Communications for Statistical Applications and Methods
2022, Vol. 29, No. 5, 513–532
https://doi.org/10.29220/CSAM.2022.29.5.513
Print ISSN 2287-7843 / Online ISSN 2383-4757

Bayes factors for accelerated life testing models

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Abstract

In this paper, the use of Bayes factors and the deviance information criterion for model selection are compared in a Bayesian accelerated life testing setup. In Bayesian accelerated life testing, the most used tool for model comparison is the deviance information criterion. An alternative and more formal approach is to use Bayes factors to compare models. However, Bayesian accelerated life testing models with more than one stressor often have mathematically intractable posterior distributions and Markov chain Monte Carlo methods are employed to obtain posterior samples to base inference on. The computation of the marginal likelihood is challenging when working with such complex models. In this paper, methods for approximating the marginal likelihood and the application thereof in the accelerated life testing paradigm are explored for dual-stress models. A simulation study is also included, where Bayes factors using the different approximation methods and the deviance information are compared.

Keywords: accelerated life testing, Bayes factors, generalized Eyring model, Markov chain Monte Carlo, model comparison

1. Introduction

Bayesian model selection is an important aspect of any Bayesian analysis and aids the researcher in determining which of the proposed models should be used. Various approaches towards Bayesian model selection are discussed in Kadane and Lazar (2004). The different perspectives to model selection discussed include Bayes factors, Bayesian model averaging, methods used for Bayesian linear models, and predictive methods. The authors also state that model selection tools should rather be used to identify inappropriate models, instead of attempting so select a single best model.

Bayesian accelerated life testing (ALT) models with more than one stressor often have mathematically intractable posterior distributions and Markov chain Monte Carlo (MCMC) methods are employed to obtain posterior samples to base inference on. For this reason, and the fact that it is a standard output in many Bayesian data analysis software packages, the deviance information criterion (DIC) is a very popular tool for model comparison. It can be argued that the more formal and traditional approach for Bayesian model selection would be the use of Bayes factors (Spiegelhalter *et al.*, 2002; Upadhyay and Mukherjee, 2010).

The authors in Spiegelhalter *et al.* (2002) explain that the DIC is intended as an alternative to Bayes factors and that there are situations where the DIC may be a more appropriate tool for model

The authors' work is based on the research supported by the National Research Foundation (NRF) of South Africa. Any opinion, finding and conclusion or recommendation expressed in this material is that of the author(s) and the NRF does not accept any liability in this regard.
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Published 30 September 2022 / journal homepage: http://csam.or.kr

comparison. Bayes factors are most suited when the complete set of possible models can be specified and the true model is included in this set, whereas the DIC does not have this requirement (Bernardo and Smith, 1994; Spiegelhalter *et al.*, 2002). In a discussion on the DIC, the authors state that Bayes factors and the DIC are intended for different purposes, and can thus lead to different conclusions regarding model selection. Bayes factors consider how well the prior predicts the observed data, whereas the DIC considers how well the posterior will predict future data by means of the same mechanism that resulted in the observed data (Spiegelhalter *et al.*, 2002).

In this paper, model selection in the Bayesian ALT setup is considered by comparing a generalized Eyring-Weibull (GEW) model and a generalized Eyring-Birnbaum-Saunders (GEBS) model via the DIC and Bayes factors. Due to the mathematically intractable posterior distributions of these models, the computation of the marginal likelihood, needed for the calculation of the Bayes factor, is also complicated. Methods for approximating the marginal likelihood, without further complicating the MCMC sampling process, are discussed. The methods considered include a simple Monte Carlo estimator (SMCE), the harmonic mean estimator (HME), the Laplace-Metropolis estimator (LME), and a posterior predictive density estimator (PPDE) for posterior Bayes factors.

2. Bayesian model selection

2.1. Deviance information criterion

The DIC, proposed by Spiegelhalter *et al.* (2002), is a widely used measure for Bayesian model comparison. It is used to assess both the goodness-of-fit and the complexity of the model. The model with a significantly lower DIC value will usually be the preferred model to use, but there are other considerations as well. The authors state that there is no specific rule of thumb on what constitutes a significant difference in DIC, but that guidelines proposed by Burnham and Anderson (1998) on the Akaike information criterion (AIC) also seem to work well for the DIC.

The formulation of the DIC follows on the work of Akaike (1973) and is based on using the posterior mean deviance as a measure of fit, and a new complexity measure called the *effective number of parameters*. For a parameter vector $\boldsymbol{\theta}$, with likelihood function $L(\underline{x}|\boldsymbol{\theta})$, the deviance can be defined as

$$D(\boldsymbol{\theta}) = -2\ln\left[L\left(\underline{x}\,|\boldsymbol{\theta}\right)\right].$$

The DIC can then be calculated as

$$\text{DIC} = \overline{D} + p_D,$$

where $\overline{D}$ is the posterior mean of the deviance and p_D is the effective number of parameters given by $p_D = \overline{D} - \hat{D}(\overline{\boldsymbol{\theta}})$, where $\hat{D}(\overline{\boldsymbol{\theta}})$ is a point estimate for the deviance. It is shown in Spiegelhalter *et al.* (2002) that the DIC is approximately equivalent to the AIC for models with very weak prior information.

The DIC is a very popular choice for model comparison in the Bayesian ALT setup, particularly when working with complicated models where MCMC methods are used to obtain posterior samples for inference (see, for example, Barriga *et al.*, 2008; Soyer *et al.*, 2008; Upadhyay and Mukherjee, 2010). The reason for this is that the DIC can easily be obtained from the MCMC output (see Spiegelhalter *et al.*, 2002), and it is also provided as a standard output in some well-known Bayesian data analysis software such as WinBUGS/OpenBUGS and JAGS.

2.2. Bayes factors

The foundation for Bayesian hypothesis testing via Bayes factors was developed by Jeffreys (1961). The author referred to his methods as "significance tests" and presented them as an alternative to p-values. The approach compares predictions made by two competing scientific theories by defining statistical models for each theory and calculating the posterior probability that one of the theories is correct (Kass and Raftery, 1995).

Denote by $f(m_i|\underline{x})$ and $f(m_j|\underline{x})$ the posterior model probabilities for models m_i and m_j, respectively. The comparison of two models m_i and m_j, in the Bayesian setup, is conducted by means of the posterior odds for model m_i against model m_j. This is given by

$$PO_{ij} = \frac{f\left(m_i\,|\underline{x}\right)}{f\left(m_j\,|\underline{x}\right)} = \frac{f\left(\underline{x}\,|m_i\right)}{f\left(\underline{x}\,|m_j\right)} \times \frac{f\left(m_i\right)}{f\left(m_j\right)} = B_{ij} \times \frac{f\left(m_i\right)}{f\left(m_j\right)},$$

where $f(\underline{x}|m_g)$, $g = i, j$ are the marginal likelihoods and $f(m_g)$, $g = i, j$ are the prior model probabilities of the two models. The above expression is often summarized as

$$\text{Posterior odds} = \text{Bayes factor} \times \text{Prior odds}.$$

In most situations no prior information will be available regarding the model structure, in which case the prior model probabilities are set equal (see, for example, Ntzoufras, 2009). This results in using the Bayes factor for hypothesis testing, which is also extended to model selection. It is important to note that Bayes factors can be used to evaluate evidence *against* the null hypothesis or *in favour* of the null hypothesis, which is not possible in classical hypothesis testing.

The Bayes factor B_{ij} of model m_i versus model m_j is defined as the ratio of the marginal likelihoods $f(\underline{x}|m_i)$ and $f(\underline{x}|m_j)$. This can be written as

$$B_{ij} = \frac{f\left(\underline{x}\,|m_i\right)}{f\left(\underline{x}\,|m_j\right)} = \frac{\int f\left(\underline{x}\,|\boldsymbol{\theta}_{m_i}, m_i\right) f\left(\boldsymbol{\theta}_{m_i}\,|m_i\right) d\boldsymbol{\theta}_{m_i}}{\int f\left(\underline{x}\,|\boldsymbol{\theta}_{m_j}, m_j\right) f\left(\boldsymbol{\theta}_{m_j}\,|m_j\right) d\boldsymbol{\theta}_{m_j}}, \tag{2.1}$$

where $f(\underline{x}|\boldsymbol{\theta}_{m_g}, m_g)$, $g = i, j$ is the likelihood of model m_g with parameter vector $\boldsymbol{\theta}_{m_g}$, and $f(\boldsymbol{\theta}_{m_g}|m_g)$, $g = i, j$ is the prior imposed on $\boldsymbol{\theta}_{m_g}$ under model m_g.

