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INNOVATION FIXED POINT THEOREMS IN 0-σ-COMPLETE METRIC-LIKE SPACES WITH APPLICATION IN INTEGRAL EQUATIONS

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Abstract. In this paper, we introduce the notion of rational g-h-ϕ-weak contractions in tripled metric-like spaces and demonstrate common fixed point results for each mappings in 0-σ complete tripled metric-like spaces and some examples and application are given.

1. Introduction

Matthews introduced the notion of partial metric space as a part of the study of denotational semantics of dataflow network [7]. In such spaces, the usual metric is replaced by a partial metric with the property that the self distance for a point of space may not be zero. Further, Matthews showed that the Banach contraction principle is valid in partial metric spaces and can be applied in program verification. Later, several authors generalized the Matthews's result. Romaguera introduced the notion of 0-Cauchy sequence and 0-complete partial metric space, and proved some characterizations of partial metric spaces in terms of completeness and 0-completeness [9].

Recently, Amini-Harandi has introduced the notion of metric-like space, which is a new generalization of partial metric space [5]. Amini-Harandi defined σ-completeness of metric-like spaces. Further, Shukla et al. have introduced in the notion of 0-σ-complete metric-like space and proved some fixed point theorems in such spaces, as improvements of Amini-Harandi's results [10]. Alber and Guerre-Delabriere in [4] suggested a generalization of the Banach contraction mapping principle by introducing the concept of a weak contraction in Hilbert spaces. Rhoades extended their result to complete metric spaces [8]. Very recently, Abbas and Dorić [1], as

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well as Abbas and Ali Khan [2] have obtained common fixed points for four and two mappings, respectively, which satisfy generalized weak contractive conditions. The purpose of this paper is to present some fixed point theorems involving weakly contractive mappings in the context of metric-like spaces. The presented theorems improve the results of papers [5] and [10]. We introduce the notion of rational g-h-ω-weak contractions in metric-like spaces and prove some fixed point results for such mappings in tripled 0-σ-complete tripled metric-like spaces. Examples are given to support the usability of our results and to show that the mentioned improvements are proper.

2. Preliminaries

A self-map f of a metric space X is weakly contractive or ψ-weakly contractive, if for all $x, y \in X$,

$$d(fx, fy) \leq d(x, y) - \psi(d(x, y)),$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function with $\psi(0) = 0$, $\psi(t) > 0$ for all $t \in (0, \infty)$ and $\lim_{t\to\infty} \psi(t) = \infty$. Let f and g he self-maps on a set X. If $w = fx = g(x)$ for some $x \in X$, then w is called corresponding coincidence point of f and g, and x is called a point of coincidence of f and g. The pair $\{f, g\}$ of self-maps in weakly compatible if they commute at their coincidence points.

Definition 2.1. A metric-like on a nonempty set X is a mapping $\sigma : X \times X \times X \to [0, \infty)$ such that for all $x, y, z \in X$

(σ_1) $\sigma(x, y) = 0$ implies $x = y$;
(σ_2) $\sigma(x, y) = \sigma(y, x)$;
(σ_3) $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$.

The pair (X, σ) is called a *metric-like space*.

3. Main Results

Definition 3.1. Let f, g and h be self-maps on a set X. If $\nu = fx = gx = hx$ for some $x \in X$, then x is called a *coincidence point* of f, g and h. The triple $\{f, g, h\}$ od self maps is weakly compatible, if $\{f, g\}$ and $\{f, h\}$ and $\{g, h\}$ commute at their coincidence points.

Definition 3.2. A tripled metric-like on a nonempty set X is a mapping $\sigma : X \times X \times X \to [0, \infty)$ such that for all $x, y, z \in X$

($T\sigma_1$) $\sigma(x, y, z) = 0$ implies $x = y = z$;

($T\sigma_2$) $\sigma(x, y, z) = \sigma(x, z, y) = \sigma(z, y, x) = \sigma(y, x, z) = \sigma(z, x, y) = \sigma(y, z, x)$ for all $x, y, z \in X$;

($T\sigma_3$) $\sigma(x, x, y) = \sigma(x, y, y)$ for all $x, y \in X$

($T\sigma_4$) $\sigma(x, y, z) \leq \sigma(x, a, a) + \sigma(y, a, a) + \sigma(z, a, a)$ for all $x, y, z, a \in X$.

Example 3.3. Let $X = [0, \infty)$, define $\sigma : X \times X \times X \to [0, \infty)$ as follows $\sigma(x, y, z) = \max\{x, y, z\}$. (X, σ) is a tripled metric-like space.

Definition 3.4. Let (X, σ) be a tripled metric-like space.

(i) A sequence $\{x_n\}$ is said σ-*converge* to a point $x \in X$, if
$$\lim_{n\to\infty} \sigma(x_n, x, x) = \lim_{n\to\infty} \sigma(x_n, x_n, x) = \sigma(x, x, x);$$

(ii) A sequence $\{x_n\}$ in X is called 0-σ-*sequence*, if there exists a point $x \in X$ such that
$$\lim_{n\to\infty} \sigma(x_n, x, x) = \lim_{n\to\infty} \sigma(x_n, x_n, x) = \sigma(x, x, x) = 0;$$

(iii) A subset $A \subset X$ is called σ-*closed* if every convergent sequence in A has all of its limits in A. The subset A is called 0-*closed* if every 0-σ-converge sequence in A has a limit in A.

