J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2022.29.3.245 ISSN(Online) 2287-6081
Volume 29, Number 3 (August 2022), Pages 245–254

# EVALUATIONS OF THE ROGERS-RAMANUJAN CONTINUED FRACTION BY THETA-FUNCTION IDENTITIES REVISITED

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Abstract. In this paper, we use some theta-function identities involving certain parameters to show how to evaluate Rogers-Ramanujan continued fraction $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for $n = \frac{1}{5\cdot 4^m}$ and $\frac{1}{4^m}$, where $m$ is any positive integer. We give some explicit evaluations of them.

## 1. Introduction

The Rogers-Ramanujan continued fraction $R(q)$, for $|q| < 1$, is defined by

$$R(q) = \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \, \cdots$$

and also $S(q)$ is defined by

$$S(q) = -R(-q).$$

Ramanujan gave the first nonelementary evaluations of $R(q)$, that is,

$$R(e^{-2\pi}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2} \quad \text{and} \quad S(e^{-\pi}) = \sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}.$$

See [3, 4] for details of these evaluations.

In the 1980s, Ramanathan [9, 10, 11, 12] evaluated $R(e^{-2\pi\sqrt{n}})$ for $n = 1, 2, 4, 5, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{17}{5}, \frac{1}{10}$, and $S(e^{-\pi\sqrt{n}})$ for $n = 1, 5, \frac{1}{5}, \frac{3}{5}, \frac{7}{5}, \frac{23}{5}, \frac{39}{5}, \frac{1}{15}, \frac{1}{25}, \frac{1}{35}, \frac{1}{115}, \frac{1}{195}$.

In the mid of 1990s, Berndt and Chan [3] established values of $R(e^{-2\pi\sqrt{n}})$ for $n = 4, 9, 16, 64$ and $S(e^{-\pi\sqrt{n}})$ for $n = \frac{3}{5}, \frac{7}{5}, \frac{1}{15}, \frac{1}{35}$ by using an eta-function identity.

Received by the editors April 12, 2022. Revised June 23, 2022. Accepted June 26, 2022.
2010 *Mathematics Subject Classification.* Primary: 11F27, 33C90, Secondary: 11F20, 33C05, 33C75.
*Key words and phrases.* theta-function, modular equation, theta-function identity, Rogers-Ramanujan continued fraction.
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[a]This work was supported by the Korea Science Academy of KAIST with funds from the Ministry of Science and ICT.

Chan [5] determined the value of $S(e^{-\pi\sqrt{3}})$ by using a modular equation. Berndt, Chan, and Zhang [4] derived formulas for the explicit evaluations of $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for positive rational numbers $n$ in terms of Ramanujan-Weber class invariants. In addition, they determined the values of $R(e^{-6\pi})$ and $S(e^{-\pi\sqrt{n}})$ for $n = 3,\ 9,\ \frac{1}{5},\ \frac{3}{5},\ \frac{7}{5},\ \frac{9}{5},\ \frac{11}{5},\ \frac{13}{5},\ \frac{29}{5},\ \frac{41}{5},\ \frac{53}{5},\ \frac{89}{5},\ \frac{101}{5}$. In 1999, Chan and Tan [6] also evaluated $S(e^{-\pi\sqrt{11}})$ and $S(e^{-\pi\sqrt{19}})$.

In the beginning of 2000s, Vasuki and Mahadeva Naika [15] evaluated the values of $R(e^{-2\pi\sqrt{n}})$ for $n = \frac{1}{4},\ \frac{1}{5},\ \frac{4}{5},\ \frac{9}{5},\ \frac{1}{16},\ \frac{1}{20},\ \frac{1}{45}$ and $S(e^{-\pi\sqrt{n}})$ for $n = \frac{1}{5},\ \frac{9}{5},\ \frac{1}{45}$. Yi [16] employed eta-function identities to compute the values of $R(e^{-2\pi\sqrt{n}})$ for $n = 1,\ 2,\ 3,\ 4,\ 9,\ 16,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{2}{5},\ \frac{3}{5},\ \frac{4}{5},\ \frac{7}{5},\ \frac{8}{5},\ \frac{9}{5},\ \frac{1}{9},\ \frac{1}{10},\ \frac{1}{15},\ \frac{1}{16},\ \frac{1}{20},\ \frac{1}{25},\ \frac{1}{35},\ \frac{1}{40},\ \frac{1}{45}$ and $S(e^{-\pi\sqrt{n}})$ for $n = 1,\ 3,\ 9,\ 27,\ \frac{1}{3},\ \frac{1}{5},\ \frac{3}{5},\ \frac{7}{5},\ \frac{9}{5},\ \frac{1}{9},\ \frac{1}{15},\ \frac{1}{25},\ \frac{1}{27},\ \frac{1}{35},\ \frac{1}{45}$. In 2004, Baruah and Saikia [1] found the values of $R(e^{-2\pi\sqrt{n}})$ for $n = \frac{6}{5},\ \frac{14}{5},\ \frac{18}{5},\ \frac{3}{10},\ \frac{7}{10},\ \frac{9}{10},\ \frac{2}{15},\ \frac{1}{30},\ \frac{2}{35},\ \frac{2}{45},\ \frac{1}{70},\ \frac{1}{90}$ by employing the same argument as in [16]. In 2007, Baruah and Saikia [2] also determined the values of $R(e^{-2\pi/\sqrt{5}})$ and $S(e^{-\pi/\sqrt{5}})$.

