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MANNHEIM PARTNER P-TRAJECTORIES IN THE EUCLIDEAN 3-SPACE E^3

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Abstract. Mannheim introduced the concept of a pair of curves, called as Mannheim partner curves, in 1878. Until now, Mannheim partner curves have been studied widely in the literature. In this study, we take into account of this concept according to Positional Adapted Frame (PAF) for the particles moving in the 3-dimensional Euclidean space. We introduce a new type special trajectory pairs which are called Mannheim partner P-trajectories in the Euclidean 3-space. The relationships between the PAF elements of this pair are investigated. Also, the relations between the Serret-Frenet basis vectors of Mannheim partner P-trajectories are given. Afterwards, we obtain the necessary conditions for one of these trajectories to be an osculating curve and for other to be a rectifying curve. Moreover, we provide an example including an illustrative figure.

1. Introduction

The curve theory and the applications of it are a substantial notion for several disciplines such as differential geometry, robotics, and so on. In the existing literature, lots of studies have been done and ongoing since the topic is attached to the attention of several researchers. In differential geometry, the concept of spatial curves in 3-dimensional Euclidean space has a significant place and is one of the main topics. Curve pairs such as Mannheim curve pairs and Bertrand curve pairs also can be given as interesting and popular research areas for many mathematicians, especially geometers. The moving frames are very useful tools to study the local theory of these kinds of curve pairs.

Developing new moving frames that have a common base vector with the Serret-Frenet frame is an attractive and intriguing topic for several researchers. The discovery of the Serret-Frenet frame is a groundbreaking invention. Then several researchers were interested in the developing new moving frames such as Bishop frames (type-1, type-2, type-3) [1, 16, 19]. By the same logic with these studies, Özen and Tosun presented a new type frame which is called

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as Positional Adapted Frame (shortly PAF) for the trajectories which have non-vanishing angular momentum and are a unit speed curve in the three-dimensional Euclidean space E^3 [12]. Also, the trajectories constructed by Smarandache curves with respect to the PAF were studied by Özen and Tosun in [13]. Several studies have been presented and ongoing with respect to this new and attractive frame. For example, Gürbüz constructed the Positional Adapted Frame in Minkowski 3-space $\mathbb{R}^3_1$ with the help of the method developed in [12]. Also, in this study, the author considered the evolution of an electric field according to this frame in $\mathbb{R}^3_1$ [6]. Besides, Solouma examined the characterization of some special Smarandache trajectory curves of moving point particles by using the PAF in [17].

Mannheim partner curves are important and surprising special curves. The principal normal line of one of these partner curves coincides with the binormal line of the other partner curve at the corresponding points of these curves. Although the first study was performed in 1878 by Mannheim [2], Mannheim curves were not well known until the early 2000s. In the early 2000s, Liu and Wang considered the Mannheim partner curves [18, 9]. In Euclidean 3-space, the necessary and sufficient conditions were determined for a curve to possess a Mannheim partner curve in [9]. The authors also obtained similar conditions for the curves in Minkowski 3-space in the same study. Mannheim offsets of ruled surfaces were defined in the study [11]. Dual Mannheim curves were considered in the studies [14] and [5]. Also, the concept of Mannheim partner curves were extended to different frames. For example, Kazaz et al. [8] introduced the Mannheim partner D-curves considering the Darboux frames of surface curves. Similarly, Masal and Azak introduced the Mannheim B-curves using the Bishop frame [10].

This paper is organized as follows. In Section 2, we recall some required information with respect to the used notions and notations from beginning to end of this paper. In Section 3, we take into consideration the concept of Mannheim partner curves with regard to PAF in the Euclidean 3-space. We present a new type special trajectory pairs which are called Mannheim partner P-trajectories in 3 dimensional Euclidean space. Also, the relationships between the PAF elements of the aforementioned partners are investigated. Afterwards, the relations between the Serret-Frenet basis vectors of Mannheim partner P-trajectories are discussed. Furthermore, we determine the necessary conditions for one of these partners to be an osculating curve and for other to be a rectifying curve. Finally, we provide an illustrative example so that the readers can visualize the Mannheim partner P-trajectories.

2. Basic Concepts

In this part, we recall some required and fundamental notions and notations which are used throughout this paper.

In E^3, let $\boldsymbol{\delta} = (\delta_1, \delta_2, \delta_3), \boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$ be given. The standard inner product of them and the norm of $\boldsymbol{\delta}$ are given as $\langle \boldsymbol{\delta}, \boldsymbol{\eta} \rangle = \delta_1\eta_1 + \delta_2\eta_2 + \delta_3\eta_3$ and $\|\boldsymbol{\delta}\| = \sqrt{\langle \boldsymbol{\delta}, \boldsymbol{\delta} \rangle}$, respectively. If a differentiable curve $\alpha = \alpha(s) : I \subset \mathbb{R} \to E^3$ satisfies the condition $\left\| \frac{d\alpha}{ds} \right\| = 1$ for each $s \in I$, it is called as a unit speed curve. Then, s is said to be arc-length parameter of α. If the derivative of a differentiable curve is non-zero along this curve, it is called a regular curve. All regular curves can be re-parameterized by the arc-length of itself [15]. It should be noted that we will show the differentiation according to the arc-length parameter s with the symbol prime "$\prime$" from the beginning of this study to the end of this study.