Interpreting Bayes factors in half-units on the $\log_{10}$-scale is suggested in Jeffreys (1961), where the author provides a table for interpreting Bayes factor values. Kass and Raftery (1995) suggest a modified version of these interpretations by rather considering twice the natural logarithm of the Bayes factor. By doing this, Bayes factors are interpreted on the same scale as the likelihood ratio test statistic. The interpretation of Bayes factors suggested in Kass and Raftery (1995) are used in this paper.

In some cases, the marginal likelihoods in (2.1) can be computed analytically (see, for example, DeGroot, 1970; Zellner, 1971). More often than not, the marginal likelihood must be estimated. Ntzoufras (2009) explains that numerous other versions of Bayes factors as well as alternative model selection approaches have also been developed. Other popular types of Bayes factors include pseudo Bayes factors, resulting from the work of Geisser and Eddy (1979), posterior Bayes factors, presented in Aitkin (1991), fractional Bayes factors, introduced in O'Hagan (1995), and intrinsic Bayes factors, presented in Berger and Pericchi (1996).

Suppose we compare two models, m_i and m_j, where the prior model probabilities are denoted by $f(m_g)$, $g = i, j$. From Bayes' theorem we then have that the posterior model probability for model m_i

can be written as

$$f\left(m_i \left| \underline{x}\right.\right) = \frac{f\left(\underline{x}\left|m_i\right.\right) f\left(m_i\right)}{f\left(\underline{x}\left|m_i\right.\right) f\left(m_i\right) + f\left(\underline{x}\left|m_j\right.\right) f\left(m_j\right)}$$
$$= \frac{f\left(m_i\right) \int f\left(\underline{x}\left|\boldsymbol{\theta}_{m_i}, m_i\right.\right) f\left(\boldsymbol{\theta}_{m_i}\left|m_i\right.\right) d\boldsymbol{\theta}_{m_i}}{f\left(m_i\right) \int f\left(\underline{x}\left|\boldsymbol{\theta}_{m_i}, m_i\right.\right) f\left(\boldsymbol{\theta}_{m_i}\left|m_i\right.\right) d\boldsymbol{\theta}_{m_i} + \left[1 - f\left(m_i\right)\right] \int f\left(\underline{x}\left|\boldsymbol{\theta}_{m_j}, m_j\right.\right) f\left(\boldsymbol{\theta}_{m_j}\left|m_j\right.\right) d\boldsymbol{\theta}_{m_j}},$$

where $f(\underline{x}|m_g)$, $g = i, j$ is the marginal likelihoods, $f(\underline{x}|\boldsymbol{\theta}_{m_g}, m_g)$, $g = i, j$ denotes the likelihood of model m_g with parameter vector $\boldsymbol{\theta}_{m_g}$, and $f(\boldsymbol{\theta}_{m_g}|m_g)$, $g = i, j$ is the prior imposed on $\boldsymbol{\theta}_{m_g}$ under model m_g. The posterior model probability for model m_i can easily be computed by re-writing it in terms of Bayes factors as

$$f\left(m_i \left|\underline{x}\right.\right) = \left[1 + \frac{1 - f\left(m_i\right)}{f\left(m_i\right)} B_{ij}^{-1}\right]^{-1},$$

where B_{ij} is the Bayes factor of model m_i versus model m_j.

3. Estimating the marginal likelihood

There are various methods that can be used to estimate the marginal likelihood. In this section, we focus on methods that can easily estimate Bayes factors from the output of an MCMC algorithm. These methods include the SMCE, LME, HME, and PPDE which is used to estimate the posterior Bayes factors.

3.1. Simple Monte Carlo estimator

Since the marginal likelihood for a model m_i is given by

$$f\left(\underline{x}\left|m_i\right.\right) = \int f\left(\underline{x}\left|\boldsymbol{\theta}_{m_i}, m_i\right.\right) f\left(\boldsymbol{\theta}_{m_i}\left|m_i\right.\right) d\boldsymbol{\theta}_{m_i},$$

a straightforward and simple estimate is provided by the Monte Carlo integration estimate

$$\hat{f}_{MC}\left(\underline{x}\left|m_i\right.\right) = \frac{1}{N} \sum_{t=1}^{N} f\left(\underline{x}\left|\boldsymbol{\theta}_{m_i}^{*(t)}, m_i\right.\right),$$

where $\boldsymbol{\theta}_{m_i}^{*(1)}, \boldsymbol{\theta}_{m_i}^{*(2)}, \ldots, \boldsymbol{\theta}_{m_i}^{*(N)}$ are samples from the prior distribution of model m_i. According to Kass and Raftery (1995) this estimator can be very inefficient, particularly when the prior distribution and posterior distribution significantly differ. An example of where a considerable difference in the prior and posterior distributions can be expected, is when a flat prior is used (Ntzoufras, 2009). In such a case, very small likelihood values will be produced for most of the samples $\boldsymbol{\theta}_{m_i}^{*(t)}$ and the estimate will be dominated by only a few large likelihood values.

3.2. Laplace-Metropolis estimator

A widely used approximation for the marginal likelihood is the Laplace approximation. This is given by

$$f\left(\underline{x}\left|m_i\right.\right) \approx (2\pi)^{\frac{d_{m_i}}{2}} \left|\tilde{\boldsymbol{\Sigma}}_{m_i}\right|^{\frac{1}{2}} f\left(\underline{x}\left|\tilde{\boldsymbol{\theta}}_{m_i}, m_i\right.\right) f\left(\tilde{\boldsymbol{\theta}}_{m_i}\left|m_i\right.\right),$$

where d_{m_i} is the number of parameters for model m_i, $\tilde{\boldsymbol{\theta}}_{m_i}$ is the posterior mode of the parameters for model m_i, and $\tilde{\boldsymbol{\Sigma}}_{m_i} = [-\boldsymbol{H}_m(\tilde{\boldsymbol{\theta}}_{m_i})]^{-1}$. $\boldsymbol{H}_m(\tilde{\boldsymbol{\theta}}_{m_i})$ is the Hessian matrix of second derivatives for the log of the posterior density, $\ln[f(\boldsymbol{\theta}_{m_i}|\underline{x}, m_i)]$, evaluated at the posterior mode $\tilde{\boldsymbol{\theta}}_{m_i}$. Kass and Raftery (1995) state that the Laplace approximation works well for symmetric likelihood functions and for a parameter vector of moderate dimensionality.

Raftery (1996) and Lewis and Raftery (1997) propose an extension of the Laplace approximation, called the LME, which avoids the analytical calculation of $\tilde{\boldsymbol{\Sigma}}_{m_i}$ and $\tilde{\boldsymbol{\theta}}_{m_i}$. The posterior mean and variance-covariance matrix of the posterior sample $\boldsymbol{\theta}_{m_i}^{(1)}, \boldsymbol{\theta}_{m_i}^{(2)}, \ldots, \boldsymbol{\theta}_{m_i}^{(N)}$, generated from an MCMC algorithm, are used to estimate $\tilde{\boldsymbol{\theta}}_{m_i}$ and $\tilde{\boldsymbol{\Sigma}}_{m_i}$, respectively. The LME is given by

$$\hat{f}_{LM}\left(\underline{x}\,|m_i\right) = (2\pi)^{\frac{d_{m_i}}{2}} \left|\boldsymbol{S}_{m_i}\right|^{\frac{1}{2}} f\left(\underline{x}\left|\overline{\boldsymbol{\theta}}_{m_i}, m_i\right.\right) f\left(\overline{\boldsymbol{\theta}}_{m_i}\,|m_i\right),$$

where $\overline{\boldsymbol{\theta}}_{m_i} = 1/N \sum_{t=1}^{N} \boldsymbol{\theta}_{m_i}^{(t)}$ and $\boldsymbol{S}_{m_i}$ is a weighted variance matrix estimate (see, Lewis and Raftery, 1997, for further details).

Ntzoufras (2009) explains how the LME can be calculated from MCMC output in OpenBUGS as follows:

1. Implement the model m_i in OpenBUGS and produce posterior samples for the parameters of interest via an MCMC algorithm.

2. From the MCMC samples calculate the following estimates:
   - $\overline{\boldsymbol{\theta}}_{m_i} = (\overline{\theta}_1, \overline{\theta}_2, \ldots, \overline{\theta}_{d_{m_i}})$, which are the posterior means of the parameters of interest.
   - $\boldsymbol{s}_{\boldsymbol{\theta}_{m_i}} = (s_{\theta_1}, s_{\theta_2}, \ldots, s_{\theta_{d_{m_i}}})$, which are the posterior standard deviations of the parameters of interest.
   - $\boldsymbol{R}_{\boldsymbol{\theta}_{m_i}}$, which is the posterior correlation matrix between the parameters of interest.

