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ACCELERATED HYBRID ALGORITHMS FOR NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we introduce and study two different iterative hybrid projection algorithms for solving a fixed point problem of nonexpansive mappings. The first algorithm is generated by the combination of the inertial method and the hybrid projection method. On the other hand, the second algorithm is constructed by the convex combination of three updated vectors and the hybrid projection method. The strong convergence of the two proposed algorithms are proved under very mild assumptions on the scalar control. For illustrating the advantages of these two newly invented algorithms, we created some numerical results to compare various numerical performances of our algorithms with the algorithm proposed by Dong and Lu [11].

1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. A mapping $T : H \to H$ is said to be a nonexpansive if $\|Tx - Ty\| \leq$

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$\|x - y\|$ holds for all $x, y \in H$. The set of all fixed points of the operator T is denoted by $Fix(T) = \{x \in H : Tx = x\}$. Given C a nonempty closed convex subset of H. The metric projection of H onto C, $P_C : H \to C$ is defined by $P_C(x) = \arg\min_{c \in C} \|x - c\|$ for all $x \in H$, see more details in [3, 25] and the references cited therein.

The fixed point problem for a mapping T is defined as:

$$\text{Find} \quad x \in H \quad \text{such that} \quad x = Tx.$$

The development of iterative methods for approximating a solution of fixed point problem of nonexpansive mappings is an important and interesting task in numerical analysis and applied scientific branches. Many authors are interested in this problem because it can be applied in a variety of applications such as optimal control problems, economic modelings, inverse problem, image recovery, signal processing, game theory and data analysis, see more detail in [1, 3, 4, 8, 25] and the references cited therein. A significant body of work on iteration methods for fixed point problems has accumulated in literature (for example, see [12, 21, 22, 26, 28]).

Among the notable algorithms developed in this direction is the Mann iteration method [18], which is given as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \geq 0, \tag{1.1}$$

where $x_0 \in H$ and $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in $[0, 1]$. Reich [24] proved fundamental results of convergence, that is, if sequence $\{\alpha_n\}_{n=0}^{\infty}$ satisfies $\sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n) = +\infty$ then the sequence $\{x_n\}_{n=0}^{\infty}$ generated by Mann's algorithm (1.1) converges weakly to a fixed point of T. Later, Xu [27] constructed the iterative method by using the convex combination of three updated vectors which is called the Mann iteration process with errors as follows:

$$x_{n+1} = \alpha_n x_n + \beta_n T x_n + \gamma_n u_n,$$

where $x_0, u_0 \in H$ and $\alpha_n, \beta_n, \gamma_n \in [0, 1]$ are suitably chosen scalars satisfying $\alpha_n + \beta_n + \gamma_n = 1$.

One of the key patterns for accelerating convergence is the inertial extrapolation term $\theta_n(x_n - x_{n+1})$ that has been an important tool employed in improving the performance of algorithms and has some nice convergence characteristics. By the main feature of the inertial-type algorithms, it can use the previous iterates to construct the next one. Which the inertial-type extrapolation based on the heavy ball method of the two-order time dynamical system as an acceleration process was first proposed by Polyak [23] to solve the smooth convex minimization problem as follows:

$$\begin{cases} x_0, x_1 \in H, \\ x_{n+1} = x_n + \alpha_n(x_n - x_{n-1}) + \beta_n A x_n, \end{cases}$$

where A is a mapping on H and $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are two control sequences. Consequently, many researchers have adopted inertial-type algorithms to speed up the convergence process. We refer interested readers to [1, 5, 6, 9, 10, 15, 16] for more information.

The strong convergence is often much more desirable than the weak convergence (see [2] and references therein). Many attempts have been made to modify the Mann iteration so that the strong convergence is guaranteed. A hybrid algorithm is one of the interesting results for approximating fixed points because it is a favor to solve strong convergence (see [7, 11, 13, 14, 17, 22, 26]).

In 2003, Nakajo and Takahashi [20] introduced a hybrid algorithm for a nonexpansive mapping T as follows:

$$\begin{cases} x_0 \in H \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_0 - x_n, z - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{cases} \tag{1.2}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H, $\{\alpha_n\}_{n=0}^{\infty} \subset [0, 1)$. They showed that the hybrid algorithm defined by (1.2) converges strongly to $q = P_{Fix(T)}x_0$.

In 2015, Dong and Lu [11] proposed and studied a hybrid algorithm for a nonexpansive mapping T as follows:

$$\begin{cases} x_0, z_0 \in H \text{ chosen arbitrarily}, \\ z_{n+1} = \alpha_n z_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in H : \|z_{n+1} - z\|^2 \leq \alpha_n\|z_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2\}, \\ Q_n = \{z \in H : \langle x_n - z, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0, \sigma]$ for some $\sigma \in [0, \frac{1}{2})$. They proved that (1.3) converges strongly to $q = P_{Fix(T)}x_0$. Moreover, the numerical results of (1.3) showed more advantage than (1.2).

