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COMMON FIXED POINT FOR RECIPROCALLY CONTINUOUS AND WEAKLY COMPATIBLE MAPS IN A G-METRIC SPACE

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Abstract. A brief comparative survey of some generalizations of a metric space with three dimensional metric structures and different forms of the triangle inequality is done along with their topological properties. Then a common fixed point is obtained for reciprocally continuous and compatible self-maps in a G-metric space. Further, a common fixed point theorem is proved for a pair of weakly compatible self-maps on a G-metric space with the common limit range property.

1. Introduction

In the last few decades, fixed point theorems were developed in a metric space, normed linear space, topological space etc.The conditions on the underlying mappings are usually metrical or compact type conditions. Further,

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new algebraic structures were also formulated to improve the results. A brief comparative study of some generalizations of a metric space with three dimensional metric structures and different forms of the triangle inequality, is done along with their topological properties. Then a common fixed point theorem is proved for reciprocally continuous and compatible self-maps in a G-metric space. Further, a common fixed point theorem is proved for a pair of weakly compatible self-maps on a G-metric space with the common limit range property.

2. Spaces with three dimensional metric structures

Let X be a non-empty set and $\rho : X \times X \to [0, \infty)$ be such that

(m1) $\rho(x, x) = 0$ for all $x \in X$,
(m2) $\rho(x, y) = 0$ implies that $x = y$,
(m3) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$,
(m4) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all x, y, $z \in X$.

Then the pair (X, ρ) denotes a metric space with metric ρ. If $X = \mathbb{R}$. the metric $\rho(x, y) = |x - y|$ for all $x, y \in X$ is called the *usual metric* and it is referred to as the *distance* between the points x and y on the number line $\mathbb{R}^1$. Let $X = \mathbb{R} \times \mathbb{R}$ and $\rho(x, y) = |x - y|$ for all $x, y \in X$. Condition (m4) says that the length of one side in a triangle with vertices x, y and z never exceeds the sum of the lengths of other sides in it. Hence it is usually referred to as the *triangle inequality* of the metric ρ. The notion of metric space was first due to Frechet in 1906. Many generalizations of a metric space were developed in analysis by relaxing and/or weakening at least one of the conditions (m1) through (m4), modifying (m4) in different ways, extending ρ to three or more dimensions, appending an additional structure to X, and so on.

Gahler [12] initiated the idea of extending the domain of ρ to three dimensions, though a 2-metric as follows:

Definition 2.1. Let X be a nonempty set and $d : X \times X \times X \to \mathbb{R}$ such that

(2m1) Given a pair of distinct elements $x, y \in X$, there exists a $z \in X$ such that $d(x, y, z) > 0$,
(2m2) $d(x, y, z) = 0$ whenever at least two of x, y, z are equal in X,
(2m3) $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(z, x, y) = d(y, z, x) = d(z, y, x)$ for all $x, y, z \in X$,
(2m4) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all x, y, z, $w \in X$.

Then the function d is called a *2*-metric on X, and the pair (X, d) denotes a 2-metric space.

Remark 2.2. If $x, y \in X$ is such that $d(x, y, z) = 0$ for all $x \in X$, then $x = y$. Axiom $(2m3)$ means that the value of $d(x, y, z)$ is independent of the order of

x, y and z, and is usually known as the symmetry of d under a permutation on x, y and z.

Remark 2.3. Gahler also observed that a topology $\tau(d)$ can be generated in X by taking the collection of all 2-balls $B_d(x,r) = \{z \in X : d(x,y,z) < r\}$ as a subbasis, which is called a 2-*metric topology.* That is $(X, \tau(d)$ is a 2-metric topological space, whose members are called 2-*open sets*.

Example 2.4. Consider $X = \mathbb{R}^2$ with metric $\rho(x,y) = \|x - y\|$ for all $x, y \in X$. Define

$$d(x,y,z) = \sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}/4 \text{ for all } x, y, z \in X,$$

where $a = \rho(x,y)$, $b = \rho(y,z)$ and $c = \rho(z,x)$. Geometrically d represents the area of a triangle with vertices x, y and z in the plane, and the pair (X, d) is referred to as a 2-metric (*area metric*) space. Since the area of a triangle face of a tetrahedron does not exceed the sum of the areas of the remaining faces, Axiom $(2m4)$ is referred to as the *tetrahedron inequality.*

Remark 2.5. Axiom $(2m1)$ stipulates that each pair of distinct points is on at least one non-degenerate triangle. Also we see that $d(x,y,z)$ need not be positive even x, y and z are all distinct, since area of a degenerate triangle with vertices x, y and z is zero and the triangle looks like a straight line. Further, $d(x,y,z) = 0$ need not imply that at least two of x, y and z are equal, that is the converse of $(2m2)$ is not true. In [21], the authors observed that a 2-metric d is continuous function in any one of the coordinate variables x, y and z, without being continuous in all the three variables, and a 2-convergent sequence may fail to be 2-Cauchy.