3. Calculate the expression

$$\ln \hat{f}_{LM}\left(\underline{x}\,|m_i\right) = \frac{1}{2} d_{m_i} \ln(2\pi) + \frac{1}{2} \ln\left|\boldsymbol{R}_{\boldsymbol{\theta}_{m_i}}\right| + \sum_{l=1}^{d_{m_i}} \ln s_{\theta_l} + \sum_{k=1}^{n} \ln f\left(x_k\left|\overline{\boldsymbol{\theta}}_{m_i}, m_i\right.\right) + \ln f\left(\overline{\boldsymbol{\theta}}_{m_i}\,|m_i\right)$$

   and simply take $e^{\ln \hat{f}_{LM}(\underline{x}|m_i)}$ to get the LME for the marginal likelihood.

3.3. Harmonic mean estimator

The HME for the marginal likelihood, introduced in Newton and Raftery (1994), is given by

$$\hat{f}_{HM}\left(\underline{x}\,|m_i\right) = \left[\frac{1}{N} \sum_{t=1}^{N} \left(f\left(\underline{x}\left|\boldsymbol{\theta}_{m_i}^{(t)}, m_i\right.\right)\right)^{-1}\right]^{-1},$$

where $\boldsymbol{\theta}_{m_i}^{(1)}, \boldsymbol{\theta}_{m_i}^{(2)}, \ldots, \boldsymbol{\theta}_{m_i}^{(N)}$ are posterior samples generated by an MCMC method or other sampling technique. According to Newton and Raftery (1994), $\hat{f}_{HM}(\underline{x}|m_i)$ converges almost surely to $f(\underline{x}|m_i)$, but the HME does not, in general, satisfy a Gaussian central limit theorem. This estimator is shown to be unstable and sensitive to small likelihood values, in Raftery (1996) and Raftery *et al.* (2007), but it is simple to calculate. The HME is simulation-consistent and unbiased, but can have infinite variance resulting in unstable behaviour.

A generalized HME, which is an unbiased, consistent and more stable estimator for the marginal likelihood, is presented in Gelfand and Dey (1994). This estimator, however, requires the specification of an importance density, which must be carefully chosen and be relatively close to the posterior. Raftery *et al.* (2007) present two more methods for stabilizing the HME, and improvements on the HME via Lebesgue integration is presented in Weinberg (2012).

3.4. Posterior predictive density estimator

A variation on the traditional Bayes factor is the posterior Bayes factor, introduced in Aitkin (1991). The posterior Bayes factor of model m_i versus model m_j is based on the ratio of the posterior predictive densities of these models, for the observed data, and is given by

$$B_{ij} = \frac{f\left(\underline{x}\,|\underline{x}, m_i\right)}{f\left(\underline{x}\,|\underline{x}, m_j\right)},$$

where $f(\underline{x}|\underline{x}, m_g)$, $g = i, j$, is the posterior predictive density of model m_g, evaluated at the observed data. The posterior predictive density can be easily estimated by the posterior mean of the likelihood for the posterior samples obtained from an MCMC algorithm (Ntzoufras, 2009). The PPDE is given by

$$\hat{f}_{PPD}\left(\underline{x}\,|\underline{x}, m_i\right) = \frac{1}{N}\sum_{t=1}^{N} f\left(\underline{x}\,|\boldsymbol{\theta}_{m_i}^{(t)}, m_i\right),$$

where $\boldsymbol{\theta}_{m_i}^{(1)}, \boldsymbol{\theta}_{m_i}^{(2)}, \ldots, \boldsymbol{\theta}_{m_i}^{(N)}$ are posterior samples for the model parameters.

The use of posterior Bayes factors has been criticized due the double use of the data, which is also the case in the construction of the effective number of parameters p_D, used as a complexity measure in the DIC. Posterior Bayes factors can, however, support more complex models than traditional Bayes factors.

4. Generalized Eyring ALT models

In ALT, items are tested in an environment that is more severe than their normal operating environment, in order to induce early failures. This is performed by applying accelerated levels of stressors such as temperature, voltage or humidity to the items. The life characteristics under normal operating conditions can then be extrapolated from these accelerated failure times, by means of a functional relationship known as a time transformation function.

Consider two stressors, one thermal and one non-thermal. Indicate the k distinct accelerated levels of the stressors by $\{T_i, S_i\}, i = 1, \ldots, k$, where $T_i, i = 1, \ldots, k$ are the accelerated levels of the thermal stressor and $S_i, i = 1, \ldots, k$ are the accelerated levels of the non-thermal stressor. An item is exposed to the constant application of a specific stress level combination $\{T_i, S_i\}$.

Suppose that n_i items are tested at each of the k different stress levels and the test is truncated at time τ_i. Denote the failure times by $x_{ij}, j = 1, \ldots, n_i, i = 1, \ldots, k$. The likelihood function, in general, is then given by

$$L\left(\underline{x}\,|\boldsymbol{\Theta}\right) = \prod_{i=1}^{k}\left[\prod_{j=1}^{r_i} f\left(x_{ij}\,|\boldsymbol{\Theta}\right)\right]\left[R\left(\tau_i\right)\right]^{n_i - r_i},$$

where R denotes the reliability function and $\mathbf{\Theta}$ the parameters under investigation. Note that for complete samples $r_i = n_i$. For type-I censoring r_i is the number of failures that occur before censoring time τ_i, where $\tau_i < \infty, i = 1, \ldots, k$ are predetermined censoring times for the k different stress levels. For type-II censoring, $\tau_i = x_{i(r_i)}$, where $x_{i(r_i)}$ is the r_i^{th} ordered failure time, and $r_i, i = 1, \ldots, k$ are the pre-chosen number of failures after which censoring occurs for the k different stress levels. Smit and Raubenheimer (2021, 2022) investigated Bayesian Eyring ALT models using the Birnbaum-Saunders and Weibull distributions as the lifetime models.

4.1. The GEBS model

Let X be a continuous random variable that follows a Birnbaum-Saunders distribution with shape parameter α and scale parameter β ($\alpha > 0, \beta > 0$). The PDF of X is then given by

$$f(x|\alpha,\beta) = \frac{x+\beta}{2\sqrt{2\pi}\alpha\sqrt{\beta}\sqrt{x_i^3}} \exp\left[-\frac{1}{2\alpha^2}\left(\frac{x}{\beta} + \frac{\beta}{x} - 2\right)\right], \quad x > 0, \tag{4.1}$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution. A common practice in ALT is to assume that the Birnbaum-Saunders' scale parameter β depends on the stress levels, whereas the shape parameter α does not (see, for example, Owen and Padgett, 2000; Upadhyay and Mukherjee, 2010; Sun and Shi, 2016; Sha, 2018). The reparameterization of β given by the generalized Eyring model is

$$\beta_i = \frac{1}{T_i} \exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right), \tag{4.2}$$

where θ_1, θ_2, θ_3, and θ_4 are unknown parameters, and V_i is a function of the non-thermal stressor S_i (Escobar and Meeker, 2006). From (4.1) and (4.2) it follows that the Birnbaum-Saunders PDF of a lifetime subjected to the i^{th} stress level, can be written as

$$\begin{aligned} f(x_i|\alpha,\theta_1,\theta_2,\theta_3,\theta_4) = {} & \frac{x_i + \frac{1}{T_i}\exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right)}{2\sqrt{2\pi}\alpha\sqrt{x_i^3}\sqrt{\frac{1}{T_i}\exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right)}} \\ & \times \exp\left\{-\frac{1}{2\alpha^2}\left[x_i T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\right.\right. \\ & \left.\left. + \frac{1}{x_i T_i}\exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right) - 2\right]\right\}. \end{aligned} \tag{4.3}$$

The corresponding reliability function at some time τ is given by

$$R(\tau) = 1 - \Phi\left\{\frac{1}{\alpha}\left[\sqrt{\tau T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)} - \sqrt{\frac{1}{\tau T_i}\exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right)}\right]\right\}. \tag{4.4}$$