Firstly, we recall briefly the Serret-Frenet frame in E^3. Let us take into account of a point particle P of a constant mass moves on a unit speed curve $\alpha = \alpha(s)$. $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ represents the Serret-Frenet frame of α. $\mathbf{T}(s), \mathbf{N}(s),\ \mathbf{B}(s)$, which are called as tangent vector, principal normal vector, binormal vector, are calculated as $\mathbf{T}(s) = \alpha'(s)$, $\mathbf{N}(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}$ $\mathbf{B}(s) = \mathbf{T}(s) \wedge \mathbf{N}(s)$. In addition to this, the Serret-Frenet derivative formulas are expressed as follows:

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix}$$

where $\kappa(s) = \|\mathbf{T}'(s)\|$ is the curvature and $\tau(s) = -\langle \mathbf{B}'(s), \mathbf{N}(s) \rangle$ is the torsion [15]. Throughout this paper, we will consider the curves whose Serret-Frenet frames are well defined. That is, we will consider the curves which have non-zero curvatures.

The key element of the construction of Positional Adapted Frame is the angular momentum vector of the moving point particle about the origin. This vector is defined with the help of the vector product of the position vector $\mathbf{x} = \langle \alpha(s), \mathbf{T}(s) \rangle \mathbf{T}(s) + \langle \alpha(s), \mathbf{N}(s) \rangle \mathbf{N}(s) + \langle \alpha(s), \mathbf{B}(s) \rangle \mathbf{B}(s)$ and linear momentum vector $\mathbf{p}(t) = m\left(\frac{ds}{dt}\right)\mathbf{T}(s)$ of the moving point particle where t denotes the time. That is, the angular momentum vector is determined by $\mathbf{H}^O = m\langle \alpha(s), \mathbf{B}(s) \rangle \left(\frac{ds}{dt}\right)\mathbf{N}(s) - m\langle \alpha(s), \mathbf{N}(s) \rangle \left(\frac{ds}{dt}\right)\mathbf{B}(s)$ where the mass is shown with m. The angular momentum vector has a major importance in Newtonian mechanics. One of the most famous and important conservation laws in physics is the conservation of angular momentum. The angular momentum is always conserved in a closed system since when the net torque is zero, angular momentum is constant [4]. Suppose that the aforementioned angular momentum vector $\mathbf{H}^O$ does not equal to zero vector along $\alpha = \alpha(s)$. This supposition warrants that the functions $\langle \alpha(s), \mathbf{N}(s) \rangle$ and $\langle \alpha(s), \mathbf{B}(s) \rangle$ do not equal to zero simultaneously during the motion of the moving point particle. Hence, it can be said that the tangent line of $\alpha = \alpha(s)$ never passes through the origin. Then, the PAF can be constructed which is denoted by $\{\mathbf{T}(s), \mathbf{M}(s), \mathbf{Y}(s)\}$ along $\alpha = \alpha(s)$. Let us consider the vector whose initial point is the foot of the perpendicular (from origin to instantaneous rectifying plane) and endpoint is the foot of the perpendicular (from origin to instantaneous osculating plane).

The equivalent of this vector at the point $\alpha(s)$ contributes to determine the vector $\mathbf{Y}(s)$. Therefore, the base vector $\mathbf{Y}(s)$ of PAF is obtained as follows:

$$\mathbf{Y}(s)=\frac{\langle -\alpha(s),\mathbf{N}(s)\rangle}{\sqrt{\langle\alpha(s),\mathbf{N}(s)\rangle^2+\langle\alpha(s),\mathbf{B}(s)\rangle^2}}\mathbf{N}(s)+\frac{\langle\alpha(s),\mathbf{B}(s)\rangle}{\sqrt{\langle\alpha(s),\mathbf{N}(s)\rangle^2+\langle\alpha(s),\mathbf{B}(s)\rangle^2}}\mathbf{B}(s).$$