The above remarks and subsequent studies on 2-metric spaces disproved Gahler's claim that 2-metric spaces are natural generalizations of metric spaces. In fact, a 2-metric space is not topologically equivalent to an ordinary metric and there was no easy relationship between results obtained in 2-metric spaces and metric spaces.

As another attempt of extending a metric, Dhage [10] proposed a D-metric space as follows:

Definition 2.6. Let X be a nonempty set. A function $D : X \times X \times X \to \mathbb{R}$ is called a D-*metric* on X, if

(d1) $D(x,y,z) = 0$ if and only if $x = y = z$,
(d2) $D(x,y,z) = D(x,z,y) = D(y,x,z) = D(z,x,y) = D(y,z,x) = D(z,y,x)$ for all $x, y, z \in X$,
(d3) $D(x,y,z) \le D(x,y,w) + D(x,w,z) + D(w,y,z)$ for all $x, y, z, w \in X$.

The pair (X, D) denotes a D-metric space.

According to Dhage [10], D-metric convergence defines a Hausdorff topology, and the D-metric is continuous in all the three coordinate variables. Naidu et al. [22] established that D-metric convergence does not always define a topology, even when D-metric convergence defines a topology, it need not be Hausdorff, and, even when D-metric convergence defines a metrizable topology, the D-metric is continuous even in a single variable. In addition, they developed certain methods for constructing a D-metric space from a given metric space, and introduced strong convergence and very strong convergence in a D-metric space. Mutual implications among the three notions of convergence were also studied. In a subsequent paper, Naidu et al. [23] observed that many of Dhage's results related to open balls in D-metric spaces are false or the proofs given by him are not valid. In continuation, the results of Dhage [10], Ahmad et al. [4] and Dhage et al. [11] were modified and generalized in [24] through the convergence of a sequence, with every element of the space as a limit.

Again, with reference to a D-metric space (X, D), Dhage [10] derived the following property:

$$D(x, y, y) \leq D(x, z, z) + D(z, y, y) \text{ for all } x, y, z, w \in X, \tag{2.1}$$

and called D a symmetric D-metric, if $D(x, x, y) = D(x, y, y)$ for all $x, y \in X$. In 2003, Mustafa and Sims [18] showed that (2.1) fails to hold, when

(a) $D(x, y, z) = \rho^*(x, y) + \rho^*(y, z) + \rho^*(z, x)$,
(b) $D(x, y, z) = \max\{\rho^*(x, y), \rho^*(y, z), \rho^*(z, x)\}$,

where ρ^* is a semi-metric on X.

Consider the following statements regarding the convergence of a sequence $\{x_n\}_{n=1}^{\infty}$ in a D-metric space (X, D):

(C1) $x_n \to p$, if $\lim_{n\to\infty} D(x_n, p, p) = 0$,
(C2) $x_n \to p$, if $\lim_{n\to\infty} D(x_n, x_n, p) = 0$,
(C3) $x_n \to p$, if $\lim_{m,n\to\infty} D(x_m, x_n, p) = 0$.

It was demonstrated in [18] that (C1) and (C2) does not imply (C3); (C2) need not imply (C1) or (C3); and (C1) need not imply (C2) or (C3). Also, D need not be a continuous function of its variables with respect to convergence of type (C1) or (C3). Further, topological deviations in Dhages assumptions on balls, topologies and Cauchy sequence and its relation with convergence of above three types in a D-metric space, were proved erroneous, which led them to define a G-metric space in 2006 as follows:

Definition 2.7. ([19]) Let X be a nonempty set and $G : X \times X \times X \to [0, \infty)$ such that

(g1) $G(x, y, z) = 0$ whenever $x, y, z \in X$ are such that $x = y = z$,
(g2) $G(x, x, y) > 0$ whenever $x \neq y$,
(g3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,
(g4) $G(x, y, z) = G(\pi(x, y, z))$ for all $x, y, z \in X$,
where $\pi(x, y, z)$ is a permutation on the set $\{x, y, z\}$,
(g5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$ for all $x, y, z, w \in X$.