From (4.3) and (4.4), it follows that the likelihood function for the GEBS model can be written as

$$
\begin{aligned}
L\left(\underline{x}\,|\alpha,\theta_1,\theta_2,\theta_3,\theta_4\right) = &\left(2\sqrt{2\pi}\alpha\right)^{-\sum_{i=1}^{k} r_i} \exp\left(\frac{1}{\alpha^2}\sum_{i=1}^{k} r_i\right) \times \left[\prod_{i=1}^{k}\prod_{j=1}^{r_i} \frac{x_{ij} + \frac{1}{T_i}\exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right)}{x_{ij}^{\frac{3}{2}}}\right] \\
&\times \exp\left\{-\frac{1}{2\alpha^2}\sum_{i=1}^{k}\left[T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\sum_{j=1}^{r_i} x_{ij}\right]\right. \\
&\left.-\frac{1}{2\alpha^2}\sum_{i=1}^{k}\left[\frac{1}{T_i}\exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right)\sum_{j=1}^{r_i}\frac{1}{x_{ij}}\right]\right\} \\
&\times \prod_{i=1}^{k}\left\{\left[T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\right]^{\frac{r_i}{2}}\right. \\
&\times\left[1 - \Phi\left(\frac{1}{\alpha}\left(\sqrt{\tau_i T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)}\right.\right.\right. \\
&\left.\left.\left.\left.-\sqrt{\frac{1}{\tau_i T_i}\exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right)}\right)\right)\right]^{n_i - r_i}\right\}.
\end{aligned} \tag{4.5}
$$

Assume independent priors on the model parameters $\theta_1, \theta_2, \theta_3, \theta_4$ and α (see, for example, Upadhyay and Mukherjee, 2010), which leads to the joint prior distribution being given by

$$
\pi(\alpha,\theta_1,\theta_2,\theta_3,\theta_4) = \pi(\alpha)\,\pi(\theta_1)\,\pi(\theta_2)\,\pi(\theta_3)\,\pi(\theta_4). \tag{4.6}
$$

The joint posterior distribution is then given by

$$
\pi\left(\alpha,\theta_1,\theta_2,\theta_3,\theta_4\,\middle|\underline{x}\right) \propto L\left(\underline{x}\,|\alpha,\theta_1,\theta_2,\theta_3,\theta_4\right)\pi(\alpha,\theta_1,\theta_2,\theta_3,\theta_4).
$$

For the GEBS model, gamma priors are imposed on all the parameters, therefore

$$
\begin{aligned}
\alpha &\sim \Gamma(c_0, c_1), \quad c_0, c_1 > 0, \quad \pi(\alpha) \propto \alpha^{c_0 - 1}\exp(-c_1\alpha), \\
\theta_1 &\sim \Gamma(c_2, c_3), \quad c_2, c_3 > 0, \quad \pi(\theta_1) \propto \theta_1^{c_2 - 1}\exp(-c_3\theta_1), \\
\theta_2 &\sim \Gamma(c_4, c_5), \quad c_4, c_5 > 0, \quad \pi(\theta_2) \propto \theta_2^{c_4 - 1}\exp(-c_5\theta_2), \\
\theta_3 &\sim \Gamma(c_6, c_7), \quad c_6, c_7 > 0, \quad \pi(\theta_3) \propto \theta_3^{c_6 - 1}\exp(-c_7\theta_3), \\
\theta_4 &\sim \Gamma(c_8, c_9), \quad c_8, c_9 > 0, \quad \pi(\theta_4) \propto \theta_4^{c_8 - 1}\exp(-c_9\theta_4).
\end{aligned}
$$

The joint prior from (4.6) can now be written as

$$
\pi(\alpha,\theta_1,\theta_2,\theta_3,\theta_4) \propto \alpha^{c_0-1}\theta_1^{c_2-1}\theta_2^{c_4-1}\theta_3^{c_6-1}\theta_4^{c_8-1}\exp(-c_1\alpha - c_3\theta_1 - c_5\theta_2 - c_7\theta_3 - c_9\theta_4). \tag{4.7}
$$

The resulting joint posterior, using (4.5) and (4.7), is

$$\begin{aligned}
\pi\left(\alpha,\theta_1,\theta_2,\theta_3,\theta_4\left|\underline{x}\right.\right) \propto\ & \alpha^{c_0-1-\sum_{i=1}^{k} r_i}\theta_1^{c_2-1}\theta_2^{c_4-1}\theta_3^{c_6-1}\theta_4^{c_8-1} \\
& \times \exp\left(\frac{1}{\alpha^2}\sum_{i=1}^{k} r_i\right)\exp\left(-c_1\alpha - c_3\theta_1 - c_5\theta_2 - c_7\theta_3 - c_9\theta_4\right) \\
& \times \left[\prod_{i=1}^{k}\prod_{j=1}^{r_i}\frac{x_{ij} + \frac{1}{T_i}\exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right)}{x_{ij}^{\frac{3}{2}}}\right] \\
& \times \exp\left\{-\frac{1}{2\alpha^2}\sum_{i=1}^{k}\left[T_i\exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\sum_{j=1}^{r_i} x_{ij}\right]\right. \\
& \left. -\frac{1}{2\alpha^2}\sum_{i=1}^{k}\left[\frac{1}{T_i}\exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right)\sum_{j=1}^{r_i}\frac{1}{x_{ij}}\right]\right\} \\
& \times \prod_{i=1}^{k}\left\{\left[T_i\exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\right]^{\frac{r_i}{2}}\right. \\
& \times \left[1 - \Phi\left(\frac{1}{\alpha}\left(\sqrt{\tau_i T_i\exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)}\right.\right.\right. \\
& \left.\left.\left.\left. - \sqrt{\frac{1}{\tau_i T_i}\exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right)}\right)\right)\right]^{n_i - r_i}\right\}.
\end{aligned}$$

See Smit and Raubenheimer (2022) for further details on this model. The posterior is mathematically intractable, hence MCMC methods have to be employed to draw posterior samples to be used for inference.

4.2. The GEW model

Let X be a continuous random variable that follows a Weibull distribution with scale parameter α and shape parameter β $(\alpha > 0, \beta > 0)$. The PDF is then given by

$$f\left(x\left|\alpha,\beta\right.\right) = \alpha\beta x^{\beta-1}\exp\left(-\alpha x^{\beta}\right), \quad x \geq 0. \tag{4.8}$$

A common assumption in the literature is that the Weibull scale parameter α is then dependent on the stress levels, whereas the shape parameter β is not (see, for example, Mazzuchi *et al.*, 1997; Soyer *et al.*, 2008; Upadhyay and Mukherjee, 2010). The reparameterization of α given by the generalized Eyring model, for this formulation of the Weibull model, is

$$\alpha_i = T_i\exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right), \tag{4.9}$$

where $\theta_1, \theta_2, \theta_3$, and θ_4 are unknown model parameters, and V_i is a function of the non-thermal stressor S_i (Escobar and Meeker, 2006). For a lifetime subjected to the i^{th} level of the stressors, it follows from

(4.8) and (4.9) that the Weibull PDF can be written as

$$f(x_i|\theta_1,\theta_2,\theta_3,\theta_4,\beta) = T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\beta x_i^{\beta-1} \times \exp\left[-T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right) x_i^{\beta}\right]. \quad (4.10)$$

The corresponding reliability function at some time τ is given by

$$R(\tau) = \exp\left[-T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\tau^{\beta}\right]. \quad (4.11)$$

From (4.10) and (4.11) it follows that the likelihood function for the GEW model is given by

$$\begin{aligned}
L\left(\underline{x}|\theta_1,\theta_2,\theta_3,\theta_4,\beta\right) &= \prod_{i=1}^{k}\left[\prod_{j=1}^{r_i} f\left(x_{ij}|\theta_1,\theta_2,\theta_3,\theta_4,\beta\right)\right][R(\tau_i)]^{n_i-r_i} \\
&= \prod_{i=1}^{k} \exp\left[-(n_i-r_i)T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\tau_i^{\beta}\right] \\
&\quad \times \beta^{r_i} T_i^{r_i} \exp\left(-\theta_1 r_i - \frac{\theta_2 r_i}{T_i} - \theta_3 r_i V_i - \frac{\theta_4 r_i V_i}{T_i}\right) \\
&\quad \times \prod_{j=1}^{r_i} x_{ij}^{\beta-1} \exp\left[-T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right) x_{ij}^{\beta}\right] \\
&= \beta^{\sum_{i=1}^{k} r_i} \exp\left(-\theta_1 \sum_{i=1}^{k} r_i - \theta_2 \sum_{i=1}^{k} \frac{r_i}{T_i} - \theta_3 \sum_{i=1}^{k} r_i V_i - \theta_4 \sum_{i=1}^{k} \frac{r_i V_i}{T_i}\right) \\
&\quad \times \exp\left[-\sum_{i=1}^{k}(n_i-r_i)T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\tau_i^{\beta}\right] \quad (4.12) \\
&\quad \times \exp\left[-\sum_{i=1}^{k}\sum_{j=1}^{r_i} T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right) x_{ij}^{\beta}\right]\left[\prod_{i=1}^{k}\prod_{j=1}^{r_i} T_i x_{ij}^{\beta-1}\right].
\end{aligned}$$