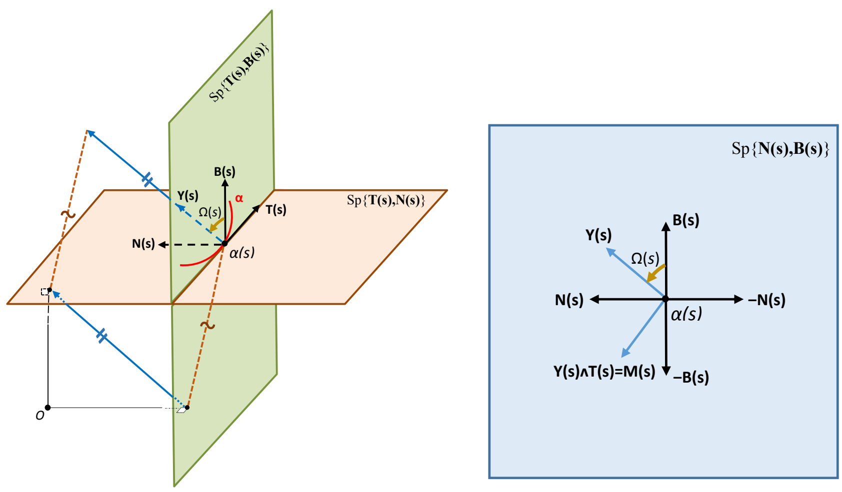
The second base vector $\mathbf{M}(s)$ of PAF is obtained with the help of the vector product $\mathbf{Y}(s)\wedge\mathbf{T}(s)$ and it is defined as follows:

$$\mathbf{M}(s)=\frac{\langle\alpha(s),\mathbf{B}(s)\rangle}{\sqrt{\langle\alpha(s),\mathbf{N}(s)\rangle^2+\langle\alpha(s),\mathbf{B}(s)\rangle^2}}\mathbf{N}(s)+\frac{\langle\alpha(s),\mathbf{N}(s)\rangle}{\sqrt{\langle\alpha(s),\mathbf{N}(s)\rangle^2+\langle\alpha(s),\mathbf{B}(s)\rangle^2}}\mathbf{B}(s).$$

The following relation between the Serret-Frenet frame and PAF can be given as:

$$\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{M}(s)\\ \mathbf{Y}(s)\end{pmatrix}=\begin{pmatrix}1&0&0\\ 0&\cos\Omega(s)&-\sin\Omega(s)\\ 0&\sin\Omega(s)&\cos\Omega(s)\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{B}(s)\end{pmatrix} \tag{1}$$

where $\Omega(s)$ is the angle between the vector $\mathbf{B}(s)$ and the vector $\mathbf{Y}(s)$ which is positively oriented from the vector $\mathbf{B}(s)$ to vector $\mathbf{Y}(s)$ (see Figure 1).



(A) (B)

FIGURE 1. An illustration for the Positional Adapted Frame [12]

Additionally, the derivative formulas of PAF can be presented as follows [12]:

$$\begin{pmatrix}\mathbf{T}'(s)\\ \mathbf{M}'(s)\\ \mathbf{Y}'(s)\end{pmatrix}=\begin{pmatrix}0&k_1(s)&k_2(s)\\ -k_1(s)&0&k_3(s)\\ -k_2(s)&-k_3(s)&0\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{M}(s)\\ \mathbf{Y}(s)\end{pmatrix}$$

where

$$k_1(s) = \kappa(s)\cos\Omega(s)$$
$$k_2(s) = \kappa(s)\sin\Omega(s)$$
$$k_3(s) = \tau(s) - \Omega'(s).$$

Besides, the following equations hold:

$$\frac{k_2(s)}{k_1(s)} = \tan\Omega(s)$$
$$k_1(s) = \sqrt{k_1^2(s) + k_2^2(s)}\cos\Omega(s)$$
$$k_2(s) = \sqrt{k_1^2(s) + k_2^2(s)}\sin\Omega(s).$$

In the following equation, the method for calculation of angle $\Omega(s)$ is given:

$$\Omega(s) = \begin{cases} \arctan\left(-\frac{\langle \alpha(s), \mathbf{N}(s)\rangle}{\langle \alpha(s), \mathbf{B}(s)\rangle}\right) & if \;\; \langle \alpha(s), \mathbf{B}(s)\rangle > 0 \\ \arctan\left(-\frac{\langle \alpha(s), \mathbf{N}(s)\rangle}{\langle \alpha(s), \mathbf{B}(s)\rangle}\right) + \pi & if \;\; \langle \alpha(s), \mathbf{B}(s)\rangle < 0 \\ -\frac{\pi}{2} \;\; if \;\; \langle \alpha(s), \mathbf{B}(s)\rangle = 0\,, \;\; \langle \alpha(s), \mathbf{N}(s)\rangle > 0 \\ \frac{\pi}{2} \;\; if \;\; \langle \alpha(s), \mathbf{B}(s)\rangle = 0\,, \;\; \langle \alpha(s), \mathbf{N}(s)\rangle < 0. \end{cases}$$

The orthonormal set $\{\mathbf{T}(s), \mathbf{M}(s), \mathbf{Y}(s)\}$ is called Positional Adapted Frame (PAF) and any elements of the set $\{\mathbf{T}(s), \mathbf{M}(s), \mathbf{Y}(s), k_1(s), k_2(s), k_3(s)\}$ is said to be PAF apparatus of $\alpha = \alpha(s)$ [12]. Finally we give the definitions of the rectifying and osculating curves in 3-dimensional Euclidean space since we will discuss these topics in the next section. A curve $\beta = \beta(s)$ is called as osculating curve (or rectifying curve) if its position vector always lies in its osculating plane (or rectifying plane). One can find more details on this topic in [3, 7].

