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THE POLYANALYTIC SUB-FOCK REPRODUCING KERNELS WITH CERTAIN POSITIVE INTEGER POWERS

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Abstract. We consider a closed subspace $\widetilde{\mathcal{A}}_q^{\alpha,m}(\mathbb{C})$ of the Fock space $\mathcal{A}_q^{\alpha,m}(\mathbb{C})$ of q-analytic functions with the weight $\phi(z) = -\alpha\log|z|^2 + |z|^{2m}$ for any positive integer m. We obtain the corresponding reproducing kernel $K^{\mathbb{C}}_{\alpha,q,m}(z,w)$ using the weighted Laguerre polynomials and the Mittag-Leffler functions. Finally, we investigate the necessary and sufficient condition on (α, q, m) such that $K^{\mathbb{C}}_{\alpha,q,m}(z,w)$ is zero-free.

1. Introduction

Let ϕ be a subharmonic function on $\mathbb{C}$. The space L^2_ϕ is the set of all measurable functions f on $\mathbb{C}$ such that

$$\|f\|^2_\phi := \int_{\mathbb{C}} |f(z)|^2 e^{-\phi(z)} dA(z) < \infty,$$

where dA is the Lebesgue area measure on $\mathbb{C}$. Then the Fock space F^2_ϕ is defined by $F^2_\phi := L^2_\phi \cap \mathcal{O}(\mathbb{C})$, where $\mathcal{O}(\mathbb{C})$ is the set of all entire functions. If $\phi(z) = |z|^2$, then F^2_ϕ is the classical Fock space.

Recently one can see many important researches on polyanalytic functions. A function f is called q-analytic if it satisfies the generalized Cauchy-Riemann equation

$$\frac{\partial^q f}{\partial \bar{z}^q}(z) = 0 \text{ for all } z \in \mathbb{C}.$$

If f is q-analytic for some q, then f is called polyanalytic. It is easily seen that 1-analytic function is analytic. Many researches on the polyanalytic functions have been made in [4, 7, 11, 17, 18, 23].

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In 2019, Hachadi and Youssfi [9] studied the Fock space $F^{2,q}_\phi$ which is the set of all q-analytic functions with $\|f\|_\phi < \infty$ when $\phi(z) = -\alpha \log |z|^2 + |z|^2$ with $\alpha > -1$. Moreover, they obtained the formula of the reproducing kernel for $F^{2,q}_\phi$ when $\phi(z) = -\alpha \log |z|^2 + |z|^2$ with $\alpha > -1$. Using the explicit formula, Park investigated the zeros and asymptotic behavior of the polyanalytic reproducing kernel in [16].

Let m be any positive integer. For any measurable function f on $\mathbb{C}$, consider

$$\|f\|^2_{\alpha,m} := \int_{\mathbb{C}} |f(z)|^2 d\nu_{\alpha,m}(z),$$

where

$$d\nu_{\alpha,m}(z) := c_{\alpha,m}|z|^{2\alpha}e^{-|z|^{2m}}dA(z) = c_{\alpha,m}e^{-\phi(z)}dA(z), \quad \alpha > -1.$$

Here $c_{\alpha,m}$ is a normalizing constant so that $d\nu_{\alpha,m}(z)$ is a probability measure on $\mathbb{C}$. We now define a generalized Fock space $\mathcal{A}^{\alpha,m}_q(\mathbb{C})$ by the space of all q-analytic functions f on $\mathbb{C}$ with $\|f\|_{\alpha,m} < \infty$.

In this paper we deal with a closed subspace $\widetilde{\mathcal{A}}^{\alpha,m}_q(\mathbb{C})$ of $\mathcal{A}^{\alpha,m}_q(\mathbb{C})$, which is defined as the set of all elements of the form $f(z^m)$ for all $f \in \mathcal{A}^{\alpha,m}_q(\mathbb{C})$. If $m = 1$, then our closed subspace $\widetilde{\mathcal{A}}^{\alpha,m}_q(\mathbb{C})$ of the space $\mathcal{A}^{\alpha,m}_q(\mathbb{C})$ coincides with the Fock space $F^{2,q}_\phi$ in [9]. Thus our result is a generalization of [9]. If α tends to 0, then our generalized Fock space $\mathcal{A}^{\alpha,m}_q(\mathbb{C})$ belongs to the weighted Fock space studied in [19, 20]. Their weighted Fock space with a more general condition on m has attracted considerable attention, due to the boundedness and compactness issues of generalized Hankel operators in operator theory. We expect that the results in this paper provide further research problems to various directions on operator theory and complex analysis in higher dimension of several variables. One can see the recent results in [5, 6, 12, 24].