In 2013, Paek and Yi [8] evaluated $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for $n = \frac{4}{5},\ \frac{16}{5}$ by using modular equations of degree 5. Saikia [13] evaluated explicit values of $R(e^{-2\pi\sqrt{n}})$ for $n = 2,\ \frac{1}{2},\ \frac{2}{25},\ \frac{1}{50}$ and both $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for $n = 3,\ \frac{1}{3},\ \frac{2}{5},\ \frac{4}{5},\ \frac{11}{5},\ \frac{13}{5},\ \frac{17}{5},\ \frac{29}{5},\ \frac{41}{5},\ \frac{53}{5},\ \frac{85}{5},\ \frac{101}{5},\ \frac{1}{10},\ \frac{1}{20},\ \frac{3}{25},\ \frac{1}{55},\ \frac{1}{65},\ \frac{1}{75},\ \frac{1}{85},\ \frac{1}{145},\ \frac{1}{205},\ \frac{1}{265},\ \frac{1}{445},\ \frac{1}{505}$ by using parametrization of Ramanujan's theta-functions. Saikia [14] also evaluated the values of $R(e^{-2\pi\sqrt{n}})$ for $n = 2,\ 3,\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{25},\ \frac{3}{25},\ \frac{1}{50},\ \frac{1}{75}$ and $S(e^{-\pi\sqrt{n}})$ for $n = 3,\ \frac{1}{3},\ \frac{3}{25},\ \frac{1}{75}$. Recently, Paek [7] used some theta-function identities to evaluate $R(e^{-2\pi\sqrt{n}})$ for $n = \frac{3}{5\cdot 4^m},\ \frac{9}{5\cdot 4^m},\ \frac{1}{15\cdot 4^m},\ \frac{1}{45\cdot 4^m}$ and $S(e^{-\pi\sqrt{n}})$ for $n = \frac{3}{5\cdot 4^{m-1}},\ \frac{9}{5\cdot 4^{m-1}},\ \frac{1}{15\cdot 4^{m-1}},\ \frac{1}{45\cdot 4^{m-1}}$, where $m$ is any positive integer.