Motivated by the research works as in the above direction, we present two new accelerated hybrid algorithms for solving a fixed point problem of nonexpansive mappings. Moreover, we create some numerical results to compare various numerical performances of our algorithms with the algorithm of Dong and Lu [11].

2. Preliminaries

In this section, we provide some notations and tools in a real Hilbert space H. We will use the symbols $\rightharpoonup$ for weak convergence and $\to$ for strong convergence and define the set $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of the sequence $\{x_n\}$.

The following lemma for the geometric properties, is useful for the proofs of the results in this paper. It is very easy to prove that for the Hilbert spaces.

Lemma 2.1. *Let H be real Hilbert space. Then the following equalities are hold.*

(i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$, *for all* $x, y \in H$.

(ii) $\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2$ *for all* $\alpha, \beta, \gamma \in [0,1]$ *with* $\alpha+\beta+\gamma = 1$ *and for all* $x, y, z \in H$.

In particular, if $\gamma = 0$ *then the following identity holds:*

(iii) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$ *for all* $\alpha \in [0,1]$ *and for all* $x, y \in H$.

Lemma 2.2. ([25]) *Let C be a closed convex subset of a real Hilbert space H and let P_C be the (metric or nearest point) projection from H onto C (i.e., for $x \in H, P_Cx$ is the only point in C such that $\|x-P_Cx\| = \inf\{\|x-z\| : z \in C\}$). Given $x \in H$ and $z \in C$. Then $z = P_Cx$ if and only if*

$$\langle x-z, y-z\rangle \leq 0 \text{ for all } y \in C.$$

Lemma 2.3. ([3]) *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to H$ be a nonexpansive mapping. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in C and $x \in H$ such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$ as $n \to +\infty$. Then $x \in Fix(T)$.*

Lemma 2.4. ([19]) *Let C be a closed convex subset of a real Hilbert space H. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in H and $u \in H$. Let $q = P_Cu$. If $\{x_n\}_{n=0}^{\infty}$ is a sequence such that $\omega_w(x_n) \subset C$ and satisfies the condition:*

$$\|x_n - u\| \leq \|u-q\| \text{ for all } n.$$

Then $x_n \to q$ as $n \to \infty$.

3. Main results

In this section, we propose two new accelerated hybrid algorithms to solve a fixed point problem of nonexpansive mappings in real Hilbert spaces. The strong convergence results of these two algorithms are proved.

Before going to the main theorems, we would like to provide the following lemma to help the proof easier.

Lemma 3.1. *Let H be a real Hilbert space and given $x, y, z, w \in H$, $t \in [0,1]$ and $a \in \mathbb{R}$. Then the set*

$$K := \{v \in H : \|x - v\|^2 \le t\|y - v\|^2 + (1-t)\|z - v\|^2 + \langle w, v\rangle + a\}$$

is closed and convex.

Proof. It can be observed from the definition of K,

$$\|x - v\|^2 \le t\|y - v\|^2 + (1-t)\|z - v\|^2 + \langle w, v\rangle + a.$$

It implies that

$$\begin{aligned}\|x\|^2 + 2\langle x, v\rangle + \|v\|^2 \le\, & t\left(\|y\|^2 + 2\langle y, v\rangle + \|v\|^2\right) \\ & + (1-t)\left(\|z\|^2 + 2\langle z, v\rangle + \|v\|^2\right) + \langle w, v\rangle + a.\end{aligned}$$

Hence we have

$$\|x\|^2 + 2\langle x, v\rangle \le \left(t\|y\|^2 + (1-t)\|z\|^2\right) + \langle 2(ty + (1-t)z) + w, v\rangle + a.$$

Therefore, we obtain

$$\langle 2x - 2(ty + (1-t)z) - w, v\rangle \le \left(t\|y\|^2 + (1-t)\|z\|^2\right) + a - \|x\|^2.$$

It is not hard to verify by using the linearity of inner product to ensure that K is closed and convex. $\square$

Theorem 3.2. *Let $T : H \to H$ be a nonexpansive mapping such that $Fix(T) \neq \varnothing$. For the the control sequences $\{\theta_n\}_{n=0}^{\infty} \subset [0,1]$ and $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1)$, define a sequence $\{x_n\}_{n=0}^{\infty}$ by the following:*

$$\begin{cases} x_0, x_1, z_1 \in H \text{ chosen arbitrarily}, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_{n+1} = \alpha_n z_n + (1-\alpha_n)Ty_n, \\ C_n = \{z \in H : \|z_{n+1} - z\|^2 \le \alpha_n\|z_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 \\ \qquad +(1-\alpha_n)(2\theta_n\langle x_n - z, x_n - x_{n-1}\rangle \\ \qquad +\theta_n^2\|x_n - x_{n-1}\|^2 - \alpha_n\|z_n - Ty_n\|^2)\}, \\ Q_n = \{z \in H : \langle x_n - z, x_n - x_0\rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \ge 1. \end{cases} \tag{3.1}$$

Then the iterative sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to $P_{Fix(T)}x_0$.