In Section 2, we review the notations on the generalized Fock spaces and certain subspaces of them. We also explain the important properties on the orthogonal polynomials and the Mittag-Leffler functions. In Section 3, we compute the polyanalytic reproducing kernel $K^{\mathbb{C}}_{\alpha,q,m}(z,w)$ for $\widetilde{\mathcal{A}}^{\alpha,m}_q(\mathbb{C})$ when $\phi(z) = -\alpha \log |z|^2 + |z|^{2m}$ for any $\alpha > -1$ and positive integer m as our first main result (see Theorem 3.2). If $m \geq 2$, then we need the technical idea to express the kernel in terms of the weighted Laguerre polynomials and the Mittag-Leffler functions. In Section 4, we study the existence of zeros of the polyanalytic reproducing kernels using the properties of orthogonal polynomials. As our second main result, we show that $K^{\mathbb{C}}_{\alpha,q,m}(z,w)$ is zero-free if and only if $(\alpha, q, m) = (0, 1, 1)$ (see Theorem 4.3).

2. Notations and basic materials

In this section, we shall exploit the generalized Fock space and its certain subspace which will be considered in this paper. We also prepare some notations and facts on orthogonal polynomials and the Mittag-Leffler functions. We follow the notations introduced in [9].

2.1. Generalized polyanalytic Fock spaces

Let us recall that the generalized Fock space $\mathcal{A}_q^{\alpha,m}(\mathbb{C})$ is defined as the space of all q-analytic functions f on $\mathbb{C}$ with $\|f\|_{\alpha,m} < \infty$. In this case, we have

$$d\mu(t) = \frac{m}{\Gamma\left(\frac{\alpha+1}{m}\right)} t^{\alpha} e^{-t^m} dt$$

and the moment sequence is given by

$$s_d = \int_0^\infty t^d d\mu(t) = \frac{m}{\Gamma\left(\frac{\alpha+1}{m}\right)} \int_0^\infty t^{d+\alpha} e^{-t^m} dt = \frac{\Gamma\left(\frac{d+\alpha+1}{m}\right)}{\Gamma\left(\frac{\alpha+1}{m}\right)}.$$

For each pair (d, n) of non-negative integers and fixed positive integers m, q such that $nm \leq q - 1$, we denote $\mathcal{P}_{(n,m)}(\mu)$ by the subspace of $L^2(x^d d\mu(x))$ consisting of all polynomials of degree km $(k = 0, 1, \ldots, n)$ with the inner product

$$\langle f, g\rangle := \int_0^\infty f(x)g(x)x^d d\mu(x).$$

Denote $Q_{d,n,m}(x, y)$ by the reproducing kernel for $\mathcal{P}_{(n,m)}(\mu)$.

Remark 2.1. *If $m = 1$, then we have $Q_{d,n,1}(x, y) = Q_{d,n}(x, y)$ (See (3.7) in* [9]*). What is more, the construction of $Q_{d,n}$ in* [9] *involves the use of the generalized Laguerre polynomials. Indeed, if we consider the polynomials of degree at most nm in the definition of $\mathcal{P}_{(n,m)}(\mu)$, we should find the associated orthogonal polynomials instead of the generalized Laguerre polynomials.*

Similarly as in [9], we define a function

$$F_{\alpha,q,m}(\lambda, x, y) := \sum_{d=0}^{\infty} \lambda^d Q_{d,N,m}(x, y) + \sum_{d=1}^{N} \overline{\lambda}^d Q_{d,N-d,m}(x, y), \tag{2.1}$$

where we use the notation $N := \lfloor \frac{q-1}{m} \rfloor$ throughout the paper.

We recall that the space $\widetilde{\mathcal{A}}_q^{\alpha,m}(\mathbb{C})$ is defined as the set of all elements of the form $f(z^m)$ for all $f \in \mathcal{A}_q^{\alpha,m}(\mathbb{C})$. Let $K^{\mathbb{C}}_{\alpha,q,m}(z, w)$ be the reproducing kernel for the sub-Fock space $\widetilde{\mathcal{A}}_q^{\alpha,m}(\mathbb{C})$. By the similar arguments in the proof of Theorem 4.3 in [9], we have the following.

Proposition 2.2. *The reproducing kernel $K^{\mathbb{C}}_{\alpha,q,m}(z, w)$ is written by*

$$K^{\mathbb{C}}_{\alpha,q,m}(z, w) = F_{\alpha,q,m}((z\overline{w})^m, |z|^2, |w|^2).$$

2.2. Orthogonal polynomials

Suppose that $w(x)$ satisfies $w(x) > 0$ on (a, b) with $\int_a^b w(x)x^n dx < \infty$ for all $n \in \mathbb{N}$. We define a sequence of *orthogonal polynomials* $p_n(x)$ of degree n such that

$$\int_a^b p_n(x)p_{n'}(x)w(x)dx = h_n\delta_{n,n'},$$

where h_n is a positive constant and $\delta_{n,n'}$ is the Kronecker delta. At first we introduce an important fact on the existence of zeros of all orthogonal polynomials.