Thus $R(e^{-2\pi\sqrt{n}})$ were evaluated for $n = 1,\ 2,\ 3,\ 4,\ 5,\ 9,\ 16,\ 64,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{2}{5},\ \frac{3}{5},\ \frac{4}{5},\ \frac{6}{5},\ \frac{7}{5},\ \frac{8}{5},\ \frac{9}{5},\ \frac{11}{5},\ \frac{13}{5},\ \frac{14}{5},\ \frac{16}{5},\ \frac{17}{5},\ \frac{18}{5},\ \frac{29}{5},\ \frac{41}{5},\ \frac{53}{5},\ \frac{101}{5},\ \frac{1}{9},\ \frac{1}{10},\ \frac{3}{10},\ \frac{7}{10},\ \frac{9}{10},\ \frac{1}{15},\ \frac{2}{15},\ \frac{1}{16},\ \frac{1}{20},\ \frac{3}{20},\ \frac{9}{20},\ \frac{1}{25},\ \frac{2}{25},\ \frac{3}{25},\ \frac{1}{30},\ \frac{1}{35},\ \frac{2}{35},\ \frac{1}{40},\ \frac{1}{45},\ \frac{2}{45},\ \frac{1}{50},\ \frac{1}{55},\ \frac{1}{60},\ \frac{1}{65},\ \frac{1}{70},\ \frac{1}{75},\ \frac{3}{80},\ \frac{9}{80},\ \frac{1}{85},\ \frac{1}{90},\ \frac{1}{145},\ \frac{1}{180},\ \frac{1}{205},\ \frac{1}{265},\ \frac{1}{240},\ \frac{3}{320},\ \frac{1}{445},\ \frac{1}{505},\ \frac{1}{720},\ \frac{1}{960}$. In addition, $S(e^{-\pi\sqrt{n}})$ were evaluated for $n = 1,\ 3,\ 5,\ 9,\ 11,\ 19,\ 27,\ \frac{1}{3},\ \frac{1}{5},\ \frac{2}{5},\ \frac{3}{5},\ \frac{4}{5},\ \frac{7}{5},\ \frac{9}{5},\ \frac{11}{5},\ \frac{13}{5},\ \frac{16}{5},\ \frac{17}{5},\ \frac{23}{5},\ \frac{29}{5},\ \frac{39}{5},\ \frac{41}{5},\ \frac{53}{5},\ \frac{89}{5},\ \frac{101}{5},\ \frac{1}{9},\ \frac{1}{10},\ \frac{1}{15},\ \frac{1}{20},\ \frac{3}{20},\ \frac{9}{20},\ \frac{1}{25},\ \frac{3}{25},\ \frac{1}{27},\ \frac{1}{35},\ \frac{1}{45},\ \frac{1}{55},\ \frac{1}{60},\ \frac{1}{65},\ \frac{1}{75},\ \frac{1}{115},\ \frac{1}{145},\ \frac{1}{180},\ \frac{1}{195},\ \frac{1}{205},\ \frac{1}{265},\ \frac{1}{445},\ \frac{1}{505}$.

In this paper, we use some theta-function identities involving parameters $h_{5,n}$ and $h'_{5,n}$ defined in (1.1) and (1.2) below to show how to evaluate $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for $n = \frac{1}{5\cdot 4^m}$ and $\frac{1}{4^m}$, where $m$ is any positive integer. Furthermore, we establish some explicit evaluations of them.

For $|q|<1$, Ramanujan's theta-functions $\varphi$ is defined by

$$\varphi(q)=\sum_{n=-\infty}^{\infty} q^{n^2}.$$

Let $k$ and $n$ be any positive real numbers. Define $h_{k,n}$ and $h'_{k,n}$ by

$$h_{k,n}=\frac{\varphi(q)}{k^{1/4}\varphi(q^k)}, \quad \textit{where} \quad q=e^{-\pi\sqrt{n/k}}, \tag{1.1}$$

and

$$h'_{k,n}=\frac{\varphi(-q)}{k^{1/4}\varphi(-q^k)}, \quad \textit{where} \quad q=e^{-2\pi\sqrt{n/k}}. \tag{1.2}$$

We use the following formulas in [13, Theorem 3.5] to evaluate $R(e^{-2\pi\sqrt{n/5}})$ and $S(e^{-\pi\sqrt{n/5}})$ in terms of $h'_{5,n}$ and $h_{5,n}$, respectively.

$$\frac{1}{R^5(e^{-2\pi\sqrt{n/5}})}-11-R^5(e^{-2\pi\sqrt{n/5}})=5\,h'^4_{5,n}\left(\frac{\sqrt{5}\,h'^2_{5,n}-5}{\sqrt{5}\,h'^2_{5,n}-1}\right) \tag{1.3}$$

and

$$\frac{1}{S^5(e^{-\pi\sqrt{n/5}})}+11-S^5(e^{-\pi\sqrt{n/5}})=5\,h^4_{5,n}\left(\frac{5-\sqrt{5}\,h^2_{5,n}}{\sqrt{5}\,h^2_{5,n}-1}\right). \tag{1.4}$$

From (1.3) and (1.4), we have

$$R^5(e^{-2\pi\sqrt{n/5}})=\sqrt{a^2+1}-a, \ \textit{where} \ 2a=11+5\,h'^4_{5,n}\left(\frac{\sqrt{5}\,h'^2_{5,n}-5}{\sqrt{5}\,h'^2_{5,n}-1}\right) \tag{1.5}$$

and

$$S^5(e^{-\pi\sqrt{n/5}})=\sqrt{b^2+1}-b, \ \textit{where} \ 2b=-11+5\,h^4_{5,n}\left(\frac{5-\sqrt{5}\,h^2_{5,n}}{\sqrt{5}\,h^2_{5,n}-1}\right). \tag{1.6}$$

Consequently, in order to find the values of $R(e^{-2\pi\sqrt{n/5}})$ and $S(e^{-\pi\sqrt{n/5}})$, it suffices to compute $h'^2_{5,n}$ and $h^2_{5,n}$, respectively. Thus, in this paper, we use theta-function identities involving $h'_{5,n}$ and $h_{5,n}$ to find some new explicit values of the Rogers-Ramanujan continued fraction.