Theorem 2.1. [12] *Let $\alpha = \alpha(s)$ be the unit speed parameterization of the trajectory. Then α is a rectifying curve if and only if $k_2 = 0$.*

Theorem 2.2. [12] *Let $\alpha = \alpha(s)$ be the unit speed parameterization of the trajectory. Then α is an osculating curve if and only if $k_1 = 0$.*

For more detailed information with respect to the PAF in Euclidean 3-space or Lorentzian 3-space, we can refer to the studies [12, 13, 6, 17].

3. Mannheim Partner P-Trajectories

In this section, Mannheim partner P-trajectories will be defined and some characterizations of these trajectories will be investigated.

Definition 3.1. *Let Q and Q^* be the point particles of constant masses which move in the 3-dimensional Euclidean space. Denote the unit speed parameterization of the trajectories of Q and Q^* by $\alpha = \alpha(s)$ and $\alpha^* = \alpha^*(s^*)$ respectively. Show the PAF apparatus of the trajectories α and α^* with $\{\mathbf{T}, \mathbf{M}, \mathbf{Y}, k_1, k_2, k_3\}$ and $\{\mathbf{T}^*, \mathbf{M}^*, \mathbf{Y}^*, k_1^*, k_2^*, k_3^*\}$ respectively. If the PAF vector $\mathbf{M}$ coincides with the PAF vector $\mathbf{Y}^*$ at the corresponding points of the trajectory curves α and α^*, then α is called as a Mannheim partner P-trajectory of α^*. Also, the pair $\{\alpha, \alpha^*\}$ is said to be a Mannheim P-pair.*

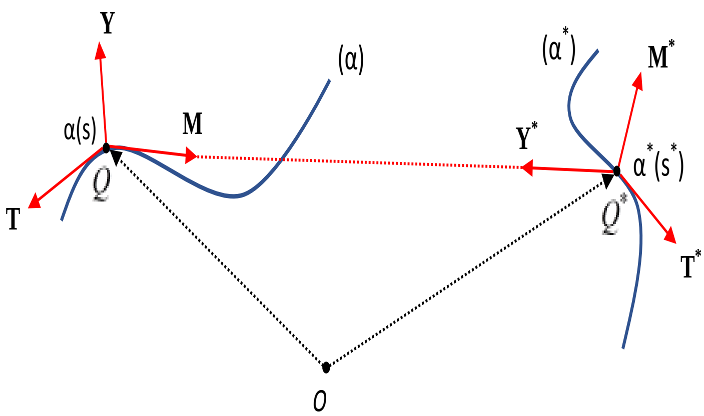


Figure 2. Mannheim partner P-trajectories

Let β be the angle between the tangent vectors $\mathbf{T}$ and $\mathbf{T}^*$. In that case, the following matrix equation can be easily given

$$\begin{pmatrix}\mathbf{T}\\ \mathbf{M}\\ \mathbf{Y}\end{pmatrix} = \begin{pmatrix}\cos\beta & \sin\beta & 0\\ 0 & 0 & 1\\ -\sin\beta & \cos\beta & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}^*\\ \mathbf{M}^*\\ \mathbf{Y}^*\end{pmatrix} \tag{2}$$

by taking into consideration the definition of Mannheim P-pair.

Theorem 3.2. *Let $\{\alpha = \alpha(s), \ \alpha^* = \alpha^*(s)\}$ be any Mannheim P-pair in E^3. Then, the distance between the corresponding points of α and α^* is constant.*

Proof. We can easily write

$$\alpha(s) = \alpha^*(s^*) + \chi(s^*)\,\mathbf{Y}^*(s^*) \tag{3}$$

where χ is a real valued smooth function of s^*(see Figure 2). The equation

$$\mathbf{T}\frac{ds}{ds^*} = (1 - \chi k_2^*)\,\mathbf{T}^* - \chi k_3^*\mathbf{M}^* + \chi'\mathbf{Y}^* \tag{4}$$

is obtained by taking the derivative of the equation (3) with respect to s* and using the PAF derivative formulas. Since $\mathbf{T}$, $\mathbf{T}^*$ and $\mathbf{M}^*$ are orthogonal to $\mathbf{Y}^*$ and also $\mathbf{Y}^*$ is a unit vector, we get $\chi' = 0$ by means of the inner product. Hence, χ is a non-zero constant and the equation (4) takes the following form:

$$\mathbf{T}\frac{ds}{ds^*} = (1-\chi k_2^*)\,\mathbf{T}^* - \chi k_3^* \mathbf{M}^*. \tag{5}$$

