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A NOTE ON LOCAL CALIBRATIONS OF ALMOST COMPLEX STRUCTURES

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Abstract. In this paper, we study the obstruction on the jets of an almost complex structure J to the existence of a symplectic form ω such that J is compatible with ω. We describe some almost complex structures on $\mathbb{R}^4$ and on $\mathbb{R}^6$, respectively, that cannot be calibrated by any symplectic forms. In particular, these examples pertain to the model almost complex structure on $\mathbb{R}^4$ in [3], and the simple model structure on $\mathbb{R}^6$ in [7].

1. Introduction

Let (M, ω) be a symplectic manifold. Then it is well-known that there exists a canonical almost complex structure J associated with any given Riemannian metric g on M and J is compatible with ω (or ω-calibrated)(cf. [1] and the references therein). From this fact, the following question naturally arises: *Is the converse of this fact is true? Namely, given an almost complex structure J on M, does there exist a symplectic form ω such that J is compatible with ω?* In [8], Migliorini and Tomassini addressed the compatibility in this question. They considered the question locally and observed obstructions on the jets of J to the existence of such a symplectic form ω. In addition, they gave a negative answer for certain almost complex structures on $\mathbb{R}^{2n}$, with $2n \geq 6$. However, computer software was necessary to provide concrete examples.

In [10], Tomassini built on the previous work and provided a more effective way to check examples of almost complex structures which are not compatible with any symplectic forms on $\mathbb{R}^6$ and, more interestingly, on $\mathbb{R}^4$. The examples in [10] are worthy of attraction due to the fact that these descend to almost complex structures with the same property on the *Iwasawa manifold*, the *torus* $\mathbb{T}^4$, and the *Kodaira-Thurston manifold* (cf. [2] and [9]).

In this paper, we focus our attention to this local calibration problem for the *model almost complex structure* on $\mathbb{R}^4$ as defined by Gaussier and Sukhov

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in [3], and for the *simple model structure* as defined by Lee in [7]. These examples are originally presented for the study of *Wong-Rosay theorem* in almost complex manifolds (see [3] and [7]). Moreover, these examples are also meaningful in perspective of the partial integrability of almost complex structures as an algebraic generalization of the celebrated *Newlander-Nirenberg theorem* (see [5]). In addition, the model almost complex structure on $\mathbb{R}^4$ pertains to *infinitesimal automorphisms* on almost complex manifolds (see [6]).

This paper is organized as follows. In Section 2, we first review some basic notations and facts on the symplectic geometry in exploiting the local calibrations of almost complex structures. We then prepare some technical results needed in the proofs of Theorems 3.1 and 3.2. In Section 3, we shall provide the proofs of our main results.

2. Preliminaries

We begin with this section by collecting some basic notions on the symplectic forms and local calibrations. We refer to [1] for more details on symplectic geometry. Let M be a smooth manifold and ω be a differential 2-form on M. Let $\tilde{\omega}_p$ be the corresponding linear mapping $\tilde{\omega}_p : T_pM \to T_p^*M$ which is defined by $\tilde{\omega}_p(Y)(X) = \omega_p(X, Y)$. A differential 2-form ω on M is said to be *symplectic* if it satisfies

(i) ω is closed, that is $d\omega = 0$,

(ii) ω_p is non-degenerate at each point $p \in M$, that is, for any non-zero $Y \in \Gamma(T_pM)$ there exists an $X \in \Gamma(T_pM)$ such that $\omega_p(X, Y) \neq 0$.

A *symplectic manifold* is a pair (M, ω) where M is a smooth manifold and ω is a symplectic form. We call a pair (M, J) an *almost complex manifold* if M is a smooth manifold of dimension $2n$ and J is a smooth tensor field of type $(1, 1)$ with $J^2 = -\mathrm{Id}$, that is, for each $p \in M$, $J_p : T_pM \to T_pM$ is linear and $J_p \circ J_p = -\mathrm{Id}$.

Definition 2.1. *Let (M, J) be an almost complex manifold and ω be a symplectic form. An almost complex structure is* compatible with ω *(or ω-*calibrated*) if it satisfies*

(i) $\omega(JX, JY) = \omega(X, Y)$ *for any vector fields X, Y on M,*

(ii) $\omega(JX, X) > 0$ *for any non-zero vector field X on M.*

It is well-known that the existence of a closed form ω such that J is ω-calibrated is equivalent to the existence of an almost Kähler metric on M (cf. [8]). For examples of almost Kähler structures that are not Kähler, we refer to [4] and the references therein.

Proposition 2.2 ([1])**.** *Let (M, ω) be a symplectic manifold and g be a Riemannian metric on M. Then there exists a canonical almost complex structure J on M which is compatible with ω.*

As we mentioned in Section 1, we shall consider so called *local calibrations of almost complex structures*, the converse of this proposition.