Remark 3.5. Every σ-closed subset of X is necessarily 0-closed but the converse is not necessarily true. For instance, let $X = \mathbb{R}^+$, $A = [0, 2) \subset X$ and the tripled metric-like on X be define by $\sigma(x, y, z) = \max\{x, y, z\}$ for all x, y, z in X. Then A is not a σ-closed subset of X, for any sequence $\{x_n\} \subset A$, we have $\lim_{n\to\infty} \sigma(x_n, 2, 2) = \sigma(2, 2, 2)$, i.e, $x_n \to 2 \notin A$ as $n \to \infty$, but A is 0-closed.

Definition 3.6. Let (X, σ) be a tripled metric-like space.

(i) A sequence $\{x_n\}$ is said to 0-σ-*Cauchy sequence*, if $\lim_{n,m\to\infty} \sigma(x_n, x_m, x_m) = 0$;

(ii) The space (X, σ) is said to be 0-σ-*complete*, if every 0-σ-Cauchy sequence in X, σ-converges to a point $x \in X$ such that $\sigma(x, x, x) = 0$.

Remark 3.7. Let (X, σ) be a tripled metric-like space, and let $\{x_n\}$ be a sequence in X such that
$$\lim_{n\to\infty} \sigma(x_{n+1}, x_n, x_n) = \lim_{n\to\infty} \sigma(x_n, x_{n+1}, x_{n+1}) = 0.$$

If $\{x_n\}$ is not a 0-σ-Cauchy sequence in (X,σ), then there exist $\varepsilon > 0$ and two sequence $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$, and the following four sequence

$$\{\sigma(x_{m_k-1}, x_{n_k-1}, x_{n_k-1})\}, \{\sigma(x_{m_k+1}, x_{n_k+1}, x_{n_k+1})\}, \{\sigma(x_{m_k-2}, x_{n_k-2}, x_{n_k-2})\}$$

and $\{\sigma(x_{m_k}, x_{n_k}, x_{n_k})\}$, tend to ε when $k \to \infty$.

In particular discussion, we denote by Ω, the class of all functions $\phi : [0,\infty) \to [0,\infty)$ such that ϕ is lower semi-continuous with $\phi(t) = 0$ if and only if $t = 0$.

Definition 3.8. Let (X,σ) be a tripled metric-like space and $f, g, h : X \to X$ be three self-mappings. The mapping f is called a *rational g-h-ϕ-weak contraction* if there exists $\phi \in \Omega$ such that condition

$$\sigma(fx, fy, fz) \leq R_{f,g,h}(x,y,z) - \phi(R_{f,g,h}(x,y,z)) \tag{3.1}$$

is satisfied for all $x, y, z \in X$, where

$$\begin{aligned} R_{f,g,h}(x,y,z) = \max\Big\{&\sigma(gx,gy,gz), \sigma(hx,hy,hz), \sigma(fx,gx,hx), \\ &\sigma(fy,gy,hy), \sigma(fz,gz,hz), \\ &\frac{\sigma(fx,gx,hx)\sigma(fy,gy,hy)\sigma(fz,gz,hz)}{1+\sigma(gx,gy,gz)\sigma(hx,hy,hz)}\Big\}. \end{aligned} \tag{3.2}$$

Lemma 3.9. *Let (X,σ) be a tripled metric-like space and $f, g, h : X \to X$ be three self-mappings such that f is a rational g-h-ϕ-weak contraction. If f, g, and h have a point of coincidence $\nu \in X$, then $\sigma(\nu,\nu,\nu) = 0$.*

Proof. Let $u \in X$ be the point of coincidence of f, g and h, and ν be the corresponding coincidence point, that is, $fu = gu = hu = \nu$. Then

$$\begin{aligned} R_{f,g,h}(u,u,u) = \max\Big\{&\sigma(gu,gu,gu), \sigma(hu,hu,hu), \sigma(fu,gu,hu), \\ &\sigma(fu,gu,hu), \sigma(fu,gu,hu), \\ &\frac{\sigma(fu,gu,hu)\sigma(fu,gu,hu)\sigma(fu,gu,hu)}{1+\sigma(gu,gu,gu)\sigma(hu,hu,hu)}\Big\} \\ = \max\Big\{&\sigma(\nu,\nu,\nu), \sigma(\nu,\nu,\nu), \sigma(\nu,\nu,\nu), \sigma(\nu,\nu,\nu), \sigma(\nu,\nu,\nu), \\ &\frac{\sigma(\nu,\nu,\nu)\sigma(\nu,\nu,\nu)\sigma(\nu,\nu,\nu)}{1+\sigma(\nu,\nu,\nu)\sigma(\nu,\nu,\nu)}\Big\} = \sigma(\nu,\nu,\nu). \end{aligned}$$

Using (3.1), we obtain

$$\begin{aligned}\sigma(\nu,\nu,\nu) &= \sigma(fu,fu,fu)\\ &\leq R_{f,g,h}(u,u,u-\phi\left(R_{f,g,h}(u,u,u)\right)\\ &= \sigma(\nu,\nu,\nu)-\phi(\sigma(\nu,\nu,\nu)).\end{aligned}$$

The above inequality shows that $\phi(\sigma(\nu,\nu,\nu))=0$, that is, $\sigma(\nu,\nu,\nu)=0$. □

The next theorem gives a sufficient condition for the existence of a unique common fixed point of three mappings on 0-σ-complete tripled metric-like space.