Now, let us take into consideration the distance function between two points. In that case the equality

$$d\left(\alpha\left(s\right),\ \alpha^*\left(s^*\right)\right) = \|\alpha\left(s\right) - \alpha^*\left(s^*\right)\| = \|\chi \mathbf{Y}^*\| = |\chi|$$

is found. Consequently, we can conclude that the distance between each corresponding points of α and α^* is constant. □

Theorem 3.3. *Let $\{\alpha = \alpha\left(s\right),\ \alpha^* = \alpha^*\left(s\right)\}$ be a Mannheim P-pair in E^3. In this case, the relationships between the PAF vectors of α and α^* are given as in the following:*

$$\begin{pmatrix}\mathbf{T}\\ \mathbf{M}\\ \mathbf{Y}\end{pmatrix} = \begin{pmatrix}\frac{ds^*}{ds}\left(1-\chi k_2^*\right) & -\frac{ds^*}{ds}\chi k_3^* & 0\\ 0 & 0 & 1\\ \frac{ds^*}{ds}\chi k_3^* & \frac{ds^*}{ds}\left(1-\chi k_2^*\right) & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}^*\\ \mathbf{M}^*\\ \mathbf{Y}^*\end{pmatrix}. \tag{6}$$

Proof. Assume that $\{\alpha,\ \alpha^*\}$ be a Mannheim P-pair in E^3. Then we have the equations (2) and (5) as mentioned earlier. These two equations yield the following:

$$\cos\beta\frac{ds}{ds^*}\,\mathbf{T}^* + \sin\beta\,\frac{ds}{ds^*}\mathbf{M}^* = (1-\chi k_2^*)\,\mathbf{T}^* - \chi k_3^* \mathbf{M}^*.$$

Since a vector can be written uniquely in terms of the basis vectors, we obtain

$$\left.\begin{aligned}\cos\beta &= \frac{ds^*}{ds}\left(1-\chi k_2^*\right)\\ \sin\beta &= -\frac{ds^*}{ds}\chi k_3^*\end{aligned}\right\}. \tag{7}$$

Substituting the equation (7) in the equation (2) gives us the desired result. □

Corollary 3.4. *Let $\{\alpha,\ \alpha^*\}$ be a Mannheim P-pair in E^3. In this case*

$$\tan\beta = \frac{\chi k_3^*}{\chi k_2^* - 1}$$

where β is the angle between $\mathbf{T}$ and $\mathbf{T}^*$.

Theorem 3.5. *Let $\{\alpha = \alpha\left(s\right),\ \alpha^* = \alpha^*\left(s^*\right)\}$ be a Mannheim P-pair in E^3 and their Serret-Frenet apparatus be $\{\mathbf{T},\ \mathbf{N},\ \mathbf{B},\ \kappa,\ \tau\}$ and $\{\mathbf{T}^*,\ \mathbf{N}^*,\ \mathbf{B}^*,\ \kappa^*,\ \tau^*\}$, respectively. In that case, the relationships between the Serret-Frenet vectors of this pair are as in the following:*

$$
\begin{aligned}
\mathbf{T}^* =& \frac{ds^*}{ds}\left(1-\chi k_2^*\right)\mathbf{T} + \frac{ds^*}{ds}\chi k_3^* \sin\Omega \mathbf{N} + \frac{ds^*}{ds}\chi k_3^* \cos\Omega \mathbf{B} \\
\mathbf{N}^* =& -\frac{ds^*}{ds}\chi k_3^* \cos\Omega^* \mathbf{T} + \left(\sin\Omega^*\cos\Omega + \frac{ds^*}{ds}\left(1-\chi k_2^*\right)\cos\Omega^*\sin\Omega\right)\mathbf{N} \\
&+ \left(-\sin\Omega^*\sin\Omega + \frac{ds^*}{ds}\left(1-\chi k_2^*\right)\cos\Omega^*\cos\Omega\right)\mathbf{B} \\
\mathbf{B}^* =& \frac{ds^*}{ds}\chi k_3^* \sin\Omega^* \mathbf{T} + \left(\cos\Omega^*\cos\Omega - \frac{ds^*}{ds}\left(1-\chi k_2^*\right)\sin\Omega^*\sin\Omega\right)\mathbf{N} \\
&+ \left(-\cos\Omega^*\sin\Omega - \frac{ds^*}{ds}\left(1-\chi k_2^*\right)\sin\Omega^*\cos\Omega\right)\mathbf{B}
\end{aligned}
$$

where Ω is the angle between the vectors $\mathbf{B}$ and $\mathbf{Y}$ and also, Ω^* is the angle between the vectors $\mathbf{B}^*$ and $\mathbf{Y}^*$.

