EVALUATIONS OF THE ROGERS-RAMANUJAN CONTINUED FRACTION BY THETA-FUNCTION IDENTITIES REVISITED

JINHEE YI$^a$ and DAE HYUN PAEK$^{b,*}$

Abstract. In this paper, we use some theta-function identities involving certain parameters to show how to evaluate Rogers-Ramanujan continued fraction $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for $n = \frac{1}{15}$ and $\frac{1}{25}$, where $m$ is any positive integer. We give some explicit evaluations of them.

1. Introduction

The Rogers-Ramanujan continued fraction $R(q)$, for $|q| < 1$, is defined by

$$R(q) = \frac{q^{1/5}}{1} + \frac{q^{2}}{1} + \frac{q^{3}}{1} + \cdots$$

and also $S(q)$ is defined by

$$S(q) = -R(-q).$$

Ramanujan gave the first nonelementary evaluations of $R(q)$, that is,

$$R(e^{-2\pi}) = \sqrt{\frac{5 + \sqrt{5}}{2}} - \frac{\sqrt{5} + 1}{2} \quad \text{and} \quad S(e^{-\pi}) = \sqrt{\frac{5 - \sqrt{5}}{2}} - \frac{\sqrt{5} - 1}{2}.$$

See [3, 4] for details of these evaluations.

In the 1980s, Ramanathan [9, 10, 11, 12] evaluated $R(e^{-2\pi\sqrt{n}})$ for $n = 1, 2, 4, 5, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{17}{5}, \frac{1}{10}$, and $S(e^{-\pi\sqrt{n}})$ for $n = 1, 5, \frac{1}{5}, \frac{1}{3}, \frac{7}{5}, \frac{23}{5}, \frac{39}{5}, \frac{1}{15}, \frac{1}{35}, \frac{1}{115}, \frac{1}{195}$.

In the mid of 1990s, Berndt and Chan [3] established values of $R(e^{-2\pi\sqrt{n}})$ for $n = 4, 9, 16, 64$ and $S(e^{-\pi\sqrt{n}})$ for $n = \frac{3}{5}, \frac{7}{5}, \frac{1}{15}, \frac{1}{35}$ by using an eta-function identity.

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$^*$Corresponding author.

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Chan [5] determined the value of $S(e^{-\pi\sqrt{3}})$ by using a modular equation. Berndt, Chan, and Zhang [4] derived formulas for the explicit evaluations of $R(e^{-2\pi\sqrt{5}})$ and $S(e^{-\pi\sqrt{5}})$ for positive rational numbers $n$ in terms of Ramanujan-Weber class invariants. In addition, they determined the values of $R(e^{-6\pi})$ and $S(e^{-\pi\sqrt{5}})$ for $n = 3, 9, \frac{1}{5}, \frac{3}{5}, \frac{7}{5}, \frac{9}{5}, \frac{11}{5}, \frac{13}{5}, \frac{29}{5}, \frac{41}{5}, \frac{53}{5}, \frac{89}{5}, \frac{101}{5}$. In 1999, Chan and Tan [6] also evaluated $S(e^{-\pi\sqrt{17}})$ and $S(e^{-\pi\sqrt{19}})$.

In the beginning of 2000s, Vasuki and Mahadeva Naika [15] evaluated the values of $R(e^{-2\pi\sqrt{5}})$ for $n = \frac{1}{4}, \frac{1}{5}, \frac{1}{5}, \frac{1}{16}, \frac{1}{20}, \frac{1}{45}$ and $S(e^{-\pi\sqrt{5}})$ for $n = \frac{1}{5}, \frac{9}{5}, \frac{11}{5}, \frac{29}{5}, \frac{41}{5}, \frac{53}{5}, \frac{89}{5}, \frac{101}{5}$. Yi [16] employed eta-function identities to compute the values of $R(e^{-2\pi\sqrt{5}})$ for $n = 1, 2, 3, 4, 9, 16, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}, \frac{2}{5}, \frac{4}{5}, \frac{7}{5}, \frac{9}{5}, \frac{11}{5}, \frac{13}{5}, \frac{29}{5}, \frac{41}{5}, \frac{53}{5}, \frac{89}{5}, \frac{101}{5}, \frac{1}{16}, \frac{1}{20}, \frac{1}{25}, \frac{1}{35}, \frac{1}{45}$ and $S(e^{-\pi\sqrt{5}})$ for $n = 1, 3, 9, 27, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \frac{3}{5}, \frac{7}{5}, \frac{9}{5}, \frac{1}{15}, \frac{1}{25}, \frac{1}{35}, \frac{1}{45}$. In 2004, Baruah and Saikia [1] found the values of $R(e^{-2\pi\sqrt{5}})$ for $n = \frac{5}{15}, \frac{1}{9}, \frac{11}{15}, \frac{1}{9}, \frac{2}{15}, \frac{2}{25}, \frac{1}{45}$, by employing the same argument as in [16]. In 2007, Baruah and Saikia [2] also determined the values of $R(e^{-2\pi/\sqrt{5}})$ and $S(e^{-\pi/\sqrt{5}})$.