Then G is called a G-metric on X and the pair (X, G) denotes a G-metric space. Axiom (g5) is known as the *rectangle inequality* of G. A G-metric space (X, G) is said to be *symmetric*, if $G(x, y, y) = G(x, x, y)$ for all $x, y \in X$ (see [9, 30, 42]).

Example 2.8. Let (X, ρ) be a metric space and $a = \rho(x, y)$, $b = \rho(y, z)$ and $c = \rho(z, x)$ be the sides of a triangle Δxyz with vertices x, y and z in the plane. The perimeter of Δxyz is a G-metric on X, and (X, G) is a symmetric G-metric space.

Example 2.9. ([19]) Let (X, ρ) be a metric space and
(a) $G_a(x, y, z) = [\rho(x, y) + \rho(y, z) + \rho(z, x)]/3$,
(b) $G_m(x, y, z) = \max\{\rho(x, y), \rho(y, z), \rho(z, x)\}$.

Then (X, G_a) and (X, G_m) are symmetric G-metric spaces. Conversely, if (X, G) is a G-metric space, then $\rho_G(x, y) = G(x, y, y) + G(x, x, y)$ for all $x, y \in X$ is a metric on X, associated with the G-metric G. While, Example 1 of [19] gives a nonsymmetric G-metric space, which does not arise from any metric space.

We use the following two properties in our proofs:

Property 2.10. *In any G-metric space (X, G), we have*

$$G(x, y, y) \leq 2G(x, x, y) \quad \text{for all } x, y \in X. \tag{2.2}$$

Proof. Writing $w = y$ and $z = x$ in (g5), we obtain that

$$G(x, y, x) \leq G(x, y, y) + G(y, y, x),$$

which in view of (g4), gives (2.2). □

Property 2.11. *In a G-metric space (X, G), if $x, y \in X$ are such that $G(x, y, y) = 0$, then $x = y$.*

Proof. If $x \neq y$, then from (g2), it follows that $G(x, x, y) > 0$ and hence (2.2) would give a contradiction that $G(x, y, y) > 0$. Therefore, $x = y$. □

Definition 2.12. ([19]) Let (X, G) be a G-metric space. A G-ball in X is defined by $B_G(x, r) = \{y \in X : G(x, y, y) < r\}$. The family $\mathscr{B}$ of all G-balls forms a base topology, called the *G-metric topology* $\tau(G)$ on X.

It is easy to see that $B_G(x, r/3) \subset B_{\rho_G}(x, r) \subset B_G(x, r)$. As a consequence of this, the G-metric topology $\tau(G)$ coincides with the metric topology generated by ρ_G. Thus, every G-metric, being isometrically distinct, is topologically equivalent to a metric space. This allows us to readily transform many concepts from metric space into the setting of G-metric space.

Definition 2.13. Let (X, G) be a G-metric space. Then a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said to be G-*convergent* with limit $p \in X$, if it converges to p in the G-metric topology $\tau(G)$.

Since G-metric topologies are also metric topologies, it follows that a G-metric function is jointly continuous in all three variables, hence separately continuous in any one, or jointly in any pair of its variables.

Lemma 2.14. *Let (X, G) be a G-metric space, then the G-metric function is jointly continuous in all three variables, hence separately continuous in any one, or jointly in any pair of its variables. Hence we have the following.*

Definition 2.15. Let (X, G) be a G-metric space. A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said to be G-Cauchy, if $\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0$, and X is said to be G-*complete*, if every G-Cauchy sequence in X converges in it.

Remark 2.16. Jleli and Samet [13] asserted that the conclusions of some fixed point theorems in a G-metric space can be obtained by some existing results in the setting of a metric (or quasi-metric) space. In fact, if $\delta(x, y) = G(x, y, y)$ for all $x, y \in X$, then $\delta(x, y) \geq 0$, $\delta(x, y) = 0$ if and only if $x = y$ and $\delta(x, y) \leq \delta(x, z) + \delta(z, y)$ for all $x, y, z \in X$. In other words, δ is a quasi-metric. Also, if $\rho(x, y) = \max\{\delta(x, y), \delta(y, x)\}$ for all $x, y \in X$, then ρ is a metric on X. Thus if any two of the three variables in are the same, many G-contraction type conditions imply those in metric (or quasi-metric) space.

With the same convention, Rhoades [32] proved that several contraction conditions in G-metric spaces are special cases of the following Ciric's quasi-contraction on a metric space:

$$\rho(fx, fy) \leq \lambda \max\big\{\rho(x, y), \rho(x, fx), \rho(y, fy), \rho(x, fy), \rho(y, fx)\}, \qquad (2.3)$$

for all $x, y \in X$, where $0 < \lambda < 1$.