## 2. Evaluations of $h'_{5,n}$ and $h_{5,n}$

In this section, we compute the values of $h'^2_{5,n}$ and $h^2_{5,n}$ for some positive real numbers $n$ to evaluate the Rogers-Ramanujan continued fraction. We begin by recalling the known values of $h'_{5,1}$ and $h'_{5,5}$ in [17] which will play key roles in evaluating the Rogers-Ramanujan continued fraction.

**Lemma 2.1** ([17, Theorems 4.16 and 5.6(v)]). *We have*

(i) $h'^2_{5,1} = \dfrac{\sqrt{\sqrt{5}+1}-\sqrt{2}}{\sqrt{\sqrt{5}-1}}$,

(ii) $h'^2_{5,5} = \dfrac{1}{4}(5+\sqrt{5})(\sqrt[4]{5}-1)^2$.

We now need a couple of theta-function identities: one shows a relation between $h'_{5,n}$ and $h'_{5,n/4}$ and the other shows a relation between $h'_{5,n}$ and $h_{5,n}$.

**Lemma 2.2.** *For any positive real number $n$, we have*

$$\frac{h'^2_{5,n/4}}{h'^2_{5,n}} + \frac{h'^2_{5,n}}{h'^2_{5,n/4}} + 4 = \sqrt{5}\left(h'^2_{5,n} + \frac{1}{h'^2_{5,n}}\right). \tag{2.1}$$

*Proof.* Let $P = \frac{\varphi(-q)}{\varphi(-q^5)}$ and $Q = \frac{\varphi(-q^2)}{\varphi(-q^{10})}$. Then $\dfrac{P^2}{Q^2} + \dfrac{Q^2}{P^2} + 4 = Q^2 + \dfrac{5}{Q^2}$ by [2, Theorem 2.17]. Rewrite $P$ and $Q$ in terms of $h'_{5,n/4}$ and $h'_{5,n}$ to complete the proof. □

**Lemma 2.3** ([8, Corollary 3.4]). *For every positive real number $n$, we have*

$$\frac{h^2_{5,n}}{h'^2_{5,n}} + \frac{h'^2_{5,n}}{h^2_{5,n}} + 4 = \sqrt{5}\left(h'^2_{5,n} + \frac{1}{h'^2_{5,n}}\right). \tag{2.2}$$

Note also that (2.2) follows from a modular equation $\dfrac{P^2}{Q^2} + \dfrac{Q^2}{P^2} + 4 = Q^2 + \dfrac{5}{Q^2}$ in [8, Theorem 3.3], where $P = \frac{\varphi(q)}{\varphi(q^5)}$ and $Q = \frac{\varphi(-q^2)}{\varphi(-q^{10})}$.

We are in position to evaluate $h'^2_{5,n}$ for $n = \frac{1}{4}$, $\frac{1}{16}$, and $\frac{1}{64}$.

**Theorem 2.4.** *We have*

(i) $h'^2_{5,1/4} = 2+\sqrt{5}-2\sqrt{2+\sqrt{5}}$,

(ii) $h'^2_{5,1/16} = \dfrac{3+2\sqrt{5}-2\sqrt{2+5\sqrt{5}}-2\sqrt{11+7\sqrt{5}-4\sqrt{22+10\sqrt{5}}}}{-2+\sqrt{5}}$,

(iii) $h'^2_{5,1/64} = \dfrac{1}{\sqrt{5}}\left(2c^2-2c+1-2\sqrt{(c^2+1)(c^2-2c)}\right)$, *where*

$$c = \frac{5+6\sqrt{5}-5\sqrt{2+5\sqrt{5}}-5\sqrt{11+7\sqrt{5}-4\sqrt{22+10\sqrt{5}}}}{5-2\sqrt{5}}.$$

*Proof.* For (i), let $n=1$ in (2.1) and put $h'^2_{5,1/4} = x$ and the value of $h'_{5,1}$ in Lemma 2.1(i) to deduce that

$$x^2-\left(3+\sqrt{5}-2\sqrt{2+\sqrt{5}}\right)x+2+\sqrt{5}-2\sqrt{2+\sqrt{5}}=0\,.$$

Solving the last equation for $x$ and using $x<1$, we complete the proof.