Theorem 3.10. *Let (X,σ) be a 0-σ-complete tripled metric-like space and f, g and $h: X\to X$ be three mappings such that f is a rational g-h-ϕ-weak contraction. If the range of g contains the range of f, $f(X)\subseteq g(X)$, and the range of h contains the range of g, $f(X)\subseteq g(X)$, and $f(X)$ or $g(X)$ or $h(X)$ is a 0-closed subset of X, then f, g, and h have a unique point of coincidence in X. Moreover, if f, g and h are weakly compatible, then f, g and h have a unique common fixed point ν and $\sigma(\nu,\nu,\nu)=0$.*

Proof. Let x_0 be an arbitrary point in X and choose an $x_1\in X$ such that $fx_0 = y_0 = gx_1$ and choose an $x_2\in X$ such that $gx_1 = hx_2$. Choose an $x_3\in X$ such that $fx_1 = y_1 = gx_2 = hx_3$. Continuing this process, having chosen $x_n\in X$, we obtain $x_{n+1}, x_{n+2}\in X$ such that

$$fx_n = y_n = gx_{n+1} = hx_{n+2}.$$

Thus we obtain a Jungle sequence

$$\{y_n\}_{n\in\mathbb{N}} = \{gx_{n+1}\}_{n\in\mathbb{N}} = \{hx_{n+2}\}_{n\in\mathbb{N}}.$$

First, we show that existence of a point of coincidence of f, g and h. If $y_{n-2}=y_{n-1}=y_n$ for some $n\in\mathbb{N}$, then $gx_n = fx_n = hx_n$ is a point of coincidence. Therefore in further calculations, we assume that $y_{n-1}\neq y_n$, $y_n\neq y_{n-2}$ and $y_{n-1}\neq y_{n-2}$ for all $n\geq 1$. We shall show that $\{y_n\}$ is a 0-σ-Cauchy sequence in X. Since

$$\sigma(gx_n,gx_{n+1},gx_{n+2}) = \sigma(y_{n-1},y_n,y_{n+1})>0,$$

for every $n\in\mathbb{N}$, we have

$$\begin{aligned}&R_{f,g,h}(x_n,x_{n+1},x_{n+2})\\ &\quad=\max\Big\{\sigma(gx_n,gx_{n+1},gx_{n+2}),\sigma(hx_n,hx_{n+1},hx_{n+2}),\sigma(fx_n,gx_n,hx_n),\\ &\qquad\sigma(fx_{n+1},gx_{n+1},hx_{n+1}),\sigma(fx_{n+2},gx_{n+2},hx_{n+2}),\end{aligned}$$

$$\left.\frac{\sigma(fx_n, gx_n, hx_n)\sigma(fx_{n+1}, gx_{n+1}, hx_{n+1})\sigma(fx_{n+2}, gx_{n+2}, hx_{n+2})}{1+\sigma(gx_n, gx_{n+1}, gx_{n+2})\sigma(hx_n, hx_{n+1}, hx_{n+2})}\right\}$$

$$= \max\Big\{\sigma(y_{n-1}, y_n, y_{n+1}), \sigma(y_{n-2}, y_{n-1}, y_n), \sigma(y_n, y_{n-1}, y_{n-2}),$$

$$\sigma(y_{n+1}, y_n, y_{n-1}), \sigma(y_{n+2}, y_{n+1}, y_n),$$

$$\left.\frac{\sigma(y_n, y_{n-1}, y_{n-2})\sigma(y_{n+1}, y_n, y_{n-1})\sigma(y_{n+2}, y_{n+1}, y_n)}{1+\sigma(y_{n-1}, y_n, y_{n+1})\sigma(y_{n-2}, y_{n-1}, y_n)}\right\}.$$

Thus

$$R_{f,g,h}(x_n, x_{n+1}, x_{n+2}) = \max\Big\{\sigma(y_{n-1}, y_n, y_{n+1}), \sigma(y_{n-2}, y_{n-1}, y_n), \sigma(y_n, y_{n+1}, y_{n+2})\Big\}.$$

We have

$$\begin{aligned}\sigma(y_{n-1}, y_n, y_{n+1}) &\leq \max\Big\{\sigma(y_{n-1}, y_n, y_{n+1}), \sigma(y_{n-2}, y_{n-1}, y_n), \sigma(y_n, y_{n+1}, y_{n+2})\Big\} \\ &\quad - \phi\left(R_{f,g,h}(x_n, x_{n+1}, x_{n+2})\right) \\ &< \max\Big\{\sigma(y_{n-1}, y_n, y_{n+1}), \sigma(y_{n-2}, y_{n-1}, y_n), \sigma(y_n, y_{n+1}, y_{n+2})\Big\}.\end{aligned}$$