Proof. First, we will show that $Fix(T) \subset C_n$ for all $n \geq 0$. Using Lemma 2.1, we get that for all $p \in Fix(T)$,

$$\begin{aligned}\|z_{n+1}-p\|^2 &= \|\alpha_n(z_n-p)+(1-\alpha_n)(Ty_n-p)\|^2\\ &= \alpha_n\|z_n-p\|^2+(1-\alpha_n)\|Ty_n-p\|^2-\alpha_n(1-\alpha_n)\|z_n-Ty_n\|^2\\ &\leq \alpha_n\|z_n-p\|^2+(1-\alpha_n)\|y_n-p\|^2-\alpha_n(1-\alpha_n)\|z_n-Ty_n\|^2. \quad (3.2)\end{aligned}$$

It can be observed that

$$\begin{aligned}\|y_n-p\|^2 &= \|(x_n-p)+\theta_n(x_n-x_{n-1})\|^2\\ &= \|x_n-p\|^2+2\theta_n\langle x_n-p, x_n-x_{n-1}\rangle+\theta_n^2\|x_n-x_{n-1}\|^2. \quad (3.3)\end{aligned}$$

Substituting (3.3) into (3.2), we obtain that

$$\begin{aligned}\|z_{n+1}-p\|^2 &\leq \alpha_n\|z_n-p\|^2+(1-\alpha_n)\Big(\|x_n-p\|^2+2\theta_n\langle x_n-p, x_n-x_{n-1}\rangle\\ &\quad +\theta_n^2\|x_n-x_{n-1}\|^2\Big)-\alpha_n(1-\alpha_n)\|z_n-Ty_n\|^2\\ &= \alpha_n\|z_n-p\|^2+(1-\alpha_n)\|x_n-p\|^2\\ &\quad +(1-\alpha_n)\left(2\theta_n\langle x_n-p, x_n-x_{n-1}\rangle+\theta_n^2\|x_n-x_{n-1}\|^2\right.\\ &\quad \left.-\alpha_n\|z_n-Ty_n\|^2\right).\end{aligned}$$

This shows that $Fix(T) \subset C_n \neq \varnothing$ for all $n \geq 0$.

Next, it is not hard to prove by using Lemma 3.1 to confirm that C_n is closed and convex.

We claim that $Fix(T) \subset Q_n$ for all $n \geq 0$. For $n = 0$, we have $Fix(T) \subset H = Q_0$. Assume that $Fix(T) \subset Q_n$. Then since $Fix(T) \subset C_n$ for all $n \geq 0$, we get that $Fix(T) \subset C_n \cap Q_n$. It follows from $x_{n+1} = P_{C_n \cap Q_n}x_0$ and by applying Lemma 2.2, we get

$$\langle x_{n+1}-z, x_{n+1}-x_0\rangle \leq 0 \text{ for all } z \in C_n \cap Q_n. \quad (3.4)$$

Since $Q_{n+1} = \{z \in H : \langle x_{n+1}-z, x_{n+1}-x_0\rangle \leq 0\}$, it yields $C_n \cap Q_n \subset Q_{n+1}$. Thus, we have $Fix(T) \subset Q_{n+1}$. By mathematical induction, we can conclude that $Fix(T) \subset Q_n$ for all $n \geq 0$.

Since $Fix(T)$ is a nonempty closed convex subset of H, there exists a unique element $q \in Fix(T)$ such that $q = P_{Fix(T)}x_0$. From the definition of Q_n actually implies $x_n = P_{Q_n}x_0$. This together with the fact that $Fix(T) \subset Q_n$ further implies $\|x_n - x_0\| \leq \|p - x_0\|$ for all $p \in Fix(T)$. Due to $q \in Fix(T)$, we get

$$\|x_n - x_0\| \leq \|q - x_0\| \text{ for all } n \in \mathbb{N} \cup \{0\}, \quad (3.5)$$

which implies that $\{x_n\}_{n=0}^{\infty}$ is bounded and thus $\omega_w(x_n) \neq \varnothing$.