Lemma 2.3 ([21])**.** *If $\{p_n(x)\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials on the interval (a, b) with respect to the weight $w(x)$, then each polynomial $p_n(x)$ of degree n has exactly n distinct real simple zeros on (a, b).*

As in [9], we shall utilize the *weighted Laguerre polynomials* $L_n^{(\alpha)}(x)$ which are orthogonal with respect to $w(x) = x^\alpha e^{-x}$ when we study the polyanalytic sub-Fock space. Precisely, $L_n^{(\alpha)}(x)$ satisfies

$$\int_0^\infty L_n^{(\alpha)}(x)L_{n'}^{(\alpha)}(x)x^\alpha e^{-x}dx = \frac{\Gamma(n+1+\alpha)}{n!}\delta_{n,n'} \tag{2.2}$$

for any positive integers n and n'. Note that $L_n^{(\alpha)}(x)$'s are polynomials of degree n and weight α and satisfy the following second order differential equation

$$xy'' + (\alpha + 1 - x)y' + ny = 0.$$

What is more, it is well-known that $L_n^{(\alpha)}(x)$ has the explicit form

$$L_n^{(\alpha)}(x) = \sum_{r=0}^{n}(-1)^r\binom{n+\alpha}{n-r}\frac{x^r}{r!}. \tag{2.3}$$

Before proceeding further, we now prepare some useful properties of the weighted Laguerre polynomials.

Lemma 2.4 ([9])**.** *The weighted Laguerre polynomials $L_n^{(\alpha)}(x)$ satisfy the following.*

(i) $L_n^{(\alpha)}(x)L_n^{(\alpha)}(y) = \dfrac{\Gamma(\alpha+n+1)}{n!}\displaystyle\sum_{r=0}^{n}\frac{(xy)^r L_{n-r}^{(\alpha+2r)}(x+y)}{r!\Gamma(\alpha+r+1)}$

(ii) $\displaystyle\sum_{n=r}^{s} L_{n-r}^{(\alpha)}(x) = L_{s-r}^{(\alpha+1)}(x)$

(iii) $\displaystyle\sum_{r=0}^{n}\frac{y^r}{r!}L_{n-r}^{(\beta+r)}(x) = L_n^{(\beta)}(x-y)$

(iv) $L_n^{(\alpha+\beta+1)}(x+y) = \displaystyle\sum_{k=0}^{n} L_{n-k}^{(\alpha)}(x)L_k^{(\beta)}(y)$

2.3. The Mittag-Leffler functions

The *Mittag-Leffler functions* $E_{\alpha,\beta}(z)$ are defined by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

for all $z \in \mathbb{C}$. For each fixed α, β, $E_{\alpha,\beta}(z)$ converges for all $z \in \mathbb{C}$ and is entire. The function $E_{\alpha,1}(z)$ is a generalization of the exponential function which was introduced in [14]. As a generalization of $E_{\alpha,1}(z)$, (two-parameter) function $E_{\alpha,\beta}(z)$ was introduced in [2]. One can see the properties of the Mittag-Leffler functions in [8] and the references therein. Note that $E_{1,1}(z) = e^z$ has no zeros. In fact, this is the very unusual case.

Lemma 2.5 ([8]). *Let $\alpha, \beta > 0$. Then $E_{\alpha,\beta}(z)$ has no zeros if and only if $\alpha = \beta = 1$. In this case, $E_{1,1}(z) = e^z$.*

Denote the generalized Mittag-Leffler function $E_k^{(\alpha)}(z)$ by

$$E_k^{(\alpha)}(z) := \frac{e^z}{k!} \frac{d^k}{dz^k} \left(z^k e^{-z} E_{1,\alpha}(z) \right),$$

where

$$E_{1,\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k + \alpha)}.$$

Note that $E_k^{(1)}(z) = e^z$ for any k and $E_0^{(\alpha)}(z) = E_{1,\alpha}(z)$ for any α. Indeed, the generalized Mittag-Leffler function can be written as the infinite sum related to the weighted Laguerre polynomials. In [9], one can see the following formula

$$E_k^{(\alpha)}(z) = \sum_{d=0}^{\infty} \frac{z^d}{\Gamma(d + \alpha)} L_k^{(d)}(z). \tag{2.4}$$

The above formula can be derived by using the explicit form (2.3) of the weighted Laguerre polynomial.

3. The polyanalytic reproducing kernel

In this section, we compute the reproducing kernel for $\widetilde{\mathcal{A}}_q^{\alpha,m}(\mathbb{C})$. At first, we need to find the subset of all polynomials which are orthogonal with respect to the weight $x^\alpha e^{-x^m}$ for any positive integer m. Recall that $L_n^{(\alpha)}(x)$'s are orthogonal with respect to the weight $x^\alpha e^{-x}$. As a counterpart of this fact, we have the following lemma.

Lemma 3.1. *Let m be any positive integer. The polynomials $L_n^{(\beta)}(x^m)$'s are orthogonal with respect to the weight $x^{d+\alpha}e^{-x^m}$ and have degree mn in the variable x, where*

$$\beta := \frac{d+\alpha+1}{m} - 1. \tag{3.5}$$

Proof. By Lemma 2.4 (i), we know that $L_n^{(\alpha)}(x)$ are orthogonal with respect to the weight $x^\alpha e^{-x}$. If we use the change of variables by $y = x^m$, then by (2.2), we have

$$\begin{aligned}\int_0^\infty L_n^{(\beta)}(x^m)L_{n'}^{(\beta)}(x^m)x^{d+\alpha}e^{-x^m}dx &= \frac{1}{m}\int_0^\infty L_n^{(\beta)}(y)L_{n'}^{(\beta)}(y)y^\beta e^{-y}dy \\ &= \frac{1}{m}\frac{\Gamma(n+1+\beta)}{n!}\delta_{n,n'}.\end{aligned}$$