Now, if

$$\max\Big\{\sigma(y_{n-1}, y_n, y_{n+1}), \sigma(y_{n-2}, y_{n-1}, y_n), \sigma(y_n, y_{n+1}, y_{n+2})\Big\} = \sigma(y_{n-1}, y_n, y_{n+1}),$$

then we have $\sigma(y_{n-1}, y_n, y_{n+1}) < \sigma(y_{n-1}, y_n, y_{n+1})$, which is a contradiction. If

$$\max\Big\{\sigma(y_{n-1}, y_n, y_{n+1}), \sigma(y_{n-2}, y_{n-1}, y_n), \sigma(y_n, y_{n+1}, y_{n+2})\Big\} = \sigma(y_{n-2}, y_{n-1}, y_n),$$

then we have

$$\sigma(y_{n-1}, y_n, y_{n+1}) < \sigma(y_{n-2}, y_{n-1}, y_n),$$

that is, the sequence $\{\sigma(y_{n-1}, y_n, y_{n+1})\}$ is a strictly decreasing sequence of positive members, but

$$\sigma(y_{n-1}, y_n, y_{n+1}) > \sigma(y_n, y_{n+1}, y_{n+2}).$$

Let $\lim_{n\to\infty} \sigma(y_{n-1}, y_n, y_{n+1}) = \delta \geq 0$. If $\delta > 0$ then we have

$$\begin{aligned}
&\sigma(y_{n-1}, y_n, y_{n+1})\\
&\quad = \sigma(fx_{n-1}, fx_n, fx_{n+1})\\
&\quad \leq R_{f,g,h}(x_{n-1}, x_n, x_{n+1}) - \phi\left(R_{f,g,h}(x_{n-1}, x_n, x_{n+1})\right)\\
&\quad \leq \max\Big\{\sigma(gx_{n-1}, gx_n, gx_{n+1}), \sigma(hx_{n-1}, hx_n, hx_{n+1}), \sigma(fx_{n-1}, gx_{n-1}, hx_{n-1}),\\
&\qquad \sigma(fx_n, gx_n, hx_n), \sigma(fx_{n+1}, gx_{n+1}, hx_{n+1}),\\
&\qquad \frac{\sigma(fx_{n-1}, gx_{n-1}, hx_{n-1})\sigma(fx_n, gx_n, hx_n)\sigma(fx_{n+1}, gx_{n+1}, hx_{n+1})}{1+\sigma(gx_{n-1}, gx_n, gx_{n+1})\sigma(hx_{n-1}, hx_n, hx_{n+1})}\Big\}\\
&\quad = \max\Big\{\sigma(y_{n-2}, y_{n-1}, y_n), \sigma(y_{n-3}, y_{n-2}, y_{n-1}), \sigma(y_{n-1}, y_{n-2}, y_{n-3})\\
&\qquad \sigma(y_n, y_{n-1}, y_{n-2}), \sigma(y_{n+1}, y_n, y_{n-1}),\\
&\qquad \frac{\sigma(y_{n-1}, y_{n-2}, y_{n-3})\sigma(y_n, y_{n-1}, y_{n-2})\sigma(y_{n+1}, y_n, y_{n-1})}{1+\sigma(y_{n-2}, y_{n-1}, y_n)\sigma(y_{n-3}, y_{n-2}, y_{n-1})}\Big\}.
\end{aligned}$$

Since $\phi \in \Omega$, taking the upper limit as $n \to \infty$, we obtain

$$\begin{aligned}
\delta &\leq \max\left\{\delta, \frac{\delta^3}{1+\delta^2}\right\} - \liminf_{n\to\infty} \phi\Big(\max\Big\{\sigma(y_{n-2}, y_{n-1}, y_n), \sigma(y_{n-3}, y_{n-2}, y_{n-1}),\\
&\quad \sigma(y_{n-1}, y_{n-2}, y_{n-3}), \sigma(y_n, y_{n-1}, y_{n-2}), \sigma(y_{n-1}, y_{n-2}, y_{n-3}), \sigma(y_{n+1}, y_n, y_{n-1}),\\
&\quad \frac{\sigma(y_n, y_{n-1}, y_{n-2})\sigma(y_{n+1}, y_n, y_{n-1})}{1+\sigma(y_{n-2}, y_{n-1}, y_n)\sigma(y_{n-3}, y_{n-2}, y_{n-1})}\Big\}\Big)\\
&\leq \delta - \phi(\delta) < \delta.
\end{aligned}$$