Proof. Using the equation (1), we can write the followings:

$$
\begin{pmatrix} \mathbf{T} \\ \mathbf{M} \\ \mathbf{Y} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\Omega & -\sin\Omega \\ 0 & \sin\Omega & \cos\Omega \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix} \tag{8}
$$

and

$$
\begin{pmatrix} \mathbf{T}^* \\ \mathbf{N}^* \\ \mathbf{B}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\Omega^* & \sin\Omega^* \\ 0 & -\sin\Omega^* & \cos\Omega^* \end{pmatrix} \begin{pmatrix} \mathbf{T}^* \\ \mathbf{M}^* \\ \mathbf{Y}^* \end{pmatrix}. \tag{9}
$$

Also from the equations (6) and (7), the equality

$$
\begin{pmatrix} \mathbf{T}^* \\ \mathbf{M}^* \\ \mathbf{Y}^* \end{pmatrix} = \begin{pmatrix} \frac{ds^*}{ds}\left(1-\chi k_2^*\right) & 0 & \frac{ds^*}{ds}\chi k_3^* \\ -\frac{ds^*}{ds}\chi k_3^* & 0 & \frac{ds^*}{ds}\left(1-\chi k_2^*\right) \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{M} \\ \mathbf{Y} \end{pmatrix} \tag{10}
$$

can be easily seen. If we substitute the equation (10) into the equation (9), we find

$$
\begin{pmatrix} \mathbf{T}^* \\ \mathbf{N}^* \\ \mathbf{B}^* \end{pmatrix} = \begin{pmatrix} \frac{ds^*}{ds}\left(1-\chi k_2^*\right) & 0 & \frac{ds^*}{ds}\chi k_3^* \\ -\frac{ds^*}{ds}\chi k_3^* \cos\Omega^* & \sin\Omega^* & \frac{ds^*}{ds}\left(1-\chi k_2^*\right)\cos\Omega^* \\ \frac{ds^*}{ds}\chi k_3^* \sin\Omega^* & \cos\Omega^* & -\frac{ds^*}{ds}\left(1-\chi k_2^*\right)\sin\Omega^* \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{M} \\ \mathbf{Y} \end{pmatrix}.
$$

Using the equation (8) in the last equation yields the desired result. $\square$

Theorem 3.6. *Let $\{\alpha,\ \alpha^*\}$ be a Mannheim P-pair in E^3. In this case, the relationships*

1. $k_1 = \dfrac{k_2^* - \chi (k_2^*)^2 - \chi (k_3^*)^2}{1 - 2\chi k_2^* + \chi^2\left((k_2^*)^2 + (k_3^*)^2\right)}$
2. $k_2^* = \dfrac{k_1 - \mu k_1^2 - \mu k_3^2}{1 - 2\mu k_1 + \mu^2\left(k_1^2 + k_3^2\right)}$

are satisfied between k_1, k_3, k_2^* *and* k_3^*. *Here* μ *is a constant satisfying* $|\mu| = |\chi|$.

Proof. 1. Suppose that $\{\alpha,\ \alpha^*\}$ is a Mannheim P-pair in E^3. Then we can write

$$\left(\frac{ds^*}{ds}\right)^2 \left((1-\chi k_2^*)^2 + \chi^2 (k_3^*)^2\right) = 1$$

with the aid of the equation (7) and the equality $\cos^2\beta + \sin^2\beta = 1$. This equation gives us the following

$$\left(\frac{ds}{ds^*}\right)^2 = 1 - 2\chi k_2^* + \chi^2 \left((k_2^*)^2 + (k_3^*)^2\right). \tag{11}$$

On the other hand, by differentiating the equation (5) with respect to s^* and using the PAF derivative formulas, we get

$$\begin{aligned} \frac{d^2 s}{ds^{*2}}\mathbf{T} + k_1\left(\frac{ds}{ds^*}\right)^2 \mathbf{M} + k_2\left(\frac{ds}{ds^*}\right)^2 \mathbf{Y} = &\left(-\chi (k_2^*)' + \chi k_1^* k_3^*\right)\mathbf{T}^* \\ &+ \left(k_1^*\left(1-\chi k_2^*\right) - \chi (k_3^*)'\right)\mathbf{M}^* \\ &+ \left(k_2^*\left(1-\chi k_2^*\right) - \chi (k_3^*)^2\right)\mathbf{Y}^*. \end{aligned} \tag{12}$$

From the equation (12), we can write

$$k_1\left(\frac{ds}{ds^*}\right)^2 = \left(1-\chi k_2^*\right)k_2^* - \chi (k_3^*)^2 \tag{13}$$

considering the definition of Mannheim P-pair. If the equation (11) is substituted into the equation (13), the desired result is obtained.

