \sum_{j=1}^{t} (-1)^{j-1}\tbinom{t}{j} G^{(q+j)}_{\langle 1/s\rangle,i} + (-1)^t \hat{G}_{\langle 1/s\rangle,i} \circ r_t(P_t^{-1})Y_t.
\end{aligned}$$

□

Theorem 10 can be compared to s-slope diagonal sums $D^{(q)}_{\langle s\rangle,i}$ of $C^{(q)}$ with various $q, q+1, \cdots, q+k$ ([1]). In fact, if $t = \mathrm{len}(\hat{D}_{\langle s\rangle,i}) - 1$ then $D^{(q)}_{\langle s\rangle,i} = \sum_{j=1}^{t} (-1)^{j-1}\binom{t}{j} D^{(q+j)}_{\langle s\rangle,i} + (-1)^t \omega_{s,i}(t!)$, where $\omega_{s,i} = 2$ if $i \equiv s+1 \pmod{2(i+1)}$, otherwise $\omega_{s,i} = 1$.

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Eunmi Choi
Department of Mathematics, Hannam University, Daejeon, Korea
E-mail: emc@hnu.kr