This contradiction shows that $\lim_{n\to\infty} \sigma(y_{n-1}, y_n, y_{n+1}) = 0$. We shall show that the sequence $\{y_n\}$ is a 0-σ-Cauchy sequence. Suppose on the contrary that $\{y_n\}$ is not a 0-σ-Cauchy sequence. Then by Remark 3.7 there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and

$$\begin{aligned}
\lim_{k\to\infty} \sigma(y_{m_k}, y_{n_k}, y_{n_k}) &= \lim_{k\to\infty} \sigma(y_{m_k-1}, y_{n_k-1}, y_{n_k-1})\\
&= \lim_{k\to\infty} \sigma(y_{m_k+1}, y_{n_k+1}, y_{n_k+1})\\
&= \lim_{k\to\infty} \sigma(y_{m_k-2}, y_{n_k-2}, y_{n_k-2}) = \varepsilon.
\end{aligned} \tag{3.3}$$

For any $k \in \mathbb{N}$, By definition

$$\begin{aligned}
&R_{f,g,h}(x_{m_k}, x_{n_k}, x_{n_k})\\
&\qquad = \max\Big\{\sigma(gx_{m_k}, gx_{n_k}, gx_{n_k}), \sigma(hx_{m_k}, hx_{n_k}, hx_{n_k}), \sigma(fx_{m_k}, gx_{m_k}, hx_{m_k}),\\
&\qquad\quad \sigma(fx_{n_k}, gx_{n_k}, hx_{n_k}), \sigma(fx_{n_k}, gx_{n_k}, hx_{n_k}),
\end{aligned}$$

$$\left.\frac{\sigma(fx_{m_k}, gx_{m_k}, hx_{m_k})\sigma(fx_{n_k}, gx_{n_k}, hx_{n_k})\sigma(fx_{n_k}, gx_{n_k}, hx_{n_k})}{1+\sigma(gx_{m_k}, gx_{n_k}, gx_{n_k})\sigma(hx_{m_k}, hx_{n_k}, hx_{n_k})}\right\}$$

$$\begin{aligned}
&= \max\Big\{\sigma(y_{m_k-1}, y_{n_k-1}, y_{n_k-1}), \sigma(y_{n_k-2}, y_{n_k-2}, y_{n_k-2}), \sigma(y_{m_k}, y_{m_k-1}, y_{m_k-2}),\\
&\quad \sigma(y_{n_k}, y_{n_k-1}, y_{n_k-2}), \sigma(y_{n_k}, y_{n_k-1}, y_{n_k-2}),\\
&\quad \frac{\sigma(y_{m_k}, y_{n_k-1}, y_{n_k-2})\sigma(y_{n_k}, y_{n_k-1}, y_{n_k-2})\sigma(y_{n_k}, y_{n_k-1}, y_{n_k-2})}{1+\sigma(y_{m_k-1}, y_{n_k-1}, y_{n_k-1})\sigma(y_{m_k-2}, y_{n_k-2}, y_{n_k-2})}\Big\}.
\end{aligned}$$

Therefore, it follows from (3.1) that

$$\sigma(y_{m_k}, y_{n_k}, y_{n_k}) \leq R_{f,g,h}(x_{m_k}, x_{n_k}, x_{n_k}) - \phi\left(R_{f,g,h}(x_{m_k}, x_{n_k}, x_{n_k})\right).$$

Taking the upper limit as $k \to \infty$ in the above inequality and using (3.3), we obtain

$$\varepsilon \leq \max\{0, \varepsilon\} - \phi\left(\max\{0, \varepsilon\}\right) = \varepsilon - \phi(\varepsilon) < \varepsilon.$$

This contradiction shows that $\{y_n\}$ is a 0-σ-Cauchy sequences. Suppose that $h(X)$ is 0-closed. Since X is 0-σ-complete, there exists $\nu = hu = gu \in X$ such that

$$\lim_{n,m\to\infty} \sigma(y_n, y_m, y_m) = \lim_{n\to\infty} \sigma(y_n, \nu, \nu) = \sigma(\nu, \nu, \nu) = 0. \tag{3.4}$$

We shall show that u is a coincidence point of f, g and h. Suppose that $\sigma(\nu, fu, gu) > 0$, By definition, we have

$$\begin{aligned}
R_{f,g,h}(x_n, u, u) &= \max\Big\{\sigma(gx_n, gu, gu), \sigma(hx_n, hu, hu), \sigma(fx_n, gu, hu),\\
&\quad \sigma(fu, gu, hu), \sigma(fu, gu, hu),\\
&\quad \frac{\sigma(fx_n, gx_n, hx_n), \sigma(fu, gu, hu)\sigma(fu, gu, hu)}{1+\sigma(gx_n, gu, gu)\sigma(hx_n, hu, hu)}\Big\}\\
&= \max\Big\{\sigma(y_{n-1}, \nu, \nu), \sigma(y_{n-2}, \nu, \nu), \sigma(y_n, \nu, \nu), \sigma(fu, \nu, \nu),\\
&\quad \frac{\sigma(y_n, y_{n-1}, y_{n-2})\sigma(fu, \nu, \nu)\sigma(fu, \nu, \nu)}{1+\sigma(y_n, \nu, \nu)\sigma(y_{n-2}, \nu, \nu)}\Big\}.
\end{aligned}$$