Thus the polynomials $L_n^{(\beta)}(x^m)$'s are orthogonal with respect to the weight $x^{d+\alpha}e^{-x^m}$, where β is defined as in (3.5). □

By Lemma 3.1, we have

$$Q_{d,n,m}(x,y) = m\sum_{k=0}^{n}\frac{k!}{\Gamma(k+1+\beta)}L_k^{(\beta)}(x^m)L_k^{(\beta)}(y^m), \tag{3.6}$$

where β is the same as defined in (3.5). By Proposition 2.2, the reproducing kernel $K_{\alpha,q,m}^{\mathbb{C}}(z,w)$ is written by

$$K_{\alpha,q,m}^{\mathbb{C}}(z,w) = F_{\alpha,q,m}((z\overline{w})^m, |z|^2, |w|^2),$$

where $F_{\alpha,q,m}$ is as in (2.1).

It is convenient to write

$$\lambda := (z\overline{w})^m, \quad x := |z|^2, \quad y := |w|^2. \tag{3.7}$$

Then we can write

$$F_{\alpha,q,m}(\lambda,x,y) = S_{\alpha,q,m}^{(1)} + S_{\alpha,q,m}^{(2)},$$

where

$$S_{\alpha,q,m}^{(1)} := \sum_{d=0}^{\infty}\lambda^d Q_{d,N,m}(x,y), \text{ and } S_{\alpha,q,m}^{(2)} := \sum_{d=1}^{N}\overline{\lambda}^d Q_{d,N-d,m}(x,y).$$

Now we shall express $S_{\alpha,q,m}^{(1)}$ and $S_{\alpha,q,m}^{(2)}$ in terms of the weighted Laguerre polynomials and the Mittag-Leffler functions.

3.1. Computation of $S^{(1)}_{\alpha,q,m}$

By (3.6), we have

$$S^{(1)}_{\alpha,q,m} = m\sum_{d=0}^{\infty}\lambda^d\sum_{n=0}^{N}\frac{n!}{\Gamma(\beta+n+1)}L_n^{(\beta)}(x^m)L_n^{(\beta)}(y^m)$$
$$= m\sum_{n=0}^{N}\sum_{d=0}^{\infty}\frac{n!\lambda^d}{\Gamma(\beta+n+1)}L_n^{(\beta)}(x^m)L_n^{(\beta)}(y^m).$$

By Lemma 2.4 (i) and (ii), we have

$$S^{(1)}_{\alpha,q,m} = m\sum_{n=0}^{N}\sum_{d=0}^{\infty}\sum_{r=0}^{n}\frac{\lambda^d(x^my^m)^rL_{n-r}^{(\beta+2r)}(x^m+y^m)}{r!\Gamma(\beta+r+1)}$$
$$= m\sum_{r=0}^{N}\sum_{d=0}^{\infty}\frac{\lambda^d(x^my^m)^r}{r!\Gamma(\beta+r+1)}\sum_{n=r}^{N}L_{n-r}^{(\beta+2r)}(x^m+y^m)$$
$$= m\sum_{r=0}^{N}\sum_{d=0}^{\infty}\frac{\lambda^d(x^my^m)^r}{r!\Gamma(\beta+r+1)}L_{N-r}^{(\beta+2r+1)}(x^m+y^m).$$

Here is the decisive step for this generalized polyanalytic Fock space. For any non-negative integer d, there exist unique non-negative integers c and e such that

$$d = mc + e \text{ with } 0 \le e \le m-1.$$

Then we have

$$\sum_{d=0}^{\infty}f(d) = \sum_{e=0}^{m-1}\sum_{c=0}^{\infty}f(mc+e)$$

for any function f of single variable. Note that by (3.5) it follows that

$$\beta = c + \frac{e+\alpha+1}{m} - 1. \tag{3.8}$$

Then a straightforward computation shows that

$$S^{(1)}_{\alpha,q,m} = m\sum_{r=0}^{N}\sum_{e=0}^{m-1}\frac{\lambda^e(x^my^m)^r}{r!}\sum_{c=0}^{\infty}\frac{(\lambda^m)^c}{\Gamma\left(c+r+\frac{e+\alpha+1}{m}\right)}L_{N-r}^{\left(c+2r+\frac{e+\alpha+1}{m}\right)}(x^m+y^m)$$
$$= m\sum_{r=0}^{N}\sum_{e=0}^{m-1}\frac{\lambda^e(\lambda\overline{\lambda})^r}{r!}\sum_{c=r}^{\infty}\frac{(\lambda^m)^{c-r}}{\Gamma\left(c+\frac{e+\alpha+1}{m}\right)}L_{N-r}^{\left(c+r+\frac{e+\alpha+1}{m}\right)}(x^m+y^m)$$
$$= m\sum_{r=0}^{N}\sum_{e=0}^{m-1}\frac{\lambda^{e+r-mr}\overline{\lambda}^r}{r!}\sum_{c=r}^{\infty}\frac{(\lambda^m)^c}{\Gamma\left(c+\frac{e+\alpha+1}{m}\right)}L_{N-r}^{\left(c+r+\frac{e+\alpha+1}{m}\right)}(x^m+y^m),$$

where we assumed $zw \neq 0$. If $zw = 0$, then it is easy to see that

$$S^{(1)}_{\alpha,q,m} = \frac{m}{\Gamma\left(\frac{\alpha+1}{m}\right)} L_N^{\left(\frac{\alpha+1}{m}\right)}(x^m + y^m). \tag{3.9}$$