Assume that the priors on the unknown parameters θ_1, θ_2, θ_3, θ_4 and β are independent (see, for example, Soyer *et al.*, 2008; Upadhyay and Mukherjee, 2010), and given by

$$\pi(\theta_1,\theta_2,\theta_3,\theta_4,\beta) = \pi(\theta_1)\pi(\theta_2)\pi(\theta_3)\pi(\theta_4)\pi(\beta).$$

The joint posterior distribution is then given by

$$\pi\left(\theta_1,\theta_2,\theta_3,\theta_4,\beta\,|\underline{x}\right) \propto L\left(\underline{x}|\theta_1,\theta_2,\theta_3,\theta_4,\beta\right)\pi(\theta_1,\theta_2,\theta_3,\theta_4,\beta).$$

Gamma priors are imposed on all the parameters, with

$$\begin{aligned}
\theta_1 &\sim \Gamma(c_{10},c_{11}), \quad c_{10},c_{11}>0, \quad \pi_2(\theta_1) \propto \theta_1^{c_{10}-1}\exp(-c_{11}\theta_1), \\
\theta_2 &\sim \Gamma(c_{12},c_{13}), \quad c_{12},c_{13}>0, \quad \pi_2(\theta_2) \propto \theta_2^{c_{12}-1}\exp(-c_{13}\theta_2), \\
\theta_3 &\sim \Gamma(c_{14},c_{15}), \quad c_{14},c_{15}>0, \quad \pi_2(\theta_3) \propto \theta_3^{c_{14}-1}\exp(-c_{15}\theta_3), \\
\theta_4 &\sim \Gamma(c_{16},c_{17}), \quad c_{16},c_{17}>0, \quad \pi_2(\theta_4) \propto \theta_4^{c_{16}-1}\exp(-c_{17}\theta_4), \\
\beta &\sim \Gamma(c_{18},c_{19}), \quad c_{18},c_{19}>0, \quad \pi_2(\beta) \propto \beta^{c_{18}-1}\exp(-c_{19}\beta).
\end{aligned}$$

Table 1: Failure times for some device

#	Temperature (K)	Humidity	Failure time	#	Temperature (K)	Humidity	Failure time
1	333	0.9	521	12	353	0.8	504
2	333	0.9	561	13	353	0.9	115
3	333	0.9	575	14	353	0.9	119
4	333	0.9	599	15	353	0.9	150
5	333	0.9	609	16	353	0.9	152
6	333	0.9	684	17	353	0.9	153
7	333	0.9	709	18	353	0.9	155
8	333	0.9	713	19	353	0.9	156
9	353	0.8	345	20	353	0.9	164
10	353	0.8	357	21	353	0.9	199
11	353	0.8	439				

The joint prior for the GEW model is then given by

$$\pi_2\left(\theta_1,\theta_2,\theta_3,\theta_4,\beta\right) \propto \theta_1^{c_{10}-1}\theta_2^{c_{12}-1}\theta_3^{c_{14}-1}\theta_4^{c_{16}-1}\beta^{c_{18}-1}\exp\left(-c_{11}\theta_1 - c_{13}\theta_2 - c_{15}\theta_3 - c_{17}\theta_4 - c_{19}\beta\right), \quad (4.13)$$

and, using (4.12) and (4.13), the joint posterior is given by

$$\begin{aligned}\pi_2\left(\theta_1,\theta_2,\theta_3,\theta_4,\beta\,\middle|\,\underline{x}\right) \propto\ & \theta_1^{c_{10}-1}\theta_2^{c_{12}-1}\theta_3^{c_{14}-1}\theta_4^{c_{16}-1}\beta^{c_{18}-1}\exp\left(-c_{11}\theta_1 - c_{13}\theta_2 - c_{15}\theta_3 - c_{17}\theta_4 - c_{19}\beta\right)\\ & \times\beta^{\sum_{i=1}^k r_i}\exp\left(-\theta_1\sum_{i=1}^k r_i - \theta_2\sum_{i=1}^k \frac{r_i}{T_i} - \theta_3\sum_{i=1}^k r_iV_i - \theta_4\sum_{i=1}^k\frac{r_iV_i}{T_i}\right)\\ & \times\exp\left[-\sum_{i=1}^k (n_i - r_i)\,T_i\exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3V_i - \frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\right]\\ & \times\exp\left[-\sum_{i=1}^k\sum_{j=1}^{r_i} T_i\exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3V_i - \frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\right]\left[\prod_{i=1}^k\prod_{j=1}^{r_i} T_ix_{ij}^{\beta-1}\right].\end{aligned}$$

See Smit and Raubenheimer (2021) for further details on this model. Due to the complexity of the posterior, MCMC methods are used to draw posterior samples to base inferences on.

5. Application

An ALT data set from Reliasoft (2020) is used in this application. The data represent failure times for a certain device under accelerated temperature and relative humidity stressors. The normal use conditions are a temperature of $T_u = 313K$ and a relative humidity of $S_u = 0.5$. The data set, displayed in Table 1, contains failure data on 21 devices that were tested, where testing continued until all the devices had failed. The combined effect of these stressors must be investigated by a dual-stress acceleration model, such as the generalized Eyring model.

Widely used non-informative priors, such as the maximal data information, Jeffreys, reference, and probability matching priors are not considered, due to the complexity of the GEW and GEBS models. The prior specifications for these models are given in Table 2. Three choices of hyperparameters are used for each model, in order to investigate model selection. The models in this application are denoted by GEW_{BF1}, GEW_{BF2}, GEW_{BF3}, GEBS_{BF1}, GEBS_{BF2}, and GEBS_{BF3}. Flat gamma priors are imposed on the GEW_{BF1} and GEBS_{BF1} models. Subjective gamma priors with mean 5 and different variances are used for the GEW_{BF2} and GEW_{BF3}, as well as for the GEBS_{BF2} and GEBS_{BF3}

Table 2: Prior specifications for the Bayes factors application

Model	θ_1	θ_2	θ_3	θ_4	α or β
GEW_{BF1}	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$
GEW_{BF2}	$\Gamma(5,1)$	$\Gamma(5,1)$	$\Gamma(5,1)$	$\Gamma(5,1)$	$\Gamma(5,1)$
GEW_{BF3}	$\Gamma(125,25)$	$\Gamma(125,25)$	$\Gamma(125,25)$	$\Gamma(125,25)$	$\Gamma(125,25)$
GEBS_{BF1}	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$
GEBS_{BF2}	$\Gamma(2.5,0.5)$	$\Gamma(2.5,0.5)$	$\Gamma(2.5,0.5)$	$\Gamma(2.5,0.5)$	$\Gamma(2.5,0.5)$
GEBS_{BF3}	$\Gamma(5,1)$	$\Gamma(5,1)$	$\Gamma(5,1)$	$\Gamma(5,1)$	$\Gamma(5,1)$