Remark 2.17. Write $\sigma_G(x, y) = \max\{(G(x, fx, f^2y), G(y, fy, f^2x)\}$ for all $x, y \in X$. With $y = x$, this gives $\sigma_G(x, x) = G(x, fx, f^2x)$, which will be positive for each $x \in X$ with $fx \neq x$. Thus $x = y$ does not imply that $\sigma_G(x, y) = 0$, showing that σ_G is not a metric on X.

In view of the geometry of three points instead of two points via perimeter of a triangle, from Remark 2.17, Karapinar and Ravi Paul [17] and Agarwal et al. [3] proved three generalized fixed point theorems, in which Jleli and Samet's assertion is not applicable.

As another probable modification of Dhage's D-metric space, Shaban Sedghi et al. [39] proposed a D^*- metric space as follows:

Definition 2.18. Let X be a nonempty set. A function $D : X \times X \times X \to [0, \infty)$ is called a D^*-*metric* on X, if

(d*1) $D^*(x, y, z) = 0$ if and only if $x = y = z$,
(d*2) $D^*(x, y, z) = D^*(x, z, y) = D^*(y, x, z) = D^*(z, x, y) = D^*(y, z, x) = D^*(z, y, x)$ for all $x, y, z \in X$,
(d*3) $D^*(x, y, z) \le D^*(x, y, w) + D^*(w, z, z)$ for all $x, y, z, w \in X$.

The pair (X, D^*) denotes a D^*-metric space.

Example 2.19. ([19]) Let (X, ρ) be a metric space and

(a) $D^*(x, y, z) = \rho(x, y) + \rho(y, z) + \rho(z, x)$,
(b) $D^*(x, y, z) = \max\{\rho(x, y), \rho(y, z), \rho(z, x)\}$.

Then (X, D^*) is a D^*-metric space.

Example 2.20. ([19]) Let $X = \mathbb{R}^p$. Then

$$D^*(x, y, z) = \sqrt[p]{|x - y|^p + |y - z|^p + |z - x|^p}$$

is a D^*-metric on X, where p is a positive real number.

Let (X, D^*) be a D^*-metric space. The family $\mathscr{D}^*$ of all D^*-balls of the form $B_{D^*}(x, r) = \big\{y \in X : G(x, y, y) < r\big\}$ gives a D^*-metric topology $\tau(D^*)$ on X. A sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to be D^*-*convergent* with limit $\xi \in X$, if it converges to ξ in the D^*-metric topology $\tau(D^*)$. Further, D^* is a continuous function on X^3 [39]. Replacing $[0, \infty)$ with a real Banach space in a D^*-metric space, Aage and Salunke [1] introduced a generalized D^*-metric space and proved some fixed point theorems in complete generalized D^*-metric spaces. This notion is an analogue of cone metric space.

Let X be a nonempty set. Sedghi et al. [38] introduced an S-metric space (X, S) as another generalization of D^*-metric space, where an S-metric $S : X \times X \times X \to [0, \infty)$ satisfies the following conditions:

(S1) $S(x, y, z) = 0$ if and only if $x = y = z$,
(S2) $S(x, y, z) \le S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.

From Axiom (S2) it follows that $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$ (see [16, 31]).

Definition 2.21. A sequence $\{x_n\}_{n=1}^{\infty}$ in a S-metric space (X, S) is said to be convergent, if there exists a point x in X such that $S(x_n, x_n, x) \to 0$ as $n \to \infty$.

Definition 2.22. A sequence$\{x_n\}_{n=1}^{\infty}$ in a S-metric space (X, S) is said to be S-Cauchy, if $\lim_{n,m\to\infty} S(x_n, x_n, x_m) = 0$. The space X is said to be S-complete, if every S-Cauchy sequence in X converges in it.

Remark 2.23. Though all the three generalized metrics D^*, G and S are nonnegative real-valued functions on $X \times X \times X$, it can be shown from their elementary properties that D^* and G are independent notions. Also, G and S are independent. However, every D^* is an S, but the converse is not true (cf. [38]).

In [19] the authors also proved that the product of G-metric spaces is a G-metric space, only when components are symmetric. However by omitting (g3), Roldn and Karapinar [33] introduced a G^*-metric spaces, and showed that the product of G^*-metric spaces is also a G^*-metric space. While, a *quasi G-metric* satisfies the conditions (g1), (g2) and (g4), while (g2), (g3) and (g4) hold good in a G-metric-like space (cf. Alghamdi et al., [6]). Analogous to modular metric spaces, Modular G-metric spaces were introduced and some related fixed point theorems of contractive mappings were proved by Azadifar et al [8]. Further, quasi S-metric space, GP-metric space and S_b-metric space were introduced in [36], [5] and [37] respectively.