In view of (3.4) we have for all $n \geq n_0$, $R_{f,g,h}(x_n, u, u) = \sigma(fu, \nu, \nu)$. Therefore, for all $n \geq n_0$, we have

$$\begin{aligned}
\sigma(\nu, \nu, fu) &\leq \sigma(\nu, y_n, y_n) + \sigma(\nu, y_n, y_n) + \sigma(fu, y_n, y_n)\\
&= 2\sigma(\nu, y_n, y_n) + \sigma(fu, fx_n, fx_n)\\
&= 2\sigma(\nu, y_n, y_n) + \sigma(fu, fu, fx_n)\\
&\leq 2\sigma(\nu, y_n, y_n) + R_{f,g,h}(x_n, u, u) - \phi\left(R_{f,g,h}(x_n, u, u)\right)\\
&= 2\sigma(\nu, y_n, y_n) + \sigma(fu, \nu, \nu) - \phi\left(\sigma(fu, \nu, \nu)\right).
\end{aligned}$$

Letting $n \to \infty$, we get

$$\sigma(\nu, \nu, fu) \leq 0 + \sigma(fu, \nu, \nu) - \phi\left(\sigma(fu, \nu, \nu)\right).$$

This contradiction shows that $\sigma(\nu, \nu, fu) = 0$, that is, $fu = \nu$, we deduce that $hu = gu = fu = \nu$. Thus, u is a coincidence point and ν is the corresponding point of coincidence of f, g and h. We shall show that the point of coincidence is unique. If possible, let ν' is another point of corresponding coincidence of f, g and h and $\sigma(\nu, \nu', \nu') > 0$, $\sigma(\nu', \nu, \nu) > 0$ and $\sigma(\nu', \nu', \nu') = 0$. Then there exists $u' \in X$ such that $fu' = gu' = hu' = \nu'$. By definition, we have

$$\begin{aligned} R_{f,g,h}(u, u', u') = \max\Big\{&\sigma(gu, gu', gu'), \sigma(hu, hu', hu'), \sigma(fu, gu, hu), \\ &\sigma(fu', gu', hu'), \sigma(fu', gu', hu'), \\ &\frac{\sigma(fu, gu, hu), \sigma(fu', gu', hu')\sigma(fu', gu', hu')}{1 + \sigma(gu, gu', gu')\sigma(hu, hu', hu')}\Big\} \\ = \max\Big\{&\sigma(\nu, \nu', \nu'), \sigma(\nu, \nu', \nu'), \sigma(\nu, \nu, \nu), \sigma(\nu', \nu', \nu'), \sigma(\nu', \nu', \nu'), 0\Big\} \\ = \sigma(&\nu, \nu', \nu'). \end{aligned}$$

Therefore, it follows from (3.1) that

$$\begin{aligned} \sigma(\nu, \nu', \nu') &= \sigma(fu, fu', fu') \\ &\leq R_{f,g,h}(u, u', u') - \phi\left(R_{f,g,h}(u, u', u')\right) \\ &= \sigma(\nu, \nu', \nu') - \phi\left(\sigma(\nu, \nu', \nu')\right) \\ &< \sigma(\nu, \nu', \nu'). \end{aligned}$$

This contradiction shows that $\sigma(\nu, \nu', \nu') = 0$, that is, $\nu = \nu'$. Thus ν is the unique point of corresponding coincidence of f, g and h, and the prove of the theorem is finished. □

Let (X, σ) be a tripled metric-like space and $f : X \to X$ be a mapping. The mapping f is called a rational ϕ-weak contraction, if there exists $\phi \in \Omega$ such that the condition

$$\sigma(fx, fy, fz) \leq R_f(x, y, z) - \phi(R_f(x, y, z)) \tag{3.5}$$

is satisfied for all $x, y, z \in X$, where

$$\begin{aligned} R_f(x, y, z) = \max\Big\{&\sigma(x, y, z), \sigma(fx, x, x), \sigma(fy, y, y), \sigma(fz, z, z), \\ &\frac{\sigma(fx, x, x), \sigma(fy, y, y)\sigma(fz, z, z)}{1 + [\sigma(x, y, z)]^2}\Big\}. \end{aligned}$$

Taking $g = h = I_X$ in Theorem 3.10, we obtain the following corollary.

Corollary 3.11. *Let (X, σ) be a 0-σ-complete tripled metric-like space and $f : X \to X$ be a rational ϕ-weak contraction. Then f have a unique fixed point ν and $\sigma(\nu, \nu, \nu) = 0$.*

If we take $\phi(t) = (1-k)t$ for $k \in (0,1)$ in contraction condition (3.1), we have the following corollary.

Corollary 3.12. *Let (X, σ) be a 0-σ-complete tripled metric-like space and $f, g, h : X \to X$ be three mappings such that $\sigma(fx, fy, fz) \leq kR_{f,g,h}(x, y, z)$ holds for all $x, y, z \in X$, where $0 < k < 1$ and $R_{f,g,h}(x, y, z)$ is defined by (3.2). Then f, g and h have a unique point of coincidence in X, and if they are weakly compatible, they have a unique common fixed point.*

Without essential changes, the following version of Theorem 3.10 can be proved.