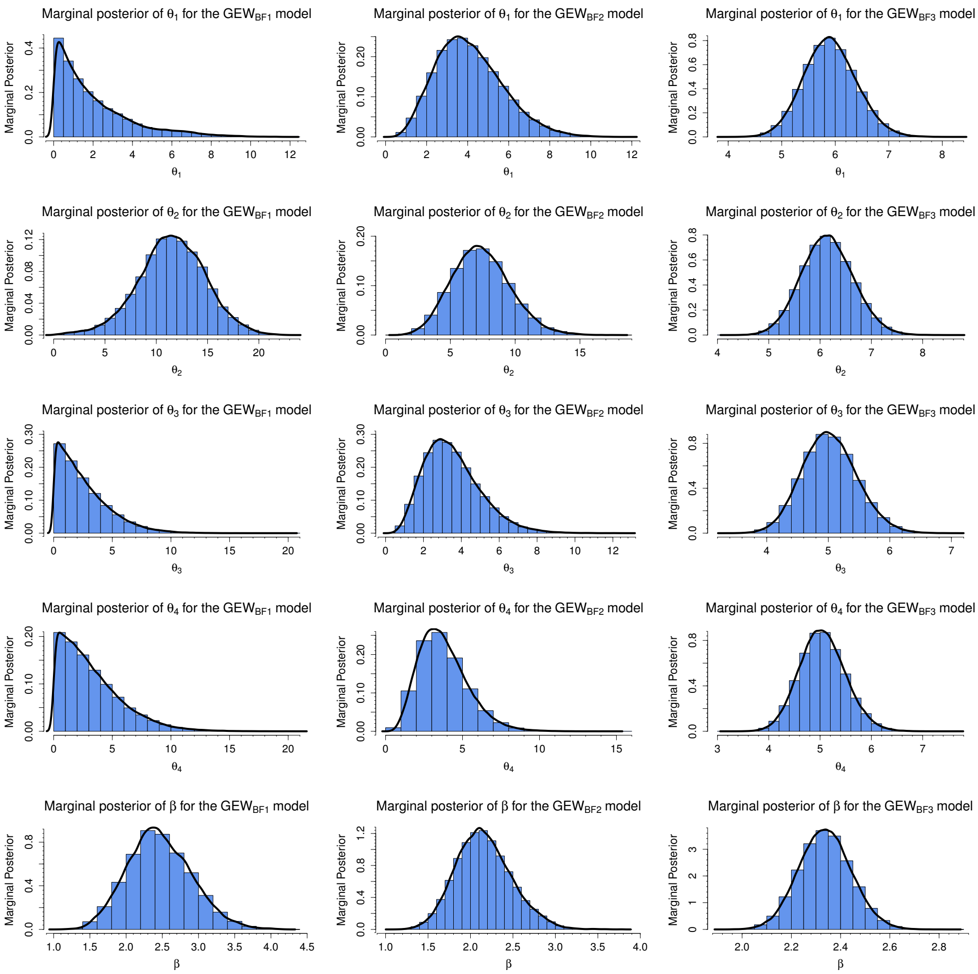

Figure 1: *Marginal posterior distributions for the GEW models.*

models. These subjective priors are chosen such that the DIC values for the models differ enough to illustrate more meaningful model selection conclusions. These choices are also made to investigate the sensitivity of the GEW and GEBS models. If a reliability engineer has prior information regarding the parameters of the models, this can easily be incorporated via the choice of the hyperparameters. As is illustrated in this application, weight is placed around a specific mean, with varying degrees of

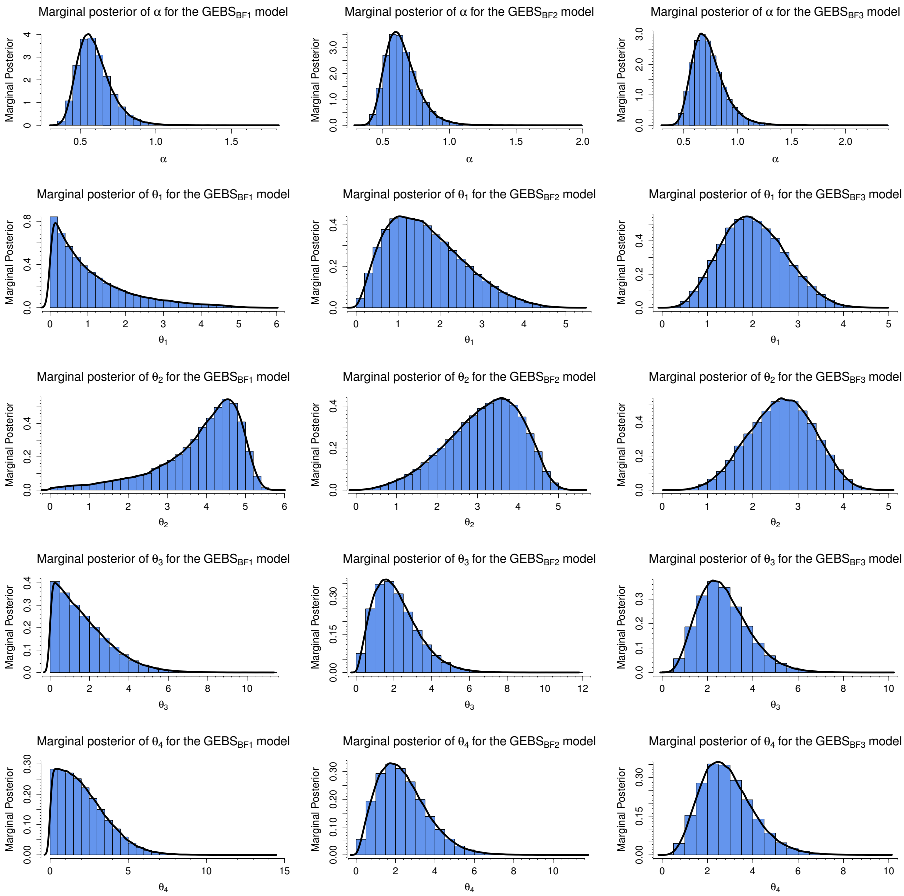

Figure 2: *Marginal posterior distributions for the GEBS models.*

certainty.

Since no prior information regarding the models are available, the prior model probabilities are set equal. This allows us to perform model selection by using only the Bayes factors. The models are implemented in OpenBUGS to generate posterior samples to base inference on. A single Markov chain is initiated for each model, with a burn-in of 50,000 iterations, after which 200,000 samples are obtained. Trace plots and the modified Gelman-Rubin statistic, proposed by Brooks and Gelman (1998), are used to verify that the Markov chains have converged before the burn-in iterations end. The Monte Carlo error is less than 5% of the sample standard deviation for all parameters in these models, indicating that enough samples have been generated.

The marginal posterior distributions for the models are given in Figures 1 and 2. It can be noted that the models where flat priors are used produce fairly skewed marginal posteriors. The marginal posteriors for the GEW$_{BF2}$, GEBS$_{BF2}$ and GEBS$_{BF3}$ models are somewhat less skewed, and those of the GEW$_{BF3}$ model are relatively symmetric.

The DIC values for the models considered in this application are given in Table 3. The GEBS$_{BF1}$ model exhibits the lowest DIC amongst the six models. It is clear that the DIC indicates a strong

Table 3: Deviance information criterion for the Bayes factors application

Model	GEW_{BF1}	GEW_{BF2}	GEW_{BF3}	$GEBS_{BF1}$	$GEBS_{BF2}$	$GEBS_{BF3}$
DIC	278.4	280.9	284.6	248.9	250.8	253.3

Table 4: Natural log of the marginal likelihood estimates for the models

Model #	Model	SMCE	LME	HME	PPDE
1	GEW_{BF1}	NA	−166.7920	−141.5369	−137.1797
2	GEW_{BF2}	−145.0426	−145.1006	−141.6805	−138.7164
3	GEW_{BF3}	NA	−176.4858	−145.1727	−141.3472
4	$GEBS_{BF1}$	NA	−170.1288	−142.2716	−136.8164
5	$GEBS_{BF2}$	NA	−147.7771	−144.2880	−137.6861
6	$GEBS_{BF3}$	−150.9820	−151.8292	−146.6435	−138.8855

preference towards the GEBS models in this case. Among the GEW models, the GEW_{BF1}, exhibits the lowest DIC. Again, it can be noted that the models with smaller variance subjective priors have higher DIC values than the models with flat priors or subjective priors with a larger variance. Considering all six models and keeping in mind that the $GEBS_{BF1}$ model has the lowest DIC, the guidelines in Burnham and Anderson (1998) indicate that $GEBS_{BF2}$ still has substantial support to be used for inference. There is considerably less support to choose $GEBS_{BF3}$, and the remaining three GEW models have almost no support to be chosen. By investigating the GEW models separately, we see that GEW_{BF1} has the lowest DIC, there is fairly substantial support for GEW_{BF2}, and considerably less support for GEW_{BF3}.

Due to the large discrepancies between many of the prior distributions in Table 2 and the marginal posteriors produced from the MCMC algorithm, only partial results are displayed for the SMCE in this application. Sampling from the prior for this estimator causes computational issues for the majority of the models. Values where computational issues are encountered are indicated by “NA”.

Table 4 shows the natural log of the marginal likelihood estimates for the models, given by the estimators. The natural log of the PPDE values, which are required to calculate the posterior Bayes factors, are also given. We number the models in this table in order to simplify the notation for the Bayes factors to follow.

The LME favours the GEW_{BF2} model, the HME favours the GEW_{BF1} model, and the PPDE favours the $GEBS_{BF1}$ model. It is interesting to note that the LME does not favour the models where flat priors are used. This can in part be explained by the skewed marginal posterior distributions produced by these models, seeing that the LME works well for symmetric distributions. The SMCE produces values close to that of the LME, where computational issues are not encountered. For the HME and PPDE there is not such a distinctive preference towards the GEBS models, as we have seen with the DIC. It must again be stressed that the DIC and Bayes factors measure model fit differently, as is explained in Spiegelhalter *et al.* (2002).