3. Reciprocally Continuous Compatible Maps in a G-metric Space

Self-maps f and r on a metric space (X, ρ) are commuting, if $frx = rfx$ for all $x \in X$. As a weaker form of it, Sessa [40] introduced weakly commuting maps f and r on X with the choice $\rho(frx, rfx) \leq \rho(fx, rx)$ for all $x \in X$. Weakly commuting maps were generalized as R-weakly commuting maps by Pant [25], which satisfy the condition:

$$\rho(frx, rfx) \leq R\rho(fx, rx) \text{ for all } x \in X \text{ for some } R > 0. \tag{3.1}$$

Writing $R = 1$ in (3.1), we get weakly commuting pair $\{f, r\}$. Splitting the condition (3.1), Pathak et al. [27] defined R-weakly commuting of types (A_g) and (A_f). In fact, self-maps f and r on X are said to be R-weakly commuting of type (A_g), if

$$\rho(frx, rrx) \leq R\rho(fx, rx) \text{ for all } x \in X \text{ for some } R > 0. \tag{3.2}$$

Interchanging the roles of f and r in (3.2), we get R-weakly commuting of type (A_f). In a comparative study of various weaker forms of commuting maps, Singh and Tomar [41] remarked that R-weak commutativity is independent of these two types. Gerald Jungck [14] introduced compatible maps as a generalization of weakly commuting maps in the following way:

Definition 3.1. Self-maps f and r on a metric space (X, ρ) are said to be compatible, if

$$\lim_{n\to\infty} \rho(frx_n, rfx_n) = 0 \tag{3.3}$$

whenever there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} rx_n = z \quad \text{for some} \quad z \in X. \tag{3.4}$$

Splitting the condition (3.3) in different ways, Pathak and Khan [28] introduced different types of compatible maps $\{f, r\}$, which are equivalent to their compatibility whenever f and r are continuous.

It is observed from [29] that a pair (f, g) of self-maps can be weakly commuting, but there may not be any sequence $\{x_n\}_{n=1}^{\infty}$ with the choice (3.4). Such maps are *vacuously* compatible. Therefore, self-maps f and g are noncompatible, if there is a sequence $\{x_n\}_{n=1}^{\infty}$ with (3.4) but $\lim_{n\to\infty} \rho(fgx_n, gfx_n) \neq 0$ or $+\infty$.

In the study of common fixed points for discontinuous maps in a metric space, the notion of reciprocal continuity was introduced as follows:

Definition 3.2. (Pant et al., [26]) Self-maps f and r on X are reciprocally continuous at $z \in X$, if for any sequence $\{x_n\}_{n=1}^{\infty} \subset X$ with the choice (3.4), we have

$$\lim_{n\to\infty} frx_n = fz \text{ and } \lim_{n\to\infty} rfx_n = rz. \tag{3.5}$$

Self-maps f and r are reciprocally continuous on X if and only if they are reciprocally continuous for all $z \in X$.

In the setting of G-metric space (X, G), Yang [43] define the following.

Definition 3.3. Self-maps f and r on $X$are said to be compatible [43], if

$$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0 \tag{3.6}$$

and

$$\lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) = 0, \tag{3.7}$$

whenever there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ with the choice (3.4).

However, in view of Property 2.2, it was shown in Proposition 4 in [35] that the conditions (3.6) and (3.7) are equivalent.

Example 3.4. (Abbas et al., [2]) Let $X = [0, 2]$ with

$$G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\} \text{ for all } x, y, z \in X. \tag{3.8}$$

Define $f, g : X \to X$ by

$$fx = \begin{cases} 2 - x, & 0 \le x \le 1, \\ x, & 1 < x \le 2, \end{cases}$$

$$gx = 1,\ x \ge 0.$$

Then (f, g) is a compatible pair.

We define reciprocal continuity on a G-metric space (X, G) as follows:

Definition 3.5. Self-maps f and r on X are reciprocally continuous at $z \in X$, if for any sequence $\{x_n\}_{n=1}^{\infty} \subset X$ with the choice (3.4), we have

$$\lim_{n\to\infty} G(frx_n, fz, fz) = 0 \text{ and } \lim_{n\to\infty} G(rfx_n, rz, rz) = 0. \tag{3.9}$$

Self-maps f and r are reciprocally continuous on X if they are reciprocally continuous for all $z \in X$.