On the other hand, by the fact that $x_{n+1} \in Q_n$, we have

$$\langle x_{n+1}-x_n, x_n-x_0\rangle \geq 0.$$

This together with Lemma 2.1 (i) implies that

$$\begin{aligned}\|x_{n+1}-x_n\|^2 &= \|(x_{n+1}-x_0)-(x_n-x_0)\|^2\\ &= \|x_{n+1}-x_0\|^2-\|x_n-x_0\|^2-2\langle x_{n+1}-x_n, x_n-x_0\rangle\\ &\le \|x_{n+1}-x_0\|^2-\|x_n-x_0\|^2. \end{aligned} \tag{3.6}$$

By (3.5) and (3.6), we obtain that

$$\begin{aligned}\sum_{n=1}^{N}\|x_{n+1}-x_n\|^2 &\le \sum_{n=1}^{N}(\|x_{n+1}-x_0\|^2-\|x_n-x_0\|^2) = \|x_N-x_0\|^2-\|x_1-x_0\|^2\\ &\le \|q-x_0\|^2-\|x_1-x_0\|^2.\end{aligned}$$

By letting $N\to\infty$, it follows that the series $\sum_{n=1}^{\infty}\|x_{n+1}-x_n\|^2$ is convergent and then we have $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$. By the definition of y_n in (3.1), we get

$$\|y_n-x_n\| = \theta_n\|x_n-x_{n-1}\|\to 0. \tag{3.7}$$

From $x_{n+1}\in C_n$, we get

$$\begin{aligned}\|z_{n+1}-x_{n+1}\|^2 &\le \alpha_n\|z_n-x_{n+1}\|^2+(1-\alpha_n)\|x_n-x_{n+1}\|^2\\ &\quad+(1-\alpha_n)\left(2\theta_n\langle x_n-x_{n+1}, x_n-x_{n-1}\rangle+\theta_n^2\|x_n-x_{n-1}\|^2\right.\\ &\quad\left.-\alpha_n\|z_n-Ty_n\|^2\right).\end{aligned} \tag{3.8}$$

By using Lemma 2.1 (iii), we have the following:

$$\begin{aligned}&\|z_{n+1}-x_{n+1}\|^2\\ &=\|\alpha_n(z_n-x_{n+1})+(1-\alpha_n)(Ty_n-x_{n+})\|^2\\ &=\alpha_n\|z_n-x_{n+1}\|^2+(1-\alpha_n)\|Ty_n-x_{n+1}\|^2-\alpha_n(1-\alpha_n)\|z_n-Ty_n\|^2.\end{aligned} \tag{3.9}$$

Substituting (3.9) into the left side of (3.8) and eliminate the same terms and then divide throughout the inequality by $(1-\alpha_n)$, we get

$$\begin{aligned}\|Ty_n-x_{n+1}\|^2 &\le \|x_n-x_{n+1}\|^2+2\theta_n\langle x_n-x_{n+1}, x_n-x_{n-1}\rangle+\theta_n^2\|x_n-x_{n-1}\|^2\\ &\to 0 \text{ as } n\to\infty.\end{aligned}$$

Note that

$$\|y_n-x_{n+1}\| \le \|x_n-x_{n+1}\|+\theta_n\|x_n-x_{n-1}\|\to 0 \text{ as } n\to\infty.$$

Therefore,

$$\|y_n-Ty_n\| \le \|y_n-x_{n+1}\|+\|x_{n+1}-Ty_n\|\to 0 \text{ as } n\to\infty.$$

By using (3.7), we get the following

$$\begin{aligned}\|x_n-Tx_n\| &\le \|x_n-y_n\|+\|y_n-Ty_n\|+\|Ty_n-Tx_n\|\\ &\le 2\|x_n-y_n\|+\|y_n-Ty_n\|\to 0 \text{ as } n\to\infty.\end{aligned} \tag{3.10}$$

From (3.10) and Lemma 2.3 guarantee that every weak limit point of $\{x_n\}$ is a fixed point of T. That is, $\omega_w(x_n) \subset Fix(T)$. And then, inequality (3.5) and Lemma 2.4 ensure the strong convergence of $\{x_n\}_{n=0}^{\infty}$ to $P_{Fix(T)}x_0$. This completes the proof. $\square$

Corollary 3.3. *Let $T : H \to H$ be a nonexpansive mapping such that $Fix(T) \neq \varnothing$. For the control sequence $\{\alpha_n\}_{n=0}^{\infty} \subset [0, 1)$, define a sequence $\{x_n\}_{n=0}^{\infty}$ by the following:*

$$\begin{cases} x_0, z_0 \in H \text{ chosen arbitrarily}, \\ z_{n+1} = \alpha_n z_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in H : \|z_{n+1} - z\|^2 \leq \alpha_n\|z_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 \\ \qquad\qquad\qquad\qquad -\alpha_n(1-\alpha_n)\|y_n - Tx_n\|^2\}, \\ Q_n = \{z \in H : \langle x_n - z, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}x_0, \quad n \geq 1. \end{cases} \tag{3.11}$$

Then the iterative sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to $P_{Fix(T)}x_0$.

Proof. If $\theta_n = 0$ for all $n \in \mathbb{N} \cup \{0\}$ in Theorem 3.2, then $y_n = x_n$, and then we have the desired result. $\square$

Note that the set C_n in Corollary 3.3 is the subset of C_n of [11, Theorem 3.1]. For this advantage, it can be said that Theorem 3.2 and Corollary 3.3 were developed to produce better results which are numerically effected in the next section.