For (ii), let $n=\frac{1}{4}$ in (2.1). Putting $h'^2_{5,1/16}=x$ and $h'^2_{5,1/4}=2+\sqrt{5}-2\sqrt{2+\sqrt{5}}$ obtained in (i), we find that

$$\left(2+\sqrt{5}+2\sqrt{2+\sqrt{5}}\right)x^2-(6+4\sqrt{5})\,x+2+\sqrt{3}+2\sqrt{2+\sqrt{5}}=0.$$

Solve the last equation for $x$ with the help of *Mathematica* and use $x<1$ to complete the proof.

The proof of (iii) is similar to that of (ii). □

By letting $n=\frac{1}{128}$ in (2.1) and using the value $h'^2_{5,1/64}$ in Theorem 2.4(iii), we can evaluate $h'^2_{5,1/128}$. Thus we can evaluate $h'^2_{5,1/4^m}$ for every positive integer $m$.

We now evaluate $h^2_{5,n}$ for $n=\frac{1}{4}$, $\frac{1}{16}$, and $\frac{1}{64}$.

**Theorem 2.5.** *We have*

(i) $$h^2_{5,1/4}=\frac{3+3\sqrt{2}+2\sqrt{5}+\sqrt{10}-4\sqrt{1+\sqrt{5}}-2\sqrt{2+\sqrt{5}}}{-2+\sqrt{5}}\,,$$

(ii) $$h^2_{5,1/16}=\sqrt{5}\,\left(10c^2-14c+5+2\sqrt{(c-1)(5c-3)(5c^2-6c+2)}\right),\ \textit{where}$$

$$c=\frac{4-2\sqrt{2+5\sqrt{5}}-\sqrt{11+7\sqrt{5}-4\sqrt{22+10\sqrt{5}}}}{5-2\sqrt{5}}\,,$$

(iii) $$h^2_{5,1/64}=\frac{1}{\sqrt{5}}\left(2c^2+6c+5+2\sqrt{(c^2+2c)(c^2+4c+5)}\right),\ \textit{where}$$

$$c=(d+1)(d+2)-\sqrt{(d^2+1)(d^2-2d)}\ \ \textit{and}$$

$$d=\frac{5+6\sqrt{5}-5\sqrt{2+5\sqrt{5}}-5\sqrt{11+7\sqrt{5}-4\sqrt{22+10\sqrt{5}}}}{5-2\sqrt{5}}\,.$$

*Proof.* For (i), let $n=\frac{1}{4}$ in (2.2) and put $h^2_{5,1/4}=x$ and $h'^2_{5,1/4}=2+\sqrt{5}-2\sqrt{2+\sqrt{5}}$ in Theorem 2.4(i) to deduce that

$$x^2-\left(32+14\sqrt{5}-4\sqrt{118+53\sqrt{5}}\right)\,x+\left(2+\sqrt{5}-2\sqrt{2+\sqrt{5}}\right)^2=0\,.$$

Solving the last equation for $x$ with the help of *Mathematica* and using $x>1$, we complete the proof.

The proofs of (ii) and (iii) are similar to that of (ii). □

We now evaluate $h'^2_{5,n}$ for $n=\frac{5}{4}$, $\frac{5}{16}$, and $\frac{5}{64}$.

**Theorem 2.6.** *We have*

(i) $h'^2_{5,5/4} = 60 - 40\sqrt[4]{5} + 27\sqrt{5} - 18\sqrt[4]{5^3}$,

(ii) $$h'^2_{5,5/16} = \frac{-95 + \sqrt{5} + 2\sqrt[4]{5^3} + 4\sqrt{10\left(76 - 4\sqrt[4]{5} - 15\sqrt{5} - 4\sqrt[4]{5^3}\right)}}{19 - 68\sqrt[4]{5} - 45\sqrt{5} + 6\sqrt[4]{5^3}},$$