Similarly as in [9], we divide $S^{(1)}_{\alpha,q,m}$ into two parts as

$$S^{(1)}_{\alpha,q,m} = S^{(1,1)}_{\alpha,q,m} - S^{(1,2)}_{\alpha,q,m},$$

where

$$\begin{aligned}
S^{(1,1)}_{\alpha,q,m} &:= m \sum_{r=0}^{N} \sum_{e=0}^{m-1} \frac{\lambda^{e+r-mr}\overline{\lambda}^r}{r!} \sum_{c=0}^{\infty} \frac{(\lambda^m)^c}{\Gamma\left(c + \frac{e+\alpha+1}{m}\right)} L_{N-r}^{\left(c+r+\frac{e+\alpha+1}{m}\right)}(x^m + y^m),\\
S^{(1,2)}_{\alpha,q,m} &:= m \sum_{r=0}^{N} \sum_{e=0}^{m-1} \frac{\lambda^{e+r-mr}\overline{\lambda}^r}{r!} \sum_{c=0}^{r-1} \frac{(\lambda^m)^c}{\Gamma\left(c + \frac{e+\alpha+1}{m}\right)} L_{N-r}^{\left(c+r+\frac{e+\alpha+1}{m}\right)}(x^m + y^m).
\end{aligned}$$

By Lemma 2.4 (iii) and (iv), we have

$$\begin{aligned}
S^{(1,1)}_{\alpha,q,m} =& m \sum_{e=0}^{m-1} \lambda^e \sum_{c=0}^{\infty} \frac{(\lambda^m)^c}{\Gamma\left(c + \frac{e+\alpha+1}{m}\right)} \sum_{r=0}^{N} \frac{\lambda^{r-mr}\overline{\lambda}^r}{r!} L_{N-r}^{\left(c+r+\frac{e+\alpha+1}{m}\right)}(x^m + y^m)\\
=& m \sum_{e=0}^{m-1} \lambda^e \sum_{c=0}^{\infty} \frac{(\lambda^m)^c}{\Gamma\left(c + \frac{e+\alpha+1}{m}\right)} L_{N}^{\left(c+\frac{e+\alpha+1}{m}\right)}(x^m + y^m - \lambda^{1-m}\overline{\lambda})\\
=& m \sum_{e=0}^{m-1} \lambda^e \sum_{c=0}^{\infty} \frac{(\lambda^m)^c}{\Gamma\left(c + \frac{e+\alpha+1}{m}\right)} \sum_{k=0}^{N} L_k^{(c)}(\lambda^m) L_{N-k}^{\left(\frac{e+\alpha+1}{m}-1\right)}(x^m + y^m - \lambda^m - \lambda^{1-m}\overline{\lambda})\\
=& m \sum_{e=0}^{m-1} \lambda^e \sum_{k=0}^{N} L_{N-k}^{\left(\frac{e+\alpha+1}{m}-1\right)}(x^m + y^m - \lambda^m - \lambda^{1-m}\overline{\lambda}) \sum_{c=0}^{\infty} \frac{(\lambda^m)^c L_k^{(c)}(\lambda^m)}{\Gamma(c + \frac{e+\alpha+1}{m})}.
\end{aligned}$$

From (3.7), one can easily obtain that

$$x^m + y^m - \lambda^m - \lambda^{1-m}\overline{\lambda} = |z|^{2m} + |w|^{2m} - (z\overline{w})^{m^2} - (z\overline{w})^{m(1-m)}(\overline{z}w)^m.$$

By (2.4), we have

$$\sum_{c=0}^{\infty} \frac{(\lambda^m)^c L_k^{(c)}(\lambda^m)}{\Gamma\left(c + \frac{e+\alpha+1}{m}\right)} = E_k^{\left(\frac{e+\alpha+1}{m}\right)}(\lambda^m).$$

It follows that

$$S^{(1,1)}_{\alpha,q,m} = m \sum_{e=0}^{m-1} \lambda^e \sum_{k=0}^{N} L_{N-k}^{\left(\frac{e+\alpha+1}{m}-1\right)}(\tau) E_k^{\left(\frac{e+\alpha+1}{m}\right)}(\lambda^m), \tag{3.10}$$

where $\tau := |z|^{2m} + |w|^{2m} - (z\overline{w})^{m^2} - (z\overline{w})^{m(1-m)}(\overline{z}w)^m$. Note that if $m = 1$, then $\tau = |z - w|^2$.