Throughout the paper, we denote a local coordinate system on $\mathbb{R}^{2n}$ by $(x^1, \ldots, x^n)$. Then the standard complex structure J_{st} on $\mathbb{C}^n$ is defined by $J_{st}\left(\frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial x^{n+k}}$ and $J_{st}\left(\frac{\partial}{\partial x^{n+k}}\right) = -\frac{\partial}{\partial x^k}$ for each $k = 1, \ldots, n$. To find obstruction for local calibrations of almost complex structures, Migliorini and Tomassini [8] proved the following theorem:

Theorem 2.3 ([8]). *Let ω_{st} be the standard symplectic form on $\mathbb{R}^{2n}$ and ω be a symplectic form on $\mathbb{R}^{2n}$ such that $\omega[0] = \omega_{st}$. Let J_{st} be the standard complex structure on $\mathbb{R}^{2n}$ and J be an almost complex structure on $\mathbb{R}^{2n}$ such that $J[0] = J_{st}$. Let f be a local diffeomorphism on $\mathbb{R}^{2n}$ such that $f^*\omega = \omega_{st}$ and $f_*[0] = \mathrm{Id}$. Then J is ω-calibrated up to the second order if and only if*

$$[J_{st}, X_h - {}^tX_h][0] = \frac{{}^t\partial J}{\partial x^h} - \frac{\partial J}{\partial x^h}[0], \quad h = 1, \ldots, n,$$

and

$$Z_{hk} - {}^tZ_{hk} + {}^tY_k J_{st} Y_k + {}^tY_k J_{st} Y_h = 0, \quad h, k = 1, \ldots, 2n,$$

where

$$\begin{aligned}
X_h &:= \frac{\partial f_*}{\partial x^h}[0],\ Y_h := [J_{st}, X_h] + B_h, \\
X_{hk} &= \frac{\partial^2 f_*}{\partial x^h \partial x^k}[0], \\
Z_{hk} &:= [J_{st}, X_{hk}] + (X_h X_k + X_k X_h) J_{st} + [B_h, X_k] + [B_k, X_h] \\
&\quad - X_k J_{st} X_h - X_h J_{st} X_k + B_{hk}, \\
B_h &:= \frac{\partial J}{\partial x_h}[0],\ B_{hk} = \frac{\partial^2 J}{\partial x_h \partial x_k}[0].
\end{aligned}$$

Remark 2.4. *The composition of Theorem 2.3 is based on the following observation. The group of the local diffeomorphisms on $\mathbb{R}^{2n}$ acts on the space of almost complex structures on $\mathbb{R}^{2n}$ in the following way:*

$$(f, J) \mapsto \tilde{J} := f_*^{-1} \circ J \circ f_*,$$

where f is a local diffeomorphism of $\mathbb{R}^{2n}$, J is an almost complex structure on $\mathbb{R}^{2n}$, and f_* is the Jacobian matrix of f. Let ω be a symplectic form on $\mathbb{R}^{2n}$ and J be an almost complex structure on $\mathbb{R}^{2n}$. Then J is ω-calibrated if and only if $\tilde{J}$ is ω_0-calibrated, ω_0 being the standard symplectic form on $\mathbb{R}^{2n}$ (see [8]).

As a corollary of Theorem 2.3, Migliorini and Tomassini proved the following (cf. Section 2.1 in [8] and Corollary 2.4 in [10]).

Corollary 2.5 ([8]). *Let J be an almost complex structure on $\mathbb{R}^6$ such that $J[0] = J_{st}$, the standard complex structure on $\mathbb{R}^6$. If there exists a symplectic* *form ω, calibrating J, then the following conditions hold:*

$$
\begin{aligned}
&-\frac{\partial}{\partial x^1}(J_{26}-J_{62})+\frac{\partial}{\partial x^2}(J_{16}-J_{61})-\frac{\partial}{\partial x^3}(J_{15}-J_{51})\\
&-\frac{\partial}{\partial x^4}(J_{23}-J_{32})+\frac{\partial}{\partial x^5}(J_{13}-J_{31})-\frac{\partial}{\partial x^6}(J_{12}-J_{21})=0;\\
&-\frac{\partial}{\partial x^1}(J_{23}-J_{32})+\frac{\partial}{\partial x^2}(J_{13}-J_{31})-\frac{\partial}{\partial x^3}(J_{12}-J_{21})\\
&+\frac{\partial}{\partial x^4}(J_{26}-J_{62})-\frac{\partial}{\partial x^5}(J_{16}-J_{61})+\frac{\partial}{\partial x^6}(J_{15}-J_{51})=0,
\end{aligned}
$$

where the derivatives of J are evaluated at the origin.