(iii) $$h'^2_{5,5/64} = \frac{\sqrt{5}}{2}\left(5c^2 - 4c + 1 - \sqrt{(c-1)(5c-1)(5c^2-2c+1)}\right), \text{ where}$$
$$c = \frac{1 + 2\sqrt[4]{5} - 19\sqrt{5} + 4\sqrt{2\left(76 - 4\sqrt[4]{5} - 15\sqrt{5} - 4\sqrt[4]{5^3}\right)}}{19 - 68\sqrt[4]{5} - 45\sqrt{5} + 6\sqrt[4]{5^3}}.$$

*Proof.* For (i), let $n = 5$ in (2.1). Putting $h'^2_{5,5/4} = x$ and $h'_{5,5} = \frac{1}{2}(\sqrt[4]{5} - 1)\sqrt{5 + \sqrt{5}}$ in Lemma 2.1(ii), we find that

$$x^2 - (65 - 40\sqrt[4]{5} + 25\sqrt{5} - 18\sqrt[4]{5^3})\,x + 5(2 + \sqrt{5}\,)(3 - 2\sqrt[4]{5}\,) = 0\,.$$

Solving the last equation for $x$ and using $x < \frac{1}{2}$, we complete the proof with the help of *Mathematica*.

For (ii) and (iii), repeat the same argument as in the proof of (i). □

By letting $n = \frac{5}{64}$ in (2.1) and using the value $h'^2_{5,5/64}$ in Theorem 2.6(iii), we can evaluate $h'^2_{5,5/256}$. Thus we can evaluate $h'^2_{5,5/4^m}$ for every positive integer $m$.

We end this section by evaluating $h^2_{5,n}$ for $n = \frac{5}{4}$, $\frac{5}{16}$, and $\frac{5}{64}$.

**Theorem 2.7.** *We have*

(i) $$h^2_{5,5/4} = \frac{10 + 80\sqrt[4]{5} - 49\sqrt{5} - 6\sqrt[4]{5^3} - 2\sqrt{5\left(363 + 710\sqrt[4]{5} + 281\sqrt{5} - 614\sqrt[4]{5^3}\right)}}{-161 + 72\sqrt{5}},$$

(ii) $$h^2_{5,5/16} = \frac{\sqrt{5}}{2}\left(5c^2 - 4c + 1 + \sqrt{(c-1)(5c-1)(5c^2-2c+1)}\right), \text{ where}$$
$$c = \frac{1 + 2\sqrt[4]{5} - 19\sqrt{5} + 4\sqrt{2\left(76 - 4\sqrt[4]{5} - 15\sqrt{5} - 4\sqrt[4]{5^3}\right)}}{19 - 68\sqrt[4]{5} - 45\sqrt{5} + 6\sqrt[4]{5^3}},$$

(iii) $$h^2_{5,5/64} = \frac{\sqrt{5}}{2}\left(5c^2 - 4c + 1 + \sqrt{(c-1)(5c-1)(5c^2-2c+1)}\right), \text{ where}$$
$$c = \frac{1}{2}\left(5d^2 - 4d + 1 + \sqrt{(d-1)(5d-1)(5d^2-2d+1)}\right) \text{ and}$$
$$d = \frac{1 + 2\sqrt[4]{5} - 19\sqrt{5} + 4\sqrt{2\left(76 - 4\sqrt[4]{5} - 15\sqrt{5} - 4\sqrt[4]{5^3}\right)}}{19 - 68\sqrt[4]{5} - 45\sqrt{5} + 6\sqrt[4]{5^3}}.$$

*Proof.* The results follow directly from (2.2) and Theorem 2.6(ii) and (iv) with the help of *Mathematica*. □

## 3. Evaluations of $R(q)$ and $S(q)$

We first evaluate $R(e^{-2\pi\sqrt{n}})$ for $n = \frac{1}{5\cdot 4^m}$, where $m$ is any positive integer. We show the cases when $m = 1$, 2, and 3.