On the other hand, we have

$$\begin{aligned} S^{(1,2)}_{\alpha,q,m} &= m\sum_{e=0}^{m-1}\lambda^e\sum_{c=0}^{N-1}\sum_{r=c+1}^{N}\frac{\lambda^{r-mr}\overline{\lambda}^r}{r!}\frac{(\lambda^m)^c}{\Gamma\left(c+\frac{e+\alpha+1}{m}\right)}L_{N-r}^{\left(c+r+\frac{e+\alpha+1}{m}\right)}(x^m+y^m) \\ &= m\sum_{e=0}^{m-1}\lambda^e\sum_{d=0}^{N-1}\sum_{r=d+1}^{N}\frac{\lambda^{r-mr}\overline{\lambda}^r}{r!}\frac{(\lambda^m)^d}{\Gamma\left(d+\frac{e+\alpha+1}{m}\right)}L_{N-r}^{\left(d+r+\frac{e+\alpha+1}{m}\right)}(x^m+y^m) \\ &= m\sum_{e=0}^{m-1}\lambda^e\sum_{r=1}^{N}\sum_{d=0}^{r-1}\frac{\lambda^{r-mr}\overline{\lambda}^r}{r!}\frac{(\lambda^m)^d}{\Gamma\left(d+\frac{e+\alpha+1}{m}\right)}L_{N-r}^{\left(d+r+\frac{e+\alpha+1}{m}\right)}(x^m+y^m). \end{aligned}$$

It follows that

$$S^{(1,2)}_{\alpha,q,m} = m\sum_{r=1}^{N}\sum_{d=0}^{r-1}\sum_{e=0}^{m-1}\frac{\lambda^{md+e+r-mr}\overline{\lambda}^r}{r!\Gamma\left(d+\frac{e+\alpha+1}{m}\right)}L_{N-r}^{\left(d+r+\frac{e+\alpha+1}{m}\right)}(x^m+y^m). \tag{3.11}$$

3.2. Computation of $S^{(2)}_{\alpha,q,m}$

Now we compute $S^{(2)}_{\alpha,q,m}$. By Lemma 2.4 (i), we have

$$\begin{aligned} S^{(2)}_{\alpha,q,m} &= m\sum_{d=1}^{N}\overline{\lambda}^d\sum_{n=0}^{N-d}\frac{n!}{\Gamma(\beta+n+1)}L_n^{(\beta)}(x^m)L_n^{(\beta)}(y^m) \\ &= m\sum_{d=1}^{N}\overline{\lambda}^d\sum_{n=0}^{N-d}\sum_{r=0}^{n}\frac{(x^my^m)^rL_{n-r}^{(\beta+2r)}(x^m+y^m)}{r!\Gamma(\beta+1+r)} \\ &= m\sum_{r=0}^{N-1}\sum_{n=r}^{N-1}\sum_{d=1}^{N-n}\frac{\lambda^r\overline{\lambda}^{d+r}L_{n-r}^{(\beta+2r)}(x^m+y^m)}{r!\Gamma(\beta+1+r)}. \end{aligned}$$

Recall that $\beta=\frac{d+\alpha+1}{m}-1$. If we write $d'=d+r$ and drop the prime, then

$$S^{(2)}_{\alpha,q,m} = m\sum_{r=0}^{N-1}\sum_{n=r}^{N-1}\sum_{d=1+r}^{N-n+r}\frac{\lambda^r\overline{\lambda}^d}{r!\Gamma\left(\frac{d-r+\alpha+1}{m}+r\right)}L_{n-r}^{\left(\frac{d-r+\alpha+1}{m}-1+2r\right)}(x^m+y^m).$$

If we exchange the order of the summation and apply Lemma 2.4 (ii), then

$$\begin{aligned} S^{(2)}_{\alpha,q,m} &= m\sum_{r=0}^{N-1}\sum_{d=1+r}^{N}\sum_{n=r}^{N+r-d}\frac{\lambda^r\overline{\lambda}^d}{r!\Gamma\left(\frac{d-r+\alpha+1}{m}+r\right)}L_{n-r}^{\left(\frac{d-r+\alpha+1}{m}-1+2r\right)}(x^m+y^m) \\ &= m\sum_{r=0}^{N-1}\sum_{d=1+r}^{N}\frac{\lambda^r\overline{\lambda}^d}{r!\Gamma\left(\frac{d-r+\alpha+1}{m}+r\right)}L_{N-d}^{\left(\frac{d-r+\alpha+1}{m}+2r\right)}(x^m+y^m). \end{aligned}$$

If we exchange r and d, then we have

$$S^{(2)}_{\alpha,q,m} = m\sum_{d=0}^{N-1}\sum_{r=1+d}^{N}\frac{\lambda^d\overline{\lambda}^r}{d!\Gamma\left(\frac{r-d+\alpha+1}{m}+d\right)}L_{N-r}^{\left(\frac{r-d+\alpha+1}{m}+2d\right)}(x^m+y^m).$$

It follows that

$$S^{(2)}_{\alpha,q,m} = m\sum_{r=1}^{N}\sum_{d=0}^{r-1}\frac{\lambda^d\overline{\lambda}^r}{d!\Gamma(\frac{r-d+\alpha+1}{m}+d)}L_{N-r}^{(\frac{r-d+\alpha+1}{m}+2d)}(x^m+y^m). \tag{3.12}$$

3.3. Forms of the polyanalytic reproducing kernels

From the previous subsections, we have

$$K^{\mathbb{C}}_{\alpha,q,m}(z,w) = S^{(1,1)}_{\alpha,q,m} - S^{(1,2)}_{\alpha,q,m} + S^{(2)}_{\alpha,q,m}. \tag{3.13}$$

Combining (3.9), (3.10), (3.11), (3.12) with (3.13), we get the following theorem.