2. We can easily see the equality

$$\alpha^*\left(s^*\right) = \alpha\left(s\right) + \mu\mathbf{M}\left(s\right)$$

where μ is a constant satisfying $|\mu| = |\chi|$ (see Figure 2). Let us take the derivative of this equation with respect to s twice. Then we obtain

$$\mathbf{T}^*\frac{ds^*}{ds} = \left(1-\mu k_1\right)\mathbf{T} + \mu k_3 \mathbf{Y}$$

and

$$\begin{aligned} \frac{d^2 s^*}{ds^2}\mathbf{T}^* + k_1^*\left(\frac{ds^*}{ds}\right)^2 \mathbf{M}^* + k_2^*\left(\frac{ds^*}{ds}\right)^2 \mathbf{Y}^* = &\left(-\mu k_1{}' - \mu k_2 k_3\right)\mathbf{T} \\ &+ \left(k_1\left(1-\mu k_1\right) - \mu k_3^2\right)\mathbf{M} \\ &+ \left(k_2\left(1-\mu k_1\right) + \mu k_3{}'\right)\mathbf{Y}. \end{aligned} \tag{14}$$

From the equation (2), it is not difficult to see $\mathbf{T}^* = \cos\beta\,\mathbf{T} - \sin\beta\,\mathbf{Y}$. Thus we get

$$\frac{ds^*}{ds}\cos\beta\,\mathbf{T} - \frac{ds^*}{ds}\sin\beta\,\mathbf{Y} = \left(1-\mu k_1\right)\mathbf{T} + \mu k_3\mathbf{Y}$$

and so $\frac{ds^*}{ds}\cos\beta = 1 - \mu k_1,\ -\frac{ds^*}{ds}\sin\beta = \mu k_3$. The last two equalities yield

$$\left(\frac{ds^*}{ds}\right)^2 = 1 - 2\mu k_1 + \mu^2\left(k_1^2 + k_3^2\right). \tag{15}$$

On the other hand, the inner product of the vectors at the right and left sides of the equation (14) with the vector $\mathbf{M}$ gives us the following

$$k_2^*\left(\frac{ds^*}{ds}\right)^2 = k_1 - \mu k_1^2 - \mu k_3^2.$$

Thus we find

$$k_2^* = \frac{k_1 - \mu k_1^2 - \mu k_3^2}{1 - 2\mu k_1 + \mu^2\left(k_1^2 + k_3^2\right)}$$

due to the equation (15).

□

Taking into consideration Theorem 2.1, Theorem 2.2, and Theorem 3.6, we can easily give the following corollaries.

Corollary 3.7. *Let $\{\alpha = \alpha(s),\ \alpha^* = \alpha^*(s^*)\}$ be a Mannheim P-pair in E^3. Then α is an osculating curve if and only if*

$$\frac{k_2^* - \chi(k_2^*)^2 - \chi(k_3^*)^2}{1 - 2\chi k_2^* + \chi^2\left((k_2^*)^2 + (k_3^*)^2\right)} = 0$$

holds.

Corollary 3.8. *Let $\{\alpha = \alpha(s),\ \alpha^* = \alpha^*(s^*)\}$ be a Mannheim P-pair in E^3. Then α^* is a rectifying curve if and only if*

$$\frac{k_1 - \mu k_1^2 - \mu k_3^2}{1 - 2\mu k_1 + \mu^2\left(k_1^2 + k_3^2\right)} = 0$$

holds.

Example 3.9. *In the Euclidean 3-space, assume that a point particle P of constant mass moves on the trajectory*

$$\begin{aligned} \alpha : \left(0, 20\sqrt{65}\right) &\to E^3 \\ s \mapsto \alpha(s) &= \left(8\cos\frac{s}{\sqrt{65}}, 8\sin\frac{s}{\sqrt{65}}, \frac{s}{\sqrt{65}}\right) \end{aligned} \tag{16}$$

which is a unit speed curve. By straightforward calculations, we get the Serret-Frenet vectors of α

$$\begin{cases} \mathbf{T}(s) = \left(-\frac{8}{\sqrt{65}}\sin\frac{s}{\sqrt{65}}, \frac{8}{\sqrt{65}}\cos\frac{s}{\sqrt{65}}, \frac{1}{\sqrt{65}}\right) \\ \mathbf{N}(s) = \left(-\cos\frac{s}{\sqrt{65}}, -\sin\frac{s}{\sqrt{65}}, 0\right) \\ \mathbf{B}(s) = \left(\frac{1}{\sqrt{65}}\sin\frac{s}{\sqrt{65}}, -\frac{1}{\sqrt{65}}\cos\frac{s}{\sqrt{65}}, \frac{8}{\sqrt{65}}\right). \end{cases}$$