**Theorem 3.1.** *We have*

(i) $R^5(e^{-\pi/\sqrt{5}})$
$$= -\frac{1}{4}\left(147 + 65\sqrt{5} - 25\sqrt{58 + 26\sqrt{5}}\right) + \frac{\sqrt[4]{5^3}\left(5 - 4\sqrt{5} + \sqrt{2 + 5\sqrt{5}}\right)}{11 - 5\sqrt{5}},$$

(ii) $R^5(e^{-\pi/2\sqrt{5}}) = \frac{1}{2}\left(-a - 11 + \sqrt{a^2 + 22a + 225}\,\right)$, *where*
$$a = \frac{10(11 - \sqrt{2} - 3\sqrt{10}\,)^2\left(30 + 31\sqrt{5} - 5\sqrt{142 + 113\sqrt{5}}\right)}{11(2 - \sqrt{5}\,)^2(14 - \sqrt{2} - 2\sqrt{5} - 3\sqrt{10}\,)},$$

(iii) $R^5(e^{-\pi/4\sqrt{5}}) = \frac{1}{2}\left(-a - 11 + \sqrt{a^2 + 22a + 125}\,\right)$, *where*
$$a = \frac{(c-2)(2c+1)\left(2c^2 - 2c + 1 - 2\sqrt{(c^2+1)(c^2-2c)}\right)^2}{c^2 - 2c - \sqrt{(c^2+1)(c^2-2c)}} \quad \textit{and}$$
$$c = \frac{5 + 6\sqrt{5} - 5\sqrt{2 + 5\sqrt{5}} - 5\sqrt{11 + 7\sqrt{5} - 4\sqrt{22 + 10\sqrt{5}}}}{5 - 2\sqrt{5}}.$$

*Proof.* The results follow directly from (1.5) and Theorem 2.4 with the help of *Mathematica*. □

See [16, Corollary 4.6(iv)] and [14, Theorem 3.13] for different proofs of Theorem 3.1(i). We next evaluate $S(e^{-\pi\sqrt{n}})$ for $n = \frac{1}{5\cdot 4^m}$, where $m$ is any positive integer. We show the cases when $m = 1$, 2 and 3.

**Theorem 3.2.** *We have*

(i) $S^5(e^{-\pi/2\sqrt{5}}) = \frac{11}{2} + \frac{b(4b-5)^2}{2(b-1)} + \sqrt{1 + \left(\frac{11}{2} + \frac{b(4b-5)^2}{2(b-1)}\right)^2}$, *where*
$$b = \frac{5\left(2 - 3\sqrt{2} - 4\sqrt{5} - \sqrt{10} + 4\sqrt{1 + \sqrt{5}} + 2\sqrt{2 + 5\sqrt{5}}\right)}{4(5 - 2\sqrt{5})},$$

(ii) $S^5(e^{-\pi/4\sqrt{5}}) = \frac{1}{2}\left(25b + 11 + 5\sqrt{25b^2 + 22b + 5}\right)$, *where*

$$b = \frac{5(2c^2+c)\left(10c^2+6c+1+2\sqrt{(5c^2+2c)(5c^2+4c+1)}\right)^2}{5c^2+2c+\sqrt{(5c^2+2c)(5c^2+4c+1)}} \quad \textit{and}$$

$$c = \frac{-1+2\sqrt{5}-\sqrt{2+5\sqrt{5}}-\sqrt{11+7\sqrt{5}-4\sqrt{22+10\sqrt{5}}}}{5-2\sqrt{5}},$$

(iii) $S^5(e^{-\pi/8\sqrt{5}}) = \frac{1}{2}\left(b+11+\sqrt{b^2+22b+125}\right)$, *where*

$$b = \frac{(2c+1)^4\left(-c^2+2c+\sqrt{(c^2+1)(c^2-2c)}\right)}{c\left(2c^2-2c+1-2\sqrt{(c^2+1)(c^2-2c)}\right)^2},$$

$$c = d^2-d-\sqrt{(d^2+1)(d^2-2d)}, \ \textit{and}$$

$$d = \frac{5+6\sqrt{5}-5\sqrt{2+5\sqrt{5}}-5\sqrt{11+7\sqrt{5}-4\sqrt{22+10\sqrt{5}}}}{5-2\sqrt{5}}.$$

*Proof.* The results are immediate consequences of (1.6) and Theorem 2.5 with the help of *Mathematica*. □

See [14, Theorem 3.13] for an alternative proof for Theorem 3.2(i). We now evaluate $R(e^{-2\pi\sqrt{n}})$ for $n = \frac{1}{4^m}$, where $m$ is any positive integer. We show the cases when $m = 1$, 2, and 3.