Theorem 3.2. *Let $\alpha > -1$ and $q, m \in \mathbb{N}$. If $zw \neq 0$, then the reproducing kernel is given by*

$$\begin{aligned} K^{\mathbb{C}}_{\alpha,q,m}(z,w) =& m\sum_{e=0}^{m-1}(z\overline{w})^{me}\sum_{k=0}^{N}L_{N-k}^{\left(\frac{e+\alpha+1}{m}-1\right)}(\tau)E_k^{\left(\frac{e+\alpha+1}{m}\right)}((z\overline{w})^{m^2}) \\ &+m\sum_{r=1}^{N}\sum_{d=0}^{r-1}\Bigg[\frac{(z\overline{w})^{md}(\overline{z}w)^{mr}}{d!\Gamma\left(\frac{r-d+\alpha+1}{m}+d\right)}L_{N-r}^{\left(\frac{r-d+\alpha+1}{m}+2d\right)}(|z|^{2m}+|w|^{2m}) \\ &-\sum_{e=0}^{m-1}\frac{(z\overline{w})^{m^2d+me+mr-m^2r}(\overline{z}w)^{mr}}{r!\Gamma\left(d+\frac{e+\alpha+1}{m}\right)}L_{N-r}^{(d+r+\frac{e+\alpha+1}{m})}(|z|^{2m}+|w|^{2m})\Bigg], \end{aligned}$$

where $\tau := |z|^{2m}+|w|^{2m}-(z\overline{w})^{m^2}-(z\overline{w})^{m(1-m)}(\overline{z}w)^m$. Moreover, if $zw=0$, then

$$K^{\mathbb{C}}_{\alpha,q,m}(z,w) = \frac{m}{\Gamma\left(\frac{\alpha+1}{m}\right)}L_N^{(\frac{\alpha+1}{m})}(x^m+y^m).$$

Remark 3.3. If $m=1$, then by (2.4),

$$\begin{aligned} K^{\mathbb{C}}_{\alpha,q,1}(z,w) =& \sum_{k=0}^{q-1}L_{q-1-k}^{(\alpha)}(|z-w|^2)E_k^{(\alpha+1)}(z\overline{w}) \\ &+\sum_{r=1}^{q-1}\sum_{d=0}^{r-1}\Bigg[\frac{(z\overline{w})^d(\overline{z}w)^r}{d!\Gamma(r+\alpha+1)}L_{q-1-r}^{(r+\alpha+1+d)}(|z|^2+|w|^2) \\ &\qquad -\frac{(z\overline{w})^d(\overline{z}w)^r}{r!\Gamma(d+\alpha+1)}L_{q-1-r}^{(d+r+\alpha+1)}(|z|^2+|w|^2)\Bigg]. \end{aligned}$$

This was already proved in [9]. Since $S^{(1,2)}_{0,q,1} = S^{(2)}_{0,q,1}$, it was already known that

$$K^{\mathbb{C}}_{0,q,1}(z,w) = S^{(1,1)}_{0,q,1} = e^{z\overline{w}}\sum_{k=0}^{q-1}L_{q-1-k}^{(0)}(|z-w|^2) = e^{z\overline{w}}L_{q-1}^{(1)}(|z-w|^2). \tag{3.14}$$

However, if $m \geq 2$, then $S^{(1,2)}_{0,q,m} \neq S^{(2)}_{0,q,m}$. For example, if $N = 1$, then

$$S^{(1,2)}_{0,q,m} = m\overline{\lambda} \sum_{e=0}^{m-1} \frac{\lambda^{e+1-m}}{\Gamma\left(\frac{e+1}{m}\right)} \text{ and } S^{(2)}_{0,q,m} = \frac{m\overline{\lambda}}{\Gamma\left(\frac{2}{m}\right)}.$$

Thus one can see that $S^{(1,2)}_{0,q,m} = S^{(2)}_{0,q,m}$ only when $m = 1$.

4. Zeros of the polyanalytic reproducing kernels

In this final section, we discuss the zeros of $K^{\mathbb{C}}_{\alpha,q,m}(z,w)$. The problem of determining the existence of zeros of the reproducing kernel is called *the Lu Qi-Keng problem* suggested in [13]. There are so many results on the Bergman kernel for bounded domains. We refer to the reader to [1, 3, 10, 15, 16, 22] and the references therein. As an application of Theorem 3.2, we shall characterize the condition of (α, q, m) such that the kernel has zeros.