The following theorem is another approach used for solving a fixed point problem of nonexpansive mappings.

Theorem 3.4. *Let $T : H \to H$ be a nonexpansive mapping such that $Fix(T) \neq \varnothing$. For the control sequences $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \subset [0, 1]$ and $\{\gamma_n\}_{n=0}^{\infty} \subset (0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, define a sequence $\{x_n\}_{n=0}^{\infty}$ by the following:*

$$\begin{cases} x_0, z_0 \in H \text{ chosen arbitrarily}, \\ z_{n+1} = \alpha_n z_n + \beta_n x_n + \gamma_n Tx_n, \\ C_n = \{z \in H : \|z_{n+1} - z\|^2 \leq \alpha_n\|z_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 \\ \qquad -\alpha_n\beta_n\|z_n - x_n\|^2 - \alpha_n\gamma_n\|z_n - Tx_n\|^2 - \beta_n\gamma_n\|x_n - Tx_n\|^2\}, \\ Q_n = \{z \in H : \langle x_n - z, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}x_0, \quad n \geq 1, \end{cases} \tag{3.12}$$

Then the iterative sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to $P_{Fix(T)}x_0$.

Proof. First, we will show that $Fix(T) \subset C_n$ for all $n \geq 0$. Using Lemma 2.1 (ii), we get that for all $p \in Fix(T)$,

$$\begin{aligned}
\|z_{n+1} - p\|^2 &= \|\alpha_n(z_n - p) + \beta_n(x_n - p) + \gamma_n(Tx_n - p)\|^2 \\
&= \alpha_n\|z_n - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|Tx_n - p\|^2 \\
&\quad - \alpha_n\beta_n\|z_n - x_n\|^2 - \alpha_n\gamma_n\|z_n - Tx_n\|^2 - \beta_n\gamma_n\|x_n - Tx_n\|^2 \\
&\leq \alpha_n\|z_n - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 \\
&\quad - \alpha_n\beta_n\|z_n - x_n\|^2 - \alpha_n\gamma_n\|z_n - Tx_n\|^2 - \beta_n\gamma_n\|x_n - Tx_n\|^2 \\
&= \alpha_n\|z_n - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - \alpha_n\beta_n\|z_n - x_n\|^2 \\
&\quad - \alpha_n\gamma_n\|z_n - Tx_n\|^2 - \beta_n\gamma_n\|x_n - Tx_n\|^2.
\end{aligned}$$

This means that $Fix(T) \subset C_n \neq \varnothing$ for all $n \geq 0$.

Next, by employing Lemma 3.1, it can be proved that C_n is closed and convex.

For proving that $Fix(T) \subset Q_n$ for all $n \geq 0$, the steps of proof are the same as in Theorem 3.2.

Since $q = P_{Fix(T)}x_0$, $x_n = P_{Q_n}x_0$ and $Fix(T) \subset Q_n$, we get that

$$\|x_n - x_0\| \leq \|q - x_0\|, \tag{3.13}$$

which implies that $\{x_n\}$ is bounded and thus $\omega_w(x_n) \neq \varnothing$.

On the other hand, from $x_{n+1} \in Q_n$, we have

$$\langle x_{n+1} - x_n, x_n - x_0\rangle \geq 0.$$

This together with Lemma 2.1 (i) implies that

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\
&\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2. \qquad (3.14)
\end{aligned}$$

By (3.13) and (3.14), we obtain that

$$\begin{aligned}
\sum_{n=1}^{N}\|x_{n+1} - x_n\|^2 &\leq \sum_{n=1}^{N}(\|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2) = \|x_N - x_0\|^2 - \|x_1 - x_0\|^2 \\
&\leq \|q - x_0\|^2 - \|x_1 - x_0\|^2.
\end{aligned}$$

By letting $N \to \infty$, it follows that the series $\sum_{n=1}^{\infty}\|x_{n+1} - x_n\|^2$ is convergent and so, we have $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

We expect that $\|Tx_n - x_n\| \to 0$. From the fact $x_{n+1} \in C_n$, we get

$$\begin{aligned}
\|z_{n+1} - x_{n+1}\|^2 &\leq \alpha_n\|z_n - x_{n+1}\|^2 + (1 - \alpha_n)\|x_n - x_{n+1}\|^2 - \alpha_n\beta_n\|z_n - x_n\|^2 \\
&\quad - \alpha_n\gamma_n\|z_n - Tx_n\|^2 - \beta_n\gamma_n\|x_n - Tx_n\|^2. \qquad (3.15)
\end{aligned}$$