Table 5 contains the Bayes factor values produced using the different estimators, for all combinations of the models used in this application. Interpretations for these Bayes factors are also provided. Only one Bayes factor can be calculated using the SMCE, where the GEW_{BF2} model enjoys very strong evidence over the $GEBS_{BF3}$ model. Note that when using the LME there is positive to very strong evidence in favour of the GEW_{BF2} model. Considering the GEBS models separately, there is very strong evidence for the $GEBS_{BF2}$ model. When using the HME, we observe negligible to very strong evidence for the GEW_{BF1} model. Among only the GEBS models, there is positive to strong evidence in favour of the $GEBS_{BF1}$ model. It is clear that the Bayes factors, making use of the HME,

Table 5: Bayes factors and interpretations for model pairs using the various estimators

ij	SMCE		LME	
	B_{ij}	Interpretation	B_{ij}	Interpretation
12	NA	NA	3.7979×10^{-10}	Very strong evidence for model 2
13	NA	NA	1.6217×10^{4}	Very strong evidence for model 1
14	NA	NA	28.1309	Strong evidence for model 1
15	NA	NA	5.5201×10^{-9}	Very strong evidence for model 5
16	NA	NA	3.1751×10^{-7}	Very strong evidence for model 6
23	NA	NA	4.2700×10^{13}	Very strong evidence for model 2
24	NA	NA	7.4069×10^{10}	Very strong evidence for model 2
25	NA	NA	14.5346	Positive evidence for model 2
26	379.6942	Very strong evidence for model 2	836.0159	Very strong evidence for model 2
34	NA	NA	0.0017	Very strong evidence for model 4
35	NA	NA	3.4039×10^{-13}	Very strong evidence for model 5
36	NA	NA	1.9579×10^{-11}	Very strong evidence for model 6
45	NA	NA	1.9623×10^{-10}	Very strong evidence for model 5
46	NA	NA	1.1287×10^{-8}	Very strong evidence for model 6
56	NA	NA	57.5192	Strong evidence for model 5
ij	HME		PPDE	
	B_{ij}	Interpretation	B_{ij}	Interpretation
12	1.1543	Negligible evidence for model 1	4.6489	Positive evidence for model 1
13	37.9317	Strong evidence for model 1	64.5548	Strong evidence for model 1
14	2.0847	Negligible evidence for model 1	0.6954	Negligible evidence for model 4
15	15.6587	Positive evidence for model 1	1.6592	Negligible evidence for model 1
16	165.1082	Very strong evidence for model 1	5.5059	Positive evidence for model 1
23	32.8600	Strong evidence for model 2	13.8860	Positive evidence for model 2
24	1.8060	Negligible evidence for model 2	0.1496	Positive evidence for model 4
25	13.5651	Positive evidence for model 2	0.3569	Negligible evidence for model 5
26	143.0325	Strong evidence for model 2	1.1843	Negligible evidence for model 2
34	0.0550	Positive evidence for model 4	0.0108	Strong evidence for model 4
35	0.4128	Negligible evidence for model 5	0.0257	Strong evidence for model 4
36	4.3528	Positive evidence for model 3	0.0853	Positive evidence for model 6
45	7.5112	Positive evidence for model 4	2.3862	Negligible evidence for model 4
46	79.2000	Strong evidence for model 4	7.9181	Positive evidence for model 4
56	10.5442	Positive evidence for model 5	3.3183	Positive evidence for model 5

also favour the models where flat priors are used. This is in agreement with the conclusions from the DIC. The PPDE values are used to calculate the posterior Bayes factors. Here, the GEBS$_{BF1}$ model is supported with negligible to strong evidence. For the GEW models separately, we see positive to strong evidence for the GEW$_{BF1}$ model. The posterior Bayes factors also favour the models where flat priors are utilized. Table 6 contains the posterior model probability (PMP) values for each pair of models, given by the estimators discussed in this paper.

6. Simulation study

In this simulation study, model selection is performed via the DIC and Bayes factors, using the discussed methods to estimate the marginal likelihood. In order to perform model selection over all iterations of the simulation runs, the results for the average DIC values and average PMP values are displayed and discussed. Two simulation settings are investigated and the *R2OpenBUGS* package is utilized.

Consider ALT performed on some product, with temperature and relative humidity as the stressors. Assume that the normal operating conditions of the product are a temperature of $T_u = 350K$ and a

Table 6: Posterior model probabilities for model pairs using the various estimators

ij	SMCE		LME		HME		PPDE	
	PMP model m_i	PMP model m_j	PMP model m_i	PMP model m_j	PMP model m_i	PMP model m_j	PMP model m_i	PMP model m_j
12	NA	NA	0.0000	1.0000	0.5358	0.4642	0.8230	0.1770
13	NA	NA	0.9999	0.0001	0.9743	0.0257	0.9847	0.0153
14	NA	NA	0.9657	0.0343	0.6758	0.3242	0.4102	0.5898
15	NA	NA	0.0000	1.0000	0.9400	0.0600	0.6240	0.3760
16	NA	NA	0.0000	1.0000	0.9940	0.0060	0.8463	0.1537
23	NA	NA	1.0000	0.0000	0.9705	0.0295	0.9328	0.0672
24	NA	NA	1.0000	0.0000	0.6436	0.3564	0.1301	0.8699
25	NA	NA	0.9356	0.0644	0.9313	0.0687	0.2630	0.7370
26	0.9974	0.0026	0.9988	0.0012	0.9931	0.0069	0.5422	0.4578
34	NA	NA	0.0017	0.9983	0.0521	0.9479	0.0107	0.9893
35	NA	NA	0.0000	1.0000	0.2922	0.7078	0.0251	0.9749
36	NA	NA	0.0000	1.0000	0.8132	0.1868	0.0786	0.9214
45	NA	NA	0.0000	1.0000	0.8825	0.1175	0.7047	0.2953
46	NA	NA	0.0000	1.0000	0.9875	0.0125	0.8879	0.1121
56	NA	NA	0.9829	0.0171	0.9134	0.0866	0.7684	0.2316

Table 7: Average deviance information criterion for the different simulation settings and sample sizes

Model #	1	2	3	4	5	6
Model	GEW_{BF1}	GEW_{BF2}	GEW_{BF3}	$GEBS_{BF1}$	$GEBS_{BF2}$	$GEBS_{BF3}$
$\Theta_W = \{\beta = 1, \theta_1 = 1, \theta_2 = 1, \theta_3 = 1, \theta_4 = 1\}$						
$n = 30$	252.0	256.1	370.9	217.8	218.5	221.2
$n = 60$	502.2	504.8	675.3	438.8	439.3	440.7
$\Theta_{BS} = \{\alpha = 3, \theta_1 = 0.5, \theta_2 = 2, \theta_3 = 3, \theta_4 = 1.5\}$						
$n = 30$	427.5	432.1	566.4	378.7	378.6	379.0
$n = 60$	853.0	855.8	1051.6	755.6	755.3	755.4

relative humidity of $S_u = 0.3$. The accelerated stress levels the product is tested under are

$$(T_i, S_i) = \{(400, 0.5);\ (450, 0.6);\ (500, 0.7)\}.$$

In the first setting, ALT data is generated from the Weibull distribution specified in (4.10), with parameter values $\Theta_W = \{\beta = 1, \theta_1 = 1, \theta_2 = 1, \theta_3 = 1, \theta_4 = 1\}$. For the second setting, ALT data is generated from the Birnbaum-Saunders distribution specified in (4.3), with parameter values $\Theta_{BS} = \{\alpha = 3, \theta_1 = 0.5, \theta_2 = 2, \theta_3 = 3, \theta_4 = 1.5\}$. In both settings the same prior distributions as in Table 2 are used. Two sample sizes, $n = \{30, 60\}$, are considered, where the samples are split evenly between the stress levels. Due to the computational intensity of the simulations, 1,000 iterations are used for each setting and sample size. A burn-in of 50,000 iterations is set, whereafter 50,000 posterior samples are generated to base inference on.

Table 7 contains the average DIC values for the settings and sample sizes used in this simulation study. It is clear that there is a strong preference towards the GEBS models, even when data is generated from the Weibull distribution. The $GEBS_{BF1}$ and $GEBS_{BF2}$ models have substantial support. Considering the GEW models only, there is substantial support for the GEW_{BF1} model. This may be an indication that the DIC might not be the best tool for model selection when fitting different distributions to data. However, considering the GEW and GEBS models separately, it does seem to produce the expected results regarding model selection. The average PMP values, using the various estimators, are displayed in Tables 8 to 11. Note that the SMCE is omitted from the simulation study, since computational issues are encountered for the majority of models and iterations.