Example 3.6. Let $X = [0, 1]$ with the G-metric given by (3.8). Define $f, g : X \to X$ by

$$fx = 0 \text{ for } x > 0,\ f(0) = 0,\ gx = x \text{ for all } x.$$

Then (f, g) is reciprocally continuous and compatible pair without a common fixed point.

Example 3.7. Let $X = [0, 1]$ with the G-metric given by (3.8). Define $f, g : X \to X$ by

$$fx = x \text{ for } x < 1,\ f(1) = 0,\ gx = x \text{ for all } x.$$

Then (f, g) is reciprocally continuous and compatible pair with $x = 0$ as a common fixed point.

Example 3.8. Let $X = [2, 20]$ with the G-metric given by (3.8). Define $f, g : X \to X$ by

$$fx = \begin{cases} 2, & x = 2 \text{ or } x > 5, \\ 6, & 2 < x \le 5, \end{cases}$$

$$gx = \begin{cases} 2, & x = 2, \\ 12, & 2 < x \le 5, \\ \frac{x+1}{3}, & x > 5. \end{cases}$$

Then (f, g) reciprocally continuous.

Now, the following is our result for reciprocally continuous self-maps on a G-metric space:

Theorem 3.9. *Let f and g be self-maps on a complete G-metric space (X, G) such that*

$$f(X) \subset g(X) \tag{3.10}$$

and

$$G^4(fx, fy, fz) \leq \lambda G(gx, gy, gz)G(gy, fy, fz)G(gy, fx, fz)G(gz, fx, fy), \tag{3.11}$$

for all $x, y, z \in X$, where $0 < \lambda < 1/4$. Suppose that $\{f, g\}$ is a reciprocally continuous and compatible pair. Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. By virtue of the inclusion (3.10), there exists a sequence of points x_1, x_2,..., x_n, ... such that

$$y_n = fx_{n-1} = gx_n,\ n = 1, 2, ... \tag{3.12}$$

We first prove that $\{y_n\}_{n=1}^{\infty}$ is a G-Cauchy sequence in X. In fact, writing with $x = x_{n-1}$ and $y = z = x_n$ in (3.11), and using (2.2), we find that

$$\begin{aligned}G^4(fx_{n-1}, fx_n, fx_n) \leq \lambda &G(gx_{n-1}, gx_n, gx_n)G(gx_n, fx_n, fx_n)\\ &\times G(gx_n, fx_{n-1}, fx_n)G(gx_n, fx_{n-1}, fx_n)\end{aligned}$$

or

$$\begin{aligned}&G^4(y_n, y_{n+1}, y_{n+1})\\ &\leq \lambda G(y_{n-1}, y_n, y_n)G(y_n, y_{n+1}, y_{n+1})G(y_n, y_n, y_{n+1})G(y_n, y_n, y_{n+1})\\ &\leq 4\lambda G(y_{n-1}, y_n, y_n)G^3(y_n, y_{n+1}, y_{n+1}).\end{aligned}$$

Simplifying this,

$$G(y_n, y_{n+1}, y_{n+1}) \leq 4\lambda G(y_{n-1}, y_n, y_n) \text{ for all } n.$$

By induction, it follows that

$$G(y_n, y_{n+1}, y_{n+1}) \leq (4\lambda)^n G(y_0, y_1, y_1) \text{ for } \ n \geq 1. \tag{3.13}$$

Now for $m > n$, by the use of $(g5)$ and (3.13), we find that

$$\begin{aligned}
&G(y_n, y_m, yx_m)\\
&\leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2})\\
&\quad + \cdots + G(y_{m-1}, y_m, y_m) \ (m-n \text{ terms})\\
&\leq [\underbrace{(4\lambda)^n + (4\lambda)^{n+1} + (4\lambda)^{n+2} + \cdots + (4\lambda)^{n+(m-n-1)}}_{m-n \text{ terms}}]G(y_0, y_1, y_1)\\
&= (4\lambda)^n[\underbrace{1 + (4\lambda) + (4\lambda)^2 + \cdots + (4\lambda)^{m-n-1}}_{m-n \text{ terms}}]G(y_0, y_1, y_1)\\
&\leq (4\lambda)^n \cdot \tfrac{1-(4\lambda)^{m-n}}{1-4\lambda} \cdot G(y_0, y_1, y_1)\\
&\leq \tfrac{(4\lambda)^n}{1-4\lambda} \cdot G(y_0, y_1, y_1). \qquad (3.14)
\end{aligned}$$

Applying the limit as $n \to \infty$, in (3.14), it follows that $G(y_n, y_m, y_m) \to 0$. This proves that $\{y_n\}_{n=1}^{\infty}$ is a G-Cauchy sequence in X.