Since $z_{n+1} = \alpha_n z_n + \beta_n x_n + \gamma_n Tx_n$, by using Lemma 2.1 (ii), it will come that

$$\begin{aligned}\|z_{n+1} - x_{n+1}\|^2 = \alpha_n\|z_n - x_{n+1}\|^2 &+ \beta_n\|x_n - x_{n+1}\|^2 \\ &+ \gamma_n\|Tx_n - x_{n+1}\|^2 - \alpha_n\beta_n\|z_n - x_n\|^2 \\ &- \alpha_n\gamma_n\|z_n - Tx_n\|^2 - \beta_n\gamma_n\|x_n - Tx_n\|^2. \end{aligned} \tag{3.16}$$

Substituting (3.16) into the left side of (3.15) and eliminate the same terms, we get that

$$\gamma_n\|Tx_n - x_{n+1}\|^2 \leq \gamma_n\|x_n - x_{n+1}\|^2.$$

By dividing the above inequality by γ_n, we obtain that

$$\|Tx_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 \to 0 \text{ as } n \to \infty.$$

Continuing simple calculations will show that

$$\|Tx_n - x_n\| \leq \|Tx_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.17}$$

From (3.17) and Lemma 2.3 guarantee that every weak limit point of $\{x_n\}$ is a fixed point of T. That is,

$$\omega_w(x_n) \subset Fix(T).$$

And then, inequality (3.13) and Lemma 2.4 ensure the strong convergence of $\{x_n\}_{n=0}^{\infty}$ to $P_{Fix(T)}x_0$. This completes the proof. □

Notice that if $\beta_n = 0$ in Theorem 3.4 then $\gamma_n = (1 - \alpha_n)$ for all $n \in \mathbb{N} \cup \{0\}$. And then for each $n \in \mathbb{N} \cup \{0\}$, the set C_n in Theorem 3.4 is the same as in Corollary 3.3. Therefore, Theorem 3.4 can be reduced to Corollary 3.3.

4. Numerical experiments

In this section, we provide a numerical example to illustrate the performance of the proposed algorithms. We first begin with presenting the specific expression of $P_{C_n \cap Q_n}x_0$ in Algorithm (3.1) and Algorithm (3.12) that were obtained from our main results. The form of hybrid algorithm is difficult to realize in actual computing programs because the specific expression of $P_{C_n \cap Q_n}x_0$ cannot be got, in general. So, He et al. [13] introduced the specific expression of $P_{C_n \cap Q_n}x_0$ and thus the hybrid method for Mann iteration process can be realized easily, where C_n and Q_n are two half-spaces.

For this reason, we follow the ideas of He et al. [13], and obtain the specific expression of $P_{C_n \cap Q_n}x_0$ of Algorithm (3.1) and Algorithm (3.12).

We now translate a new algorithm which is equivalent to Algorithm (3.1) as follows:

$$\begin{cases} x_0, x_1, z_1 \in H \text{ chosen arbitrarily}, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_{n+1} = \alpha_n z_n + (1-\alpha_n)Ty_n, \\ u_n = \alpha_n z_n + (1-\alpha_n)y_n - z_{n+1}, \\ v_n = \frac{1}{2}\Big(\alpha_n\|z_n\|^2 + (1-\alpha_n)\big(\|x_n\|^2 + 2\theta_n\langle x_n - x_{n-1}, x_n\rangle \\ \qquad\qquad +\theta_n^2\|x_n - x_{n-1}\|^2 - \alpha_n\|z_n - Ty_n\|^2\big) - \|z_{n+1}\|^2\Big), \\ C_n = \{z \in H : \langle u_n, z\rangle \le v_n\}, \\ Q_n = \{z \in H : \langle x_n - z, x_n - x_0\rangle \le 0\}, \\ x_{n+1} = p_n, \quad \text{if } p_n \in Q_n, \\ x_{n+1} = q_n, \quad \text{if } p_n \notin Q_n, \end{cases} \tag{4.1}$$

and we translate a new algorithm which equivalent to Algorithm (3.12) as follows:

$$\begin{cases} x_1, z_1 \in H \text{ chosen arbitrarily}, \\ z_{n+1} = \alpha_n z_n + \beta_n x_n + \gamma_n Tx_n, \\ u_n = \alpha_n z_n + (1-\alpha_n)x_n - z_{n+1}, \\ v_n = \frac{1}{2}\Big(\alpha_n\|z_n\|^2 + (1-\alpha_n)\|x_n\|^2 - \alpha_n\beta_n\|z_n - x_n\|^2 \\ \qquad\qquad -\alpha_n\gamma_n\|z_n - Tx_n\|^2 - \beta_n\gamma_n\|x_n - Tx_n\|^2 - \|z_{n+1}\|^2\Big), \\ C_n = \{z \in H : \langle u_n, z\rangle \le v_n\}, \\ Q_n = \{z \in H : \langle x_n - z, x_n - x_0\rangle \le 0\}, \\ x_{n+1} = p_n, \quad \text{if } p_n \in Q_n, \\ x_{n+1} = q_n, \quad \text{if } p_n \notin Q_n, \end{cases} \tag{4.2}$$