Considering setting Θ_W, the LME produces inconsistent results over the two sample sizes. For

Table 8: Average posterior model probabilities under setting Θ_W and sample size $n = 30$

ij	LME		HME		PPDE	
	PMP model m_i	PMP model m_j	PMP model m_i	PMP model m_j	PMP model m_i	PMP model m_j
12	0.0000	1.0000	0.9878	0.0122	0.7902	0.2098
13	0.9968	0.0032	1.0000	0.0000	1.0000	0.0000
14	0.8307	0.1693	0.7896	0.2104	0.7854	0.2146
15	0.0260	0.9740	0.8940	0.1060	0.8002	0.1998
16	0.1701	0.8299	0.9970	0.0030	0.8427	0.1573
23	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000
24	1.0000	0.0000	0.3250	0.6750	0.6572	0.3428
25	0.4703	0.5297	0.4590	0.5410	0.6749	0.3251
26	0.9700	0.0300	0.8916	0.1084	0.7270	0.2730
34	0.0154	0.9846	0.0002	0.9998	0.0007	0.9993
35	0.0019	0.9981	0.0005	0.9995	0.0009	0.9991
36	0.0083	0.9917	0.0013	0.9987	0.0022	0.9978
45	0.0000	1.0000	0.8081	0.1919	0.5649	0.4351
46	0.0146	0.9854	0.9974	0.0026	0.7103	0.2897
56	0.9999	0.0001	0.9881	0.0119	0.6668	0.3332

Table 9: Average posterior model probabilities under setting Θ_W and sample size $n = 60$

ij	LME		HME		PPDE	
	PMP model m_i	PMP model m_j	PMP model m_i	PMP model m_j	PMP model m_i	PMP model m_j
12	0.0000	1.0000	0.9737	0.0263	0.7025	0.2975
13	0.0288	0.9712	1.0000	0.0000	1.0000	0.0000
14	1.0000	0.0000	0.9102	0.0898	0.9109	0.0891
15	1.0000	0.0000	0.9279	0.0721	0.9148	0.0852
16	1.0000	0.0000	0.9758	0.0242	0.9238	0.0762
23	0.1973	0.8027	1.0000	0.0000	1.0000	0.0000
24	1.0000	0.0000	0.7032	0.2968	0.8775	0.1225
25	1.0000	0.0000	0.7393	0.2607	0.8823	0.1177
26	1.0000	0.0000	0.8513	0.1487	0.8934	0.1066
34	1.0000	0.0000	0.0000	1.0000	0.0000	1.0000
35	0.9981	0.0019	0.0000	1.0000	0.0003	0.9997
36	1.0000	0.0000	0.0000	1.0000	0.0009	0.9991
45	0.0000	1.0000	0.6947	0.3053	0.5617	0.4383
46	0.0239	0.9761	0.9454	0.0546	0.6564	0.3436
56	1.0000	0.0000	0.9046	0.0954	0.6126	0.3874

$n = 30$, model GEBS_{BF2} is favoured and for $n = 60$, model GEW_{BF3}. This is not the expected result considering the simulation setting, and can again be attributed to the fact that the LME only works well for symmetrical distributions. For the HME and PPDE the results are consistent over sample sizes and there is strong evidence for the GEW_{BF1} model. This is the expected result, since the parameters for this setting were chosen to be significantly different to those imposed by the subjective priors.

Considering setting Θ_{BS}, the LME again produces inconsistent results over the two sample sizes. The HME shows the strongest evidence toward the GEBS_{BF1} model, whereas the PPDE does so towards the GEBS_{BF2} model. It must however be noted that the average PMP values for both these models are close to 0.5 for both the HME and PPDE, which may indicate that they are both suitable models. This is the expected result when considering the parameters for this setting. The parameters were chosen to be different to those imposed by the subjective priors, but to a lesser degree than in setting Θ_W. Taking into account the leniency, in terms of the prior distribution variance, of the priors imposed on GEBS_{BF2}, it can be expected that either this model or the GEBS model where flat priors are imposed (GEBS_{BF1}) can be suitable for setting Θ_{BS}.

Table 10: Average posterior model probabilities under setting Θ_{BS} and sample size $n = 30$

$i\,j$	LME		HME		PPDE	
	PMP model m_i	PMP model m_j	PMP model m_i	PMP model m_j	PMP model m_i	PMP model m_j
12	0.0000	1.0000	0.9933	0.0067	0.8158	0.1842
13	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000
14	0.0128	0.9872	0.1074	0.8926	0.0887	0.9113
15	0.0000	1.0000	0.1078	0.8922	0.0930	0.9070
16	0.0000	1.0000	0.2387	0.7613	0.1065	0.8935
23	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000
24	0.9446	0.0554	0.0035	0.9965	0.0336	0.9664
25	0.0004	0.9996	0.0043	0.9957	0.0349	0.9651
26	0.0029	0.9971	0.0112	0.9888	0.0402	0.9598
34	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
35	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
36	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
45	0.0000	1.0000	0.4889	0.5111	0.5122	0.4878
46	0.0000	1.0000	0.7438	0.2562	0.5562	0.4438
56	0.9718	0.0282	0.7642	0.2358	0.5463	0.4537

Table 11: Average posterior model probabilities under setting Θ_{BS} and sample size $n = 60$

$i\,j$	LME		HME		PPDE	
	PMP model m_i	PMP model m_j	PMP model m_i	PMP model m_j	PMP model m_i	PMP model m_j
12	0.0000	1.0000	0.9841	0.0159	0.7158	0.2842
13	0.3820	0.6180	1.0000	0.0000	1.0000	0.0000
14	1.0000	0.0000	0.0237	0.9763	0.0219	0.9781
15	0.9970	0.0030	0.0241	0.9759	0.0220	0.9780
16	0.9984	0.0016	0.0398	0.9602	0.0234	0.9766
23	0.8378	0.1622	1.0000	0.0000	1.0000	0.0000
24	1.0000	0.0000	0.0008	0.9992	0.0122	0.9878
25	1.0000	0.0000	0.0007	0.9993	0.0122	0.9878
26	1.0000	0.0000	0.0021	0.9979	0.0129	0.9871
34	0.9843	0.0157	0.0000	1.0000	0.0000	1.0000
35	0.9019	0.0981	0.0000	1.0000	0.0000	1.0000
36	0.9243	0.0757	0.0000	1.0000	0.0000	1.0000
45	0.0000	1.0000	0.4879	0.5121	0.4999	0.5001
46	0.0000	1.0000	0.6143	0.3857	0.5204	0.4796
56	0.9738	0.0262	0.6268	0.3732	0.5212	0.4788

7. Conclusions

In this paper, the use of Bayes factors and the DIC for model selection are compared in a Bayesian ALT setup. Two dual-stress models, namely the GEW and GEBS models with gamma priors, are utilized for this comparison. The posterior distributions for these models can not be written in closed form, which complicates the calculation of the Bayes factors. MCMC methods are employed to generate posterior samples to base inference on. Methods for estimating the marginal likelihood, without further complicating the sampling process, is explored. These methods include the SMCE, LME, HME, and PPDE used for estimating posterior Bayes factors.

The models are applied to an ALT data set where the stressors are temperature and relative humidity. Several choices of hyperparameters are used in order to illustrate the use of the DIC and Bayes factors in model selection. It is interesting to note that the DIC shows definitive support for the GEBS models above the GEW models. The models where flat priors are imposed on the model parameters are favoured by the DIC. The different methods for estimating the marginal likelihood give variable conclusions. The SMCE causes computational issues for some of the models and only partial results are given. The LME results in Bayes factors which show virtually no evidence in favour of the models with flat priors. The Bayes factors produced by the HME and the posterior Bayes factors have results that are more comparable to the DIC. Viewing the GEW and GEBS models separately, these Bayes factors also favour the models with flat priors. The HME shows more evidence in support of the GEW models, whereas the posterior Bayes factors support the GEBS models to a greater extent. It is interesting to note that the conclusions from the posterior Bayes factors most closely relate to those made by the DIC.

A simulation study is also conducted, where two simulation settings are used as well as two sample sizes. The average DIC and average PMP values are compared for the models under the two settings. The DIC favours the GEBS models, even when data were generated from the Weibull distribution. The LME produces inconsistent average PMP results over the two sample sizes. The expected results are obtained from the HME and PPDE for both the Weibull and Birnbaum-Saunders settings.

Acknowledgements

The authors would like to thank the editor and reviewers for the comments and recommendations, which has led to a significant improvement of the paper.

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Received November 19, 2021; Revised March 28, 2022; Accepted May 19, 2022