**Theorem 3.3.** *We have*

(i) $R^5(e^{-\pi}) = \frac{1}{2}\left(-25a-11+5\sqrt{25a^2+22a+5}\right)$, *where*

$$a = \frac{5\left(3+7\sqrt{5}-2\sqrt{40+21\sqrt{5}}\right)}{123-55\sqrt{5}},$$

(ii) $R^5(e^{-\pi/2}) = -\frac{11}{2}-\frac{125a^2(a-1)}{2(5a-1)}+\sqrt{1+\left(\frac{11}{2}+\frac{125a^2(a-1)}{2(5a-1)}\right)^2}$, *where*

$$a = \frac{1+2\sqrt[4]{5}-19\sqrt{5}+4\sqrt{2(76-4\sqrt[4]{5}-15\sqrt{5}-4\sqrt[4]{5^3})}}{19-68\sqrt[4]{5}-45\sqrt{5}+6\sqrt[4]{5^3}},$$

(iii) $R^5(e^{-\pi/4}) = \frac{1}{4}(-25a-22+5\sqrt{25a^2+44a+20})$, *where*

$$a = \frac{5c(c-1)\left(5c^2-4c+1-\sqrt{(c-1)(5c-1)(5c^2-2c+1)}\right)^2}{5c^2-6c+1-\sqrt{(c-1)(5c-1)(5c^2-2c+1)}} \quad \textit{and}$$

$$c = \frac{1+2\sqrt[4]{5}-19\sqrt{5}+4\sqrt{2(76-4\sqrt[4]{5}-15\sqrt{5}-4\sqrt[4]{5^3})}}{19-68\sqrt[4]{5}-45\sqrt{5}+6\sqrt[4]{5^3}}.$$

*Proof.* The results follow directly from (1.5) and Theorem 2.6. We used *Mathematica* to verify (i)–(iii). □

See [16, Corollary 3.8(ii)] and [15, Theorem 2.1] for different proofs for Theorem 3.3(i). See also [16, Corollary 3.10(ii)] for a different proof for Theorem 3.3(ii).

We end this section by evaluating $S(e^{-\pi\sqrt{n}})$ for $n=\frac{1}{4^m}$, where $m$ is any positive integer. We show the cases when $m=1$, 2, and 3.

**Theorem 3.4.** *We have*

(i) $$S^5(e^{-\pi/2})=\frac{11}{2}+\frac{b^2(b-5)}{2(b-1)}+\sqrt{1+\left(\frac{11}{2}+\frac{b^2(b-5)}{2(b-1)}\right)^2},\ \textit{where}$$

$$b=\frac{5\left(49+6\sqrt[4]{5}-2\sqrt{5}-16\sqrt[4]{5^3}+2\sqrt{363+710\sqrt[4]{5}+281\sqrt{5}-614\sqrt[4]{5^3}}\right)}{161-72\sqrt{5}},$$

(ii) $$S^5(e^{-\pi/4})=\frac{1}{2}\left(25b+11+5\sqrt{25b^2+22b+5}\right),\ \textit{where}$$

$$b=\frac{5(c^2-c)\left(5c^2-4c+1+\sqrt{(c-1)(5c-1)(5c^2-2c+1)}\right)^2}{2\left(5c^2-6c+1+\sqrt{(c-1)(5c-1)(5c^2-2c+1)}\right)},\ \textit{and}$$

$$c=\frac{1+2\sqrt[4]{5}-19\sqrt{5}+4\sqrt{2(76-4\sqrt[4]{5}-15\sqrt{5}-4\sqrt[4]{5^3})}}{19-68\sqrt[4]{5}-45\sqrt{5}+6\sqrt[4]{5^3}},$$

(iii) $$S^5(e^{-\pi/8})=\frac{1}{2}\left(25b+11+5\sqrt{25b^2+22b+5}\right),\ \textit{where}$$

$$b=\frac{5(c^2-c)\left(5c^2-4c+1+\sqrt{(c-1)(5c-1)(5c^2-2c+1)}\right)^2}{2\left(5c^2-6c+1+\sqrt{(c-1)(5c-1)(5c^2-2c+1)}\right)},$$

$$c=\frac{1}{2}\left(5d^2-4d+1-\sqrt{(d-1)(5d-1)(5d^2-2d+1)}\right),\ \textit{and}$$

$$d=\frac{1+2\sqrt[4]{5}-19\sqrt{5}+4\sqrt{2(76-4\sqrt[4]{5}-15\sqrt{5}-4\sqrt[4]{5^3})}}{19-68\sqrt[4]{5}-45\sqrt{5}+6\sqrt[4]{5^3}}.$$

*Proof.* The results follow from (1.6) and Theorem 2.7. We used *Mathematica* to verify (i)–(iii). □

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