where

$$p_n = x_0 - \frac{\langle u_n, x_0\rangle - v_n}{\|u_n\|^2}u_n,$$
$$q_n = \Big(1 - \frac{\langle x_0 - x_n, x_n - p_n\rangle}{\langle x_0 - x_n, w_n - p_n\rangle}\Big)p_n + \frac{\langle x_0 - x_n, x_n - p_n\rangle}{\langle x_0 - x_n, w_n - p_n\rangle}w_n,$$
$$w_n = x_n - \frac{\langle u_n, x_n\rangle - v_n}{\|u_n\|^2}u_n.$$

Let $\mathbb{R}^2$ be a two-dimensional Euclidean space with the usual inner product $\langle v^{(1)}, v^{(2)}\rangle = v_1^{(1)}v_1^{(2)} + v_2^{(1)}v_2^{(2)}$ for all $v^{(1)} = (v_1^{(1)}, v_2^{(1)})^T, v^{(2)} = (v_1^{(2)}, v_2^{(2)})^T \in \mathbb{R}^2$ and the norm $\|v\| = \sqrt{v_1^2 + v_2^2}$ $(v = (v_1, v_2)^T \in \mathbb{R}^2)$. He et al. [13] defined a mapping

$$T : (v_1, v_2)^T \mapsto \Big(\sin\frac{v_1 + v_2}{\sqrt{2}}, \cos\frac{v_1 + v_2}{\sqrt{2}}\Big)^T, \tag{4.3}$$

then it is nonexpansive. It is easy to get that T has a fixed point in the unit disk which is difficult to calculate.

For supporting Theorem 3.2 and Theorem 3.4, we simulate a numerical example by using the nonexpansive mapping T defined by (4.3), Algorithm (3.1) and Algorithm (3.12) compared with Algorithm (1.3). In the Table, 'Iter.' and 'Sec.' denote the number of iterations and the CPU time in seconds, respectively. We set $x_0 = x_1 = z_1$ in Algorithm (1.3), Algorithm (3.1) and Algorithm (3.12), $\alpha_n = 0.1, \theta_n = 0.8$ for Algorithm (1.3), Algorithm (3.1) and set $\alpha_n = 0.1, \beta_n = 0.1, \gamma_n = 0.8$ for Algorithm (3.12). Denote $E(x) = \|x_n - x_{n-1}\|$. We took $E(x) < \varepsilon$ as the stopping criterion and $\varepsilon = 10^{-4}$. The algorithms were coded in Matlab R2016b and run on a personal computer.

Table 1. Comparison of Algorithm (1.3), Algorithm (3.1) and Algorithm (3.12) with different initial values.

$x_0 = x_1$	Algorithm (1.3)		Algorithm (3.1)		Algorithm (3.12)	
(z_1)	Iter.	Sec.	Iter.	Sec.	Iter.	Sec.
(4,8)	130	0.038485	6	0.008101	19	0.008332
(7,-7)	922	0.082289	6	0.008230	16	0.008659
(-2,-6)	280	0.054435	4	0.022878	78	0.016505
(-4,9)	1106	0.091497	11	0.008419	19	0.008346

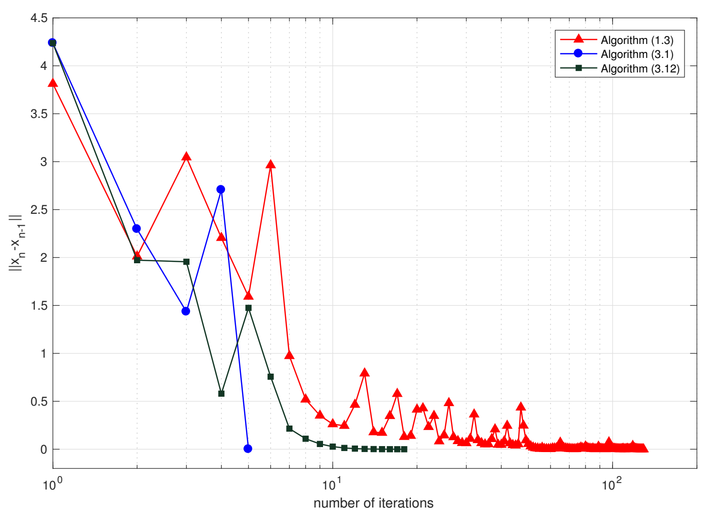

Figure 1. The results computed by Algorithm (1.3), Algorithm (3.1) and Algorithm (3.12). Case: $x_0 = x_1(z_1) = (4, 8)$.