Theorem 3.13. *Let (X, σ) be a 0-σ-complete tripled metric-like space and $f, g, h : X \to X$ be three mappings such that*

$$\psi\left(\sigma(fx, fy, fz)\right) \leq \psi\left(R_{f,g,h}(x, y, z)\right) - \phi\left(R_{f,g,h}(x, y, z)\right)$$

holds for all $x, y, z \in X$, where $\phi \in \Omega$, $\psi : [0, \infty) \to [0, \infty)$ is continuous, non-decreasing and $\psi^{-1}(\{0\}) = 0$ and $R_{r,g,h}$ is given by (3.2). If $f(X) \subseteq g(X) \subseteq h(X)$ and $h(X)$ is a 0-closed subset of X, then f, g and h have a unique point of coincidence in X. Moreover if f, g and h are weakly compatible, then f, g and h have a unique common fixed point ν and $\sigma(\nu, \nu, \nu) = 0$.

Now, we present an example to support the usability of our result Corollary 3.12.

Example 3.14. Let $X = \{1, 2, 3, 4\}$. Define $\sigma : X \times X \times X \to \mathbb{R}^+$ as follows:

$$\sigma(1,1,1) = 0, \sigma(2,2,2) = 3, \sigma(3,3,3) = 6, \sigma(4,4,4) = 7,$$
$$\sigma(1,2,3) = \sigma(2,1,3) = \sigma(3,2,1) = \sigma(1,3,2) = \sigma(3,1,2) = \sigma(2,3,1) = 9,$$
$$\sigma(1,1,2) = \sigma(1,2,2) = \sigma(2,1,1) = \sigma(1,2,1) = \sigma(2,1,2) = \sigma(2,2,1) = 9,$$
$$\sigma(1,1,3) = \sigma(3,1,1) = \sigma(1,3,1) = \sigma(1,3,3) = \sigma(3,1,3) = \sigma(3,3,1) = 5,$$
$$\sigma(1,1,4) = \sigma(1,4,1) = \sigma(4,1,1) = \sigma(1,4,4) = \sigma(4,4,1) = \sigma(4,1,4) = 6,$$
$$\sigma(2,2,3) = \sigma(3,2,2) = \sigma(2,3,2) = \sigma(2,3,3) = \sigma(3,3,2) = \sigma(3,2,3) = 5,$$

$$\begin{aligned}
&\sigma(2,3,4)=\sigma(2,4,3)=\sigma(3,2,4)=\sigma(4,3,2)=\sigma(4,2,3)=\sigma(3,4,2)=10,\\
&\sigma(1,2,4)=\sigma(1,4,2)=\sigma(4,2,1)=\sigma(2,1,4)=\sigma(4,1,2)=\sigma(2,4,1)=10,\\
&\sigma(1,3,4)=\sigma(1,4,3)=\sigma(3,1,4)=\sigma(4,3,1)=\sigma(4,1,3)=\sigma(3,4,1)=6,\\
&\sigma(3,3,4)=\sigma(3,4,3)=\sigma(4,3,3)=\sigma(3,4,4)=\sigma(4,4,3)=\sigma(4,3,4)=5,\\
&\sigma(2,2,4)=\sigma(2,4,2)=\sigma(4,2,2)=\sigma(2,4,4)=\sigma(4,4,2)=\sigma(4,2,4)=10.
\end{aligned}$$

Let $f,g,h : X \to X$ be defined by

$$f(x)=\begin{cases}1, & x=1,\\ 3, & x=2,\\ 3, & x=3,\\ 1, & x=4,\end{cases}\qquad g(x)=\begin{cases}1, & x=1,\\ 2, & x=2,\\ 3, & x=3,\\ 3, & x=4,\end{cases}\qquad h(x)=\begin{cases}1, & x=1,\\ 2, & x=2,\\ 3, & x=3,\\ 4, & x=4.\end{cases}$$

We next verify that $\{f,g,h\}$ satisfies the inequality (3.1), inequality (3.5) with $\phi(t)=\frac{t}{4}$. Let us consider the following possible cases.

Case I. If $\{x,y,z\}\subset\{2,3,4\}$ and $x=2$, $y=3$ and $z=4$, then we have $\sigma(f2,f3,f4)=\sigma(3,3,1)=5$ and

$$\begin{aligned}R_{f,g,h}(2,3,4)=\max\Big\{&\sigma(2,3,2),\sigma(3,4,3),\sigma(3,2,2),\sigma(4,3,4),\sigma(1,2,3),\\ &\frac{\sigma(3,2,2),\sigma(4,3,4)\sigma(1,2,3)}{1+\sigma(2,3,2)\sigma(3,4,3)}\Big\}=9.\end{aligned}$$

Thus $5\le 9-\phi(9)=9-\frac{9}{4}=6.75$.