Since X is G-complete,

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t, \qquad (3.15)$$

for some $t \in X$. The reciprocal continuity of $\{f, g\}$ and (3.15) imply that

$$\lim_{n\to\infty} fgx_n = ft \text{ and } \lim_{n\to\infty} gfx_n = gt. \qquad (3.16)$$

Suppose that (f, g) is compatible. Then

$$\lim_{n\to\infty} G(fga_n, gfa_n, gfa_n) = 0.$$

In view of Lemma 2.14, (3.16) implies that

$$G(fgt, ggt, ggt) = 0 \text{ so that } fgt = ggt. \qquad (3.17)$$

That is gt is a coincidence point. Now, writing $x = x_n$, $y = z = gt$ in (3.11), and using (3.17), then for all n,

$$\begin{aligned}
&G^4(fx_n, fgt, fgt\\
&\leq \lambda G(gx_n, ggt, ggt)G(ggt, fgt, fgt)G(ggt, fx_n, fz)G(ggt, fx_n, fgt). \qquad (3.18)
\end{aligned}$$

As $n \to \infty$, this gives $G(gt, fgt, fgt) = 0$ or $fgt = gt = ggt$. That is, gt is a fixed point of f and hence is a common fixed point of f and g. □

Writing $g = i_X$, the identity map on X in Theorem 3.9, we have:

Corollary 3.10. *Let f be a self-map on a complete G-metric space (X, G) satisfying the inequality*

$$G^4(fx, fy, fz) \leq \lambda G(x, y, z)G(y, fy, fz)G(y, fx, fz)G(z, fx, fy) \qquad (3.19)$$

for all $x, y, z \in X$. Then f has a unique fixed point.

It is well known that reciprocal continuity is an efficient tool to study compatible mappings. However, it is less suitable to deal with non-compatible mappings. For, if f and g are non-compatible with a common fixed point, say p, in Theorem 3.9, then there exists a sequence $\{a_n\}_{n=1}^{\infty}$ in X such that $fa_n \to t$, $gx_n \to t$ as $n \to \infty$ for some $t \in X$, but $\lim_{n\to\infty} G(fga_n, gfa_n)$ is either non-zero or ∞. Now, the reciprocal continuity of f and g implies that $fga_n \to ft$ and $gfa_n \to gt$ as $n \to \infty$. Therefore,

$$ft \neq gt. \tag{3.20}$$

While, with $x = p$ and $y = z = a_n$, (3.11) becomes

$$\begin{aligned} &G^4(fp, fa_n, fa_n) \\ &\leq \lambda G(gp, ga_n, ga_n)G(ga_n, fa_n, fa_n)G(ga_n, fp, fa_n)G(ga_n, fp, fa_n). \end{aligned}$$

On letting $n \to \infty$, this gives $G^4(p, t, t) = 0$ or $z = t$. That is $ft = fp = gp = gt$. This contradicts (3.20). Thus reciprocal continuity is not suitable in fixed point considerations of non-compatible maps.

Example 3.11. Let $X = [2, 20]$ with the G-metric given by (3.8). Define $f, g : X \to X$ by

$$fx = \begin{cases} 2, & x = 2 \text{ or } x > 5, \\ 6, & 2 << x \leq 5, \end{cases}$$

$$gx = \begin{cases} 2, & x = 2, \\ 11, & 2 < x \leq 5, \\ \frac{x+1}{3}, & x > 5. \end{cases}$$

Then (f, g) is neither compatible nor reciprocally continuous.

4. Common Fixed Point for Weakly Compatible Maps with CLRg-property

It may be noted that non-vacuously compatible, compatible maps of all types and non-compatible maps are included in the wider class of self-maps $\{f, g\}$ satisfying the property (EA), in which (3.4) holds good for some sequence $\{x_n\}_{n=1}^{\infty} \subset X$.

Definition 4.1. A point $x \in X$ is called a coincidence point for self-maps f and r if $fx = rx = y$ and y a point of coincidence of f and r. Self-maps which commute at their coincidence points are called coincidentally commuting [10]. They are also called weakly compatible maps [15].

Note that the compatibility and all its types, and R-weak commutativity and its types imply the weak compatibility [41]. Since self-maps fail to be weakly compatible only if they have a coincidence point at which they do not commute, weak compatibility is the minimal condition for the maps to have a common fixed point. The following is an easy consequence for weakly compatible maps:

Lemma 4.2. *If self-maps f and r are weakly compatible, then their point of coincidence with respect to a coincidence point is a coincidence point for them.*

From the above discussion, it follows that weak compatibility and property (EA) are weaker than the compatibility and all its types. However, both these notations are independent of each other [27].