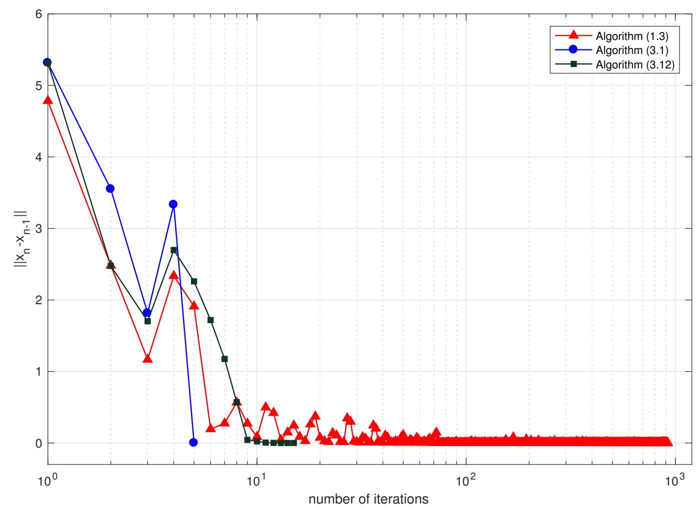

FIGURE 2. The results computed by Algorithm (1.3), Algorithm (3.1) and Algorithm (3.12). Case: $x_0 = x_1(z_1) = (7, -7)$.

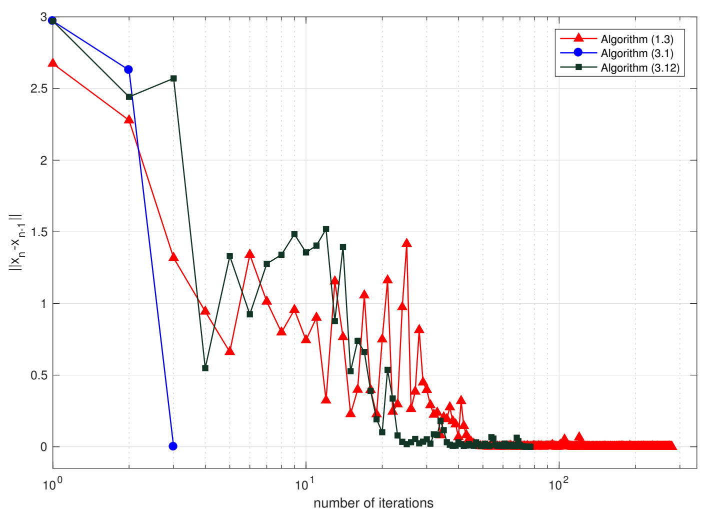

FIGURE 3. The results computed by Algorithm (1.3), Algorithm (3.1) and Algorithm (3.12). Case: $x_0 = x_1(z_1) = (-2, -6)$.

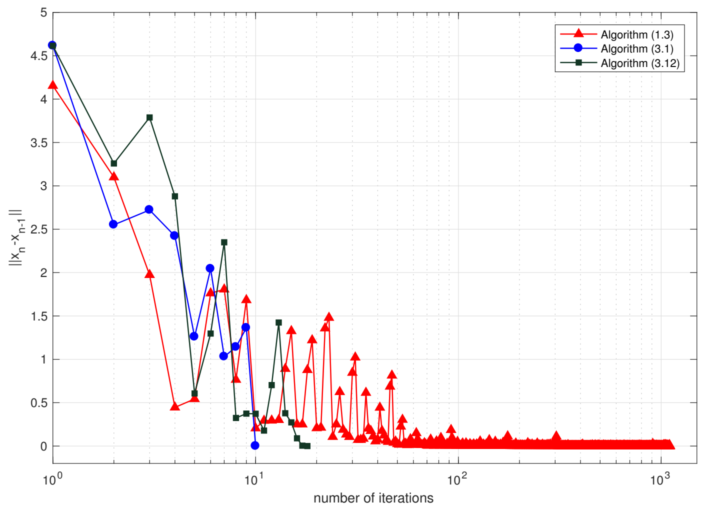

FIGURE 4. The results computed by Algorithm (1.3), Algorithm (3.1) and Algorithm (3.12). Case: $x_0 = x_1(z_1) = (-4, 9)$.

5. Conclusions

We studied strong convergence theorems for finding a fixed point of nonexpansive mappings by using the accelerated hybrid algorithm presented in Algorithm (3.1) and Algorithm (3.12), respectively. In order to present the advantage and performance of the new algorithms, the numerical results of Algorithm (1.3), Algorithm (3.1) and Algorithm (3.12) were compared. From the example, it can be seen that Algorithm (3.1) and Algorithm (3.12) showed better results than Algorithm (1.3) that was presented by Dong and Lu [11]. Moreover, Algorithm (3.1) seemed to show better performance than Algorithm (3.12). All of these can be clearly seen by Table 1, Figure 1, Figure 2, Figure 3 and Figure 4, respectively.

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