Example 4.1. From (3.14), we have $K^{\mathbb{C}}_{0,q,1}(z,w) = e^{z\overline{w}} L^{(1)}_{q-1}(|z-w|^2)$. By the explicit form (2.3), we have

$$\begin{aligned}
L^{(1)}_0(x) &= 1. \\
L^{(1)}_1(x) &= -x + 2, \\
L^{(1)}_2(x) &= \frac{1}{2}x^2 - 3x + 3, \\
L^{(1)}_3(x) &= -\frac{1}{6}x^3 + 2x^2 - 6x + 4.
\end{aligned}$$

Thus $K^{\mathbb{C}}_{0,1,1}(z,w) = e^{z\overline{w}}$ has no zeros. But, $K^{\mathbb{C}}_{0,2,1}(z,w) = e^{z\overline{w}}(2 - |z-w|^2)$ has zeros when $|z - w| = \sqrt{2}$. And also $K^{\mathbb{C}}_{0,2,1}(z,w)$ and $K^{\mathbb{C}}_{0,3,1}(z,w)$ have zeros, since $L^{(1)}_2(x) = 0$ and $L^{(1)}_3(x) = 0$ also have positive real roots.

Example 4.2. Consider $(q,m) = (2,2)$. Then $K^{\mathbb{C}}_{\alpha,2,2}(z,w) = S^{(1,1)}_{\alpha,2,2} - S^{(1,2)}_{\alpha,2,2} + S^{(2)}_{\alpha,2,2}$ and $N = 0$. By (3.10), (3.11), (3.12), we have

$$\begin{aligned}
S^{(1,1)}_{\alpha,2,2} &= 2E_0^{\left(\frac{\alpha+1}{2}\right)}((z\overline{w})^4) + 2(z\overline{w})^2 E_0^{\left(\frac{\alpha+2}{2}\right)}((z\overline{w})^4) \\
S^{(1,2)}_{\alpha,2,2} &= S^{(2)}_{\alpha,2,2} = 0.
\end{aligned}$$

Since $E^{(\alpha)}_0(x) = E_{1,\alpha}(x)$, we have

$$S^{(1,1)}_{\alpha,2,2} = 2E_{1,\frac{\alpha+1}{2}}((z\overline{w})^4) + 2(z\overline{w})^2 E_{1,\frac{\alpha+2}{2}}((z\overline{w})^4)$$

If we use the duplication formula $E_{a,b}(z) = E_{2a,b}(z^2) + zE_{2a,b+a}(z^2)$, then

$$K^{\mathbb{C}}_{\alpha,2,2}(z,w) = S^{(1,1)}_{\alpha,2,2} = 2E_{\frac{1}{2},\frac{\alpha+1}{2}}((z\overline{w})^2),$$

which has zeros for any $\alpha > -1$ by Lemma 2.3.

Indeed, one can characterize this phenomena by using Lemma 2.3.

Theorem 4.3. *$K^{\mathbb{C}}_{\alpha,q,m}(z,w)$ has no zeros if and only if $(\alpha,q,m)=(0,1,1)$.*

Proof. If $N=0$, then $S^{(1,2)}_{\alpha,q,m}=S^{(2)}_{\alpha,q,m}=0$ for any α,q,m. By Theorem 3.2, it follows that

$$K^{\mathbb{C}}_{\alpha,q,m}(z,w)=S^{(1,1)}_{\alpha,q,m}=m\sum_{e=0}^{m-1}(z\overline{w})^{me}E_0^{\left(\frac{e+\alpha+1}{m}\right)}((z\overline{w})^{m^2}).$$

Since $E_0^{(\alpha)}(x)=E_{1,\alpha}(x)$ for any α, we have

$$\begin{aligned}K^{\mathbb{C}}_{\alpha,q,m}(z,w)=&m\sum_{e=0}^{m-1}(z\overline{w})^{me}\sum_{c=0}^{\infty}\frac{(z\overline{w})^{m^2c}}{\Gamma\left(c+\frac{e+\alpha+1}{m}\right)}\\ =&m\sum_{e=0}^{m-1}\sum_{c=0}^{\infty}((z\overline{w})^m)^e\frac{((z\overline{w})^m)^{mc}}{\Gamma\left(c+\frac{e+\alpha+1}{m}\right)}\\ =&m\sum_{\ell=0}^{\infty}\frac{((z\overline{w})^m)^{\ell}}{\Gamma\left(\frac{\ell+\alpha+1}{m}\right)}\\ =&mE_{\frac{1}{m},\frac{\alpha+1}{m}}((z\overline{w})^m).\end{aligned}$$

It follows that $K^{\mathbb{C}}_{\alpha,q,m}(z,w)$ has no zeros if and only if $\alpha=0$ and $m=1$ by Lemma 2.5.

Let $N\geq 1$. By Theorem 3.2, we have

$$K^{\mathbb{C}}_{\alpha,q,m}(z,0)=\frac{m}{\Gamma(\frac{\alpha+1}{m})}L_N^{(\frac{\alpha+1}{m})}(|z|^{2m}).$$

Thus we conclude that $K^{\mathbb{C}}_{\alpha,q,m}(z,0)$ has zeros for any $\alpha>-1$ and $m\in\mathbb{N}$ by Lemma 2.3. $\square$

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