Since $\langle \alpha(s), \mathbf{B}(s)\rangle = \frac{8s}{65} > 0$ *and* $\langle \alpha(s), \mathbf{N}(s)\rangle = -8$, *we get* $\Omega(s) = \arctan\left(\frac{65}{s}\right)$.
Then, the PAF vectors can be constructed as:

$$\begin{cases} \mathbf{T}(s) = \left(-\frac{8}{\sqrt{65}}\sin\frac{s}{\sqrt{65}}, \frac{8}{\sqrt{65}}\cos\frac{s}{\sqrt{65}}, \frac{1}{\sqrt{65}}\right) \\ \mathbf{M}(s) = \begin{pmatrix} -\cos\left(\arctan\left(\frac{65}{s}\right)\right)\cos\frac{s}{\sqrt{65}} - \frac{1}{\sqrt{65}}\sin\left(\arctan\left(\frac{65}{s}\right)\right)\sin\frac{s}{\sqrt{65}}, \\ -\cos\left(\arctan\left(\frac{65}{s}\right)\right)\sin\frac{s}{\sqrt{65}} + \frac{1}{\sqrt{65}}\sin\left(\arctan\left(\frac{65}{s}\right)\right)\cos\frac{s}{\sqrt{65}}, \\ -\frac{8}{\sqrt{65}}\sin\left(\arctan\left(\frac{65}{s}\right)\right) \end{pmatrix} \\ \mathbf{Y}(s) = \begin{pmatrix} -\sin\left(\arctan\left(\frac{65}{s}\right)\right)\cos\frac{s}{\sqrt{65}} + \frac{1}{\sqrt{65}}\cos\left(\arctan\left(\frac{65}{s}\right)\right)\sin\frac{s}{\sqrt{65}}, \\ -\sin\left(\arctan\left(\frac{65}{s}\right)\right)\sin\frac{s}{\sqrt{65}} - \frac{1}{\sqrt{65}}\cos\left(\arctan\left(\frac{65}{s}\right)\right)\cos\frac{s}{\sqrt{65}}, \\ \frac{8}{\sqrt{65}}\cos\left(\arctan\left(\frac{65}{s}\right)\right) \end{pmatrix}. \end{cases}$$

In this case, Mannheim partner P-trajectory of α *can be obtained as:*

$$\alpha^*(s) = \alpha(s) + \mu\mathbf{M}(s) \tag{17}$$

$$= \begin{pmatrix} 8\cos\frac{s}{\sqrt{65}} + \mu\left(-\cos\left(\arctan\left(\frac{65}{s}\right)\right)\cos\frac{s}{\sqrt{65}} - \frac{1}{\sqrt{65}}\sin\left(\arctan\left(\frac{65}{s}\right)\right)\sin\frac{s}{\sqrt{65}}\right), \\ 8\sin\frac{s}{\sqrt{65}} + \mu\left(-\cos\left(\arctan\left(\frac{65}{s}\right)\right)\sin\frac{s}{\sqrt{65}} + \frac{1}{\sqrt{65}}\sin\left(\arctan\left(\frac{65}{s}\right)\right)\cos\frac{s}{\sqrt{65}}\right), \\ \frac{s}{\sqrt{65}} - \frac{8\mu}{\sqrt{65}}\sin\left(\arctan\left(\frac{65}{s}\right)\right) \end{pmatrix}.$$

In Figure 3, the trajectories α *(blue) and* α^* *(red), which are given in the equations* (16) *and* (17), *can be seen for the case* $\mu = 1$. *Finally we must note that the Figure 3 is drawn by using the website Wolfram Mathematica (Wolfram Cloud).*

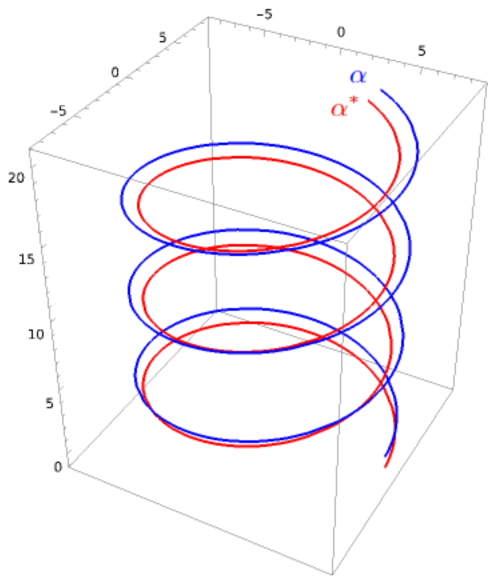


FIGURE 3. The trajectories α and α^*. In this plot $\mu = 1$.

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