In the framework of G-metric spaces, we have:

Definition 4.3. (Mustafa et al. [20]) Self-maps f and g on a G-metric space (X, G) satisfy the property (EA), if (3.4) holds good for some $\{x_n\}_{n=1}^{\infty} \subset X$.

Example 4.4. (Example 1.8, [2]) Let $X = [0, 2]$ with the G-metric given by (3.8). Define $f, g : X \to X$ by

$$fx = \begin{cases} 1, & 0 \le x \le 1, \\ \frac{2-x}{3}, & 1 < x \le 2, \end{cases}$$

$$gx = \begin{cases} \frac{3-x}{2}, & 0 \le x \le 1, \\ \frac{x}{2}, & 1 < x \le 2. \end{cases}$$

Then (f, g) is non-compatible, but commute at the coincidence point $x = 1$, hence is a weakly compatible pair. Consider a decreasing sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $x_n \to 1$ as $n \to \infty$. Then fx_n, $gx_n \to 1/2$ as $n \to \infty$. Thus (f, g) satisfies the property (EA).

Obviously, compatible and non-compatible maps satisfy the property (EA). The following is a generalization of the property (EA):

Definition 4.5. (Aydi et al., [7]) Self-maps f and g on a G-metric space (X, G) satisfy the *common limit in the range of g* property (briefly, (CLRg)-property), if there exist a sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ and a point $u \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu. \tag{4.1}$$

Given below is a lemma, proved in [35], which reveals the advantage of the (CLRg)-property over the completeness of the space and the property (EA):

Lemma 4.6. *Suppose that one of the following conditions holds good:*

(a) (f, g) *is not compatible,*

(b) (f, g) *satisfies the property (EA) and* $g(X)$ *is closed,*
(c) (f, g) *satisfies the property (EA) and* $f(X) \subset g(X)$.

Then (f, g) *satisfies the (CLRg)-property.*

We have the interesting result.

Theorem 4.7. *Let* f *and* g *be self-maps on a* G*-metric space* (X, G) *satisfying the inequality* (3.11) *where* $\lambda < 1/2$*, and the (CLRg)-property. Then they have a unique coincidence point. Further, if* (f, g) *is a weakly compatible pair, then* f *and* g *have a unique fixed point.*

Proof. Let (f, g) satisfy the (CLRg)-property with the choice (4.1). Writing $x = u$ and $y = z = y_n$ in (3.11), we see that

$$\begin{aligned} &G^4(fu, fx_n, fx_n) \\ &\leq \lambda G(gu, gx_n, gx_n)G(gx_n, fx_n, fx_n)G(gx_n, fu, fx_n)G(gx_n, fu, fx_n). \end{aligned}$$

Employing the limit as $n \to \infty$ in this, and using (4.1) and Lemma 2.14, we see that

$$G^4(fu, gu, gu) \leq \lambda G(gu, gu, gu)G(gu, gu, gu)G(gu, fu, gu)G(gu, fu, gu), \tag{4.2}$$

so that $G^4(fu, gu, gu) = 0$, which in view of Property 2.11 implies that $fu = gu$. Thus u is a coincidence point of f and g. Let v be another coincidence point of f and g, that is $fv = gv$. Now writing $x = u$ and $y = z = v$ in (3.11),

$$G^4(fu, fv, fv) \leq \lambda G(gu, gv, gv)G(gv, fv, fv)G(gv, fu, fv)G(gv, fu, fv), \tag{4.3}$$

so that

$$G^4(fu, fv, fv) \leq \lambda G(fu, fv, fv)G(fv, fv, fv)G(fv, fu, fv)G(fv, fu, fv)$$

or $fu = fv$, in view of Property 2.11. Thus the coincidence point of f and g is unique. Since f and g are weakly compatible, $fu = gu = w$ implies that $fgu = gfu$ or $fw = gw$. In other words, w is also a coincidence point of f and g. Finally, the uniqueness of the coincidence point implies that $w = fw = gw$. The uniqueness of the common fixed point follows easily from (3.11). □

In view of Lemma 4.6, the following is an immediate consequence of Theorem 4.7:

Corollary 4.8. *Let* f *and* g *be self-maps on a* G*-metric space* (X, G) *satisfying the inequality* (3.11) *and the inclusion* (3.10). *If the pair* (f, g) *satisfies the property (EA), and is weakly compatible, then* f *and* g *have a unique fixed point.*

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