Here we follow the same notation as in [10, Corollary 2.4].

Applying the same argument as in the proof of Corollary 2.5 to $\mathbb{R}^8$, one can obtain the following result.

Corollary 2.6. *Let J be an almost complex structure on $\mathbb{R}^8$ such that $J[0]=J_{st}$, the standard complex structure on $\mathbb{R}^8$. If there exists a symplectic form ω, calibrating J, then the following conditions hold:*

$$
\begin{aligned}
&-\frac{\partial}{\partial x^1}(J_{28}-J_{82})+\frac{\partial}{\partial x^2}(J_{18}-J_{81})-\frac{\partial}{\partial x^4}(J_{16}-J_{61})\\
&-\frac{\partial}{\partial x^5}(J_{24}-J_{42})+\frac{\partial}{\partial x^6}(J_{14}-J_{41})-\frac{\partial}{\partial x^8}(J_{12}-J_{21})=0;\\
&-\frac{\partial}{\partial x^1}(J_{24}-J_{42})+\frac{\partial}{\partial x^2}(J_{14}-J_{41})-\frac{\partial}{\partial x^4}(J_{12}-J_{21})\\
&+\frac{\partial}{\partial x^5}(J_{28}-J_{82})-\frac{\partial}{\partial x^6}(J_{18}-J_{81})+\frac{\partial}{\partial x^8}(J_{16}-J_{61})=0;\\
&-\frac{\partial}{\partial x^1}(J_{23}-J_{32})+\frac{\partial}{\partial x^2}(J_{13}-J_{31})-\frac{\partial}{\partial x^3}(J_{12}-J_{21})\\
&+\frac{\partial}{\partial x^5}(J_{27}-J_{72})-\frac{\partial}{\partial x^6}(J_{17}-J_{71})+\frac{\partial}{\partial x^7}(J_{16}-J_{61})=0;\\
&-\frac{\partial}{\partial x^1}(J_{27}-J_{72})+\frac{\partial}{\partial x^2}(J_{17}-J_{71})-\frac{\partial}{\partial x^3}(J_{16}-J_{61})\\
&-\frac{\partial}{\partial x^5}(J_{23}-J_{32})+\frac{\partial}{\partial x^6}(J_{13}-J_{31})-\frac{\partial}{\partial x^7}(J_{12}-J_{21})=0.
\end{aligned}
$$

3. Main results

In this section, we construct some non-calibrable almost complex structures on $\mathbb{R}^4$ and on $\mathbb{R}^6$, respectively. As we mentioned in Section 1, the examples presented in this section are closely related to the study of *model almost complex structures* and *simple model structures.* At first, for the 4-dimensional case, we get the following theorem by modifying the arguments in the proof of Theorem 3.1 in [10].

Theorem 3.1. *Let $f, g : \mathbb{R}^4 \to \mathbb{R}$ be functions such that $f(0) = g(0) = 0$. If f and g satisfy either $\frac{\partial f}{\partial x^3}(0) \neq 0$ or $\frac{\partial g}{\partial x^3}(0) \neq 0$ for local coordinates $x = (x^1, x^2, x^3, x^4)$, then the following*

$$J(x) := \begin{pmatrix} 0 & 0 & -1 & 0 \\ f(x) & 0 & -g(x) & -1 \\ 1 & 0 & 0 & 0 \\ -g(x) & 1 & -f(x) & 0 \end{pmatrix}$$

is an almost complex structure on $\mathbb{R}^4$ which cannot be calibrated by any symplectic forms.

Proof. Aiming for the contradiction, assume that the almost complex structure J is calibrated by a symplectic form ω_1 on $\mathbb{R}^4$. Then one can define an almost complex structure $\mathbb{J}$ by setting

$$\mathbb{J} := \begin{pmatrix} J & 0 \\ 0 & J_2 \end{pmatrix},$$

where J is the standard complex structure on $\mathbb{R}^6$, with local coordinates (x^5, x^6) and $\omega_2 = dx^6 \wedge dx^5$ the canonical symplectic form. Let $\omega := \omega_1 + \omega_2$. Then ω is closed and J is calibrated by ω. Now we let

$$A := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\tilde{\mathbb{J}} := \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ f & 0 & 0 & -g & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -g & 1 & 0 & -f & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then we get

$$\begin{aligned} \tilde{\mathbb{J}}_{26} - \tilde{\mathbb{J}}_{62} &= 0; \\ \tilde{\mathbb{J}}_{16} - \tilde{\mathbb{J}}_{61} &= 0; \\ \tilde{\mathbb{J}}_{15} - \tilde{\mathbb{J}}_{51} &= g; \\ \tilde{\mathbb{J}}_{23} - \tilde{\mathbb{J}}_{32} &= 0; \\ \tilde{\mathbb{J}}_{13} - \tilde{\mathbb{J}}_{31} &= 0; \\ \tilde{\mathbb{J}}_{12} - \tilde{\mathbb{J}}_{21} &= -f. \end{aligned}$$