Case II. If $\{x,y,z\}\subset\{1,3,4\}$ and $x=1$, $y=3$ and $z=4$, then we have $\sigma(f1,f3,f4)=\sigma(1,3,1)=5$ and

$$\begin{aligned}R_{f,g,h}(1,3,4)=\max\Big\{&\sigma(1,3,2),\sigma(1,4,3),\sigma(1,2,2),\sigma(4,3,4),\sigma(1,2,3),\\ &\frac{\sigma(1,2,2),\sigma(4,3,4)\sigma(1,2,3)}{1+\sigma(1,3,2)\sigma(1,4,3)}\Big\}=9.\end{aligned}$$

Thus $5\le 9-\phi(9)=9-\frac{9}{4}=6.75$.

Case III. If $\{x,y,z\}\subset\{1,1,4\}$ and $x=1$, $y=1$ and $z=4$. We have $\sigma(f1,f1,f4)=\sigma(1,1,1)=0$ and (3.1) , (3.5) trivially hold.

Case IV. If $\{x,y,z\}\subset\{1,2,3\}$ and $x=1$, $y=2$ and $z=3$. We have $\sigma(f1,f2,f3)=\sigma(1,3,3)=5$ and

$$R_{f,g,h}(1,2,3)=\max\big\{\sigma(1,2,3),\sigma(1,2,4),\sigma(1,1,1),\sigma(3,2,2),\sigma(4,3,4),0\big\}=10.$$

We have $5 \leq 10 - \phi(10) = 10 - \frac{10}{4} = 7.50$. Thus all the required hypothesis of condition (3.5) are satisfied. Then f, g and h have a unique point of coincidence in X and a unique common fixed point, here 1 is the unique fixed point.

4. An Application

In this section, we are going to apply Corollary 3.12 with $g = h = I_X$, to the study of existence and uniqueness of solutions. Denote $I = [0, 1]$ and consider the following equations system

$$\begin{cases} x''(t) = -F(t, x(t)), \\ x(0) = x(1) = 0, \\ y''(t) = -F(t, y(t)), \\ y(0) = y(1) = 0, \\ z''(t) = -F(t, z(t)), \\ z(0) = z(1) = 0, \end{cases} \tag{4.1}$$

for all $t \in I$, where $F \in C(I \times \mathbb{R}, \mathbb{R})$. It is known and easy to check that problem (4.1) is equivalent to the integral equations

$$\begin{aligned} x(t) &= \int_0^1 G(t, s)F(s, x(s))\, ds, \\ y(t) &= \int_0^1 G(t, s)F(s, y(s))\, ds, \\ z(t) &= \int_0^1 G(t, s)F(s, z(s))\, ds, \end{aligned} \tag{4.2}$$

where G is the Green function. Let $X = C(I)$ be the space of all continuous functions defined on I and $\|u\|_\infty = \sup_{t\in[0,1]} |u(t)|$ for each $u \in X$. Consider the tripled metric-like on X given by

$$\sigma(x, y, z) = \max\left\{\|x\|_\infty, \|y\|_\infty, \|z\|_\infty\right\}$$

for all $x, y, z \in X$. Thus (X, σ) is a complete tripled metric-like space. We recall that

$$\max_{t\in[0,1]} \int_0^1 G(s, t)\, ds = \frac{1}{8}.$$

Theorem 4.1. *Assume $F(s, x(s)) \leq \lambda_1\|x\|_\infty$, $F(s, y(s)) \leq \lambda_2\|y\|_\infty$ and $F(s, z(s)) \leq \lambda_3\|z\|_\infty$ for all $s \in I$ and $\lambda_1+\lambda_2+\lambda_3 < 1$. Then problem (4.2) has a unique solution $u \in X = C(I, \mathbb{R})$.*

Proof. Define the self-map $f : X \to X$ by

$$fx(t) = \int_0^1 G(t,s)F(s,x(s))\,ds,$$
$$fy(t) = \int_0^1 G(t,s)F(s,y(s))\,ds,$$
$$fz(t) = \int_0^1 G(t,s)F(s,z(s))\,ds.$$

We have

$$\|fx\|_\infty = \max_{t\in[0,1]} \int_0^1 G(s,t)F(s,x(s))\,ds = \frac{1}{8}F(s,x(s)) \leq \frac{1}{8}\lambda_1\|x\|_\infty,$$

$\|fy\|_\infty \leq \frac{1}{8}\lambda_2\|y\|_\infty$, and $\|fz\|_\infty \leq \frac{1}{8}\lambda_3\|z\|_\infty$. Thus

$$\begin{aligned}\sigma(fx,fy,fz) &= \max\left\{\|fx\|_\infty, \|fy\|_\infty, \|fz\|_\infty\right\}\\ &\leq \frac{1}{8}(\lambda_1+\lambda_2+\lambda_3)\max\left\{\|x\|_\infty, \|y\|_\infty, \|z\|_\infty\right\}\\ &\leq \frac{1}{8}\sigma(x,y,z)\\ &\leq \frac{1}{8}R_f(x,y,z).\end{aligned}$$

By choosing $k = \frac{1}{8}$ we have $\sigma(fx,fy,fz) \leq kR_f(x,y,z)$. Then f has a unique fixed point $u \in X$, that is, problem (4.2) has a unique solution $u \in C^2(I)$. □

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