Hence the structure $\tilde{\mathbb{J}}$ does not satisfy at least one of the conditions in Corollary 2.5, which is a contradiction. This completes the proof. □

Now we recall that the simple model structures [7] are defined by the following setting: We denote by $\tilde{z} = (z^1, z^2)$ local coordinates in $\mathbb{C}^2$. Then, in $\mathbb{C}^3$, a *simple model structure* is defined by

$$J = \begin{pmatrix} J_{st}^{(2)} & 0 \\ B_{\tilde{z}}^J & J_{st}^{(1)} \end{pmatrix},$$

where $J_{st}^{(\ell)}$ is the standard complex structure on $\mathbb{C}^\ell$ for each $\ell = 1, 2$, and

$$\left(B^J(\tilde{z})\right)_{\mathbb{C}} = \left(b_{1,1}^J \bar{z}^1 + b_{1,2}^J \bar{z}^2, b_{2,1}^J \bar{z}^1 + b_{2,2}^J \bar{z}^2\right) := \left(\overline{B}_J^1(\tilde{z}), \overline{B}_J^2(\tilde{z})\right)$$

for $b_{j,k}^J \in \mathbb{C}$, $j, k = 1, 2$. If this simple model structure is calibrated by a symplectic form ω_1 on $\mathbb{R}^6$, then one can consider an almost complex structure on $\mathbb{R}^6$ defined by

$$\mathbb{J} := \begin{pmatrix} J & 0 \\ 0 & J_2 \end{pmatrix},$$

where J_2 is the standard complex structure on $\mathbb{R}^2$, with local coordinates (x^7, x^8) and $\omega_2 = dx^8 \wedge dx^7$ the canonical symplectic form. Let $\omega := \omega_1 + \omega_2$. Then ω is closed and $\mathbb{J}$ is calibrated by ω. Let us consider an extension of the six-dimensional simple model structure to an eight-dimensional almost complex structure, which is given as follows.

$$\begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ \mathrm{Re}(B_J^1(\tilde{z})) & \mathrm{Re}(B_J^2(\tilde{z})) & 0 & -\mathrm{Im}(B_J^1(\tilde{z})) & -\mathrm{Im}(B_J^1(\tilde{z})) & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\mathrm{Im}(B_J^1(\tilde{z})) & -\mathrm{Im}(B_J^2(\tilde{z})) & 1 & -\mathrm{Re}(B_J^1(\tilde{z})) & -\mathrm{Re}(B_J^2(\tilde{z})) & 0 \end{pmatrix}$$

We denote by $\tilde{\mathbb{J}}$ the corresponding extended almost complex structure on $\mathbb{R}^8$. Applying Corollary 2.6 and the above argument to $\tilde{\mathbb{J}}$, we obtain the following theorem.

Theorem 3.2. *For $\tilde{z} = (z^1, z^2)$, $z^1 := x^1 + \imath x^5$, $z^2 := x^2 + \imath x^6$, we let*

$$\begin{cases} B_J^1(\tilde{z}) = \bar{b}_{1,1}^J z^1 + \bar{b}_{1,2}^J z^2; \\ B_J^2(\tilde{z}) = \bar{b}_{2,1}^J z^1 + \bar{b}_{2,2}^J z^2, \end{cases}$$

such that $b^J_{2,1} \neq b^J_{1,2}$, where $b_{j,k} \in \mathbb{C}$ for $j, k = 1, 2$. Then the almost complex structure

$$\begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ \mathrm{Re}(B^1_J(\tilde{z})) & \mathrm{Re}(B^2_J(\tilde{z})) & 0 & -\mathrm{Im}(B^1_J(\tilde{z})) & -\mathrm{Im}(B^1_J(\tilde{z})) & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\mathrm{Im}(B^1_J(\tilde{z})) & -\mathrm{Im}(B^2_J(\tilde{z})) & 1 & -\mathrm{Re}(B^1_J(\tilde{z})) & -\mathrm{Re}(B^2_J(\tilde{z})) & 0 \end{pmatrix}$$

cannot be calibrated by any symplectic forms on $\mathbb{R}^6$.

Remark 3.3. *The assumption of this theorem is consistent with the obstruction to the integrability of a given almost complex structure.*

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