In 2013, Paek and Yi [8] evaluated $R(e^{-2\pi\sqrt{5}})$ and $S(e^{-\pi\sqrt{5}})$ for $n = \frac{4}{5}, \frac{16}{5}$ by using modular equations of degree 5. Saikia [13] evaluated explicit values of $R(e^{-2\pi\sqrt{5}})$ for $n = 2, \frac{1}{2}, \frac{2}{25}, \frac{1}{45}$ and both $R(e^{-2\pi\sqrt{5}})$ and $S(e^{-\pi\sqrt{5}})$ for $n = 3, \frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{11}{5}, \frac{13}{5}, \frac{17}{5}, \frac{29}{5}, \frac{41}{5}, \frac{53}{5}, \frac{89}{5}, \frac{101}{5}, \frac{1}{16}, \frac{1}{20}, \frac{1}{25}, \frac{1}{35}, \frac{1}{45}$ by using parametrization of Ramanujan’s theta-functions. Saikia [14] also evaluated the values of $R(e^{-2\pi\sqrt{5}})$ for $n = 2, 3, \frac{1}{2}, \frac{2}{25}, \frac{1}{3}, \frac{165}{7}, \frac{1}{45}$ and $S(e^{-\pi\sqrt{5}})$ for $n = 3, \frac{1}{5}, \frac{2}{25}, \frac{1}{45}$. Recently, Paek [7] used some theta-function identities to evaluate $R(e^{-2\pi\sqrt{5}})$ for $n = \frac{3}{5}, \frac{8}{5}, \frac{13}{5}, \frac{14}{5}, \frac{19}{5}, \frac{20}{5}, \frac{41}{5}, \frac{53}{5}, \frac{101}{5}, \frac{1}{9}, \frac{1}{13}, \frac{1}{15}, \frac{3}{15}, \frac{2}{15}, \frac{1}{45}, \frac{1}{45}$, and $S(e^{-\pi\sqrt{5}})$ for $n = \frac{3}{5}, \frac{8}{5}, \frac{13}{5}, \frac{14}{5}, \frac{19}{5}, \frac{20}{5}, \frac{41}{5}, \frac{53}{5}, \frac{101}{5}, \frac{1}{9}, \frac{1}{13}, \frac{1}{15}, \frac{3}{15}, \frac{2}{15}, \frac{1}{45}, \frac{1}{45}$, where $m$ is any positive integer.

Thus $R(e^{-2\pi\sqrt{5}})$ were evaluated for $n = 1, 2, 3, 4, 5, 9, 16, 64, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5}, \frac{7}{5}, \frac{9}{5}, \frac{11}{5}, \frac{13}{5}, \frac{14}{5}, \frac{16}{5}, \frac{17}{5}, \frac{18}{5}, \frac{20}{5}, \frac{41}{5}, \frac{53}{5}, \frac{101}{5}, \frac{1}{9}, \frac{1}{13}, \frac{1}{15}, \frac{3}{15}, \frac{2}{15}, \frac{1}{45}, \frac{1}{45}$, $\frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{7}{5}, \frac{9}{5}, \frac{11}{5}, \frac{13}{5}, \frac{16}{5}, \frac{23}{5}, \frac{29}{5}, \frac{39}{5}, \frac{41}{5}, \frac{53}{5}, \frac{89}{5}, \frac{101}{5}, \frac{1}{9}, \frac{1}{13}, \frac{1}{15}, \frac{3}{15}, \frac{2}{15}, \frac{1}{45}, \frac{1}{45}$, in addition, $S(e^{-\pi\sqrt{5}})$ were evaluated for $n = 1, 3, 5, 9, 11, 19, 27, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \frac{3}{5}, \frac{4}{5}, \frac{7}{5}, \frac{9}{5}, \frac{11}{5}, \frac{13}{5}, \frac{16}{5}, \frac{23}{5}, \frac{29}{5}, \frac{39}{5}, \frac{41}{5}, \frac{53}{5}, \frac{89}{5}, \frac{101}{5}, \frac{1}{9}, \frac{1}{13}, \frac{1}{15}, \frac{3}{15}, \frac{2}{15}, \frac{1}{45}, \frac{1}{45}$, $\frac{1}{3}, \frac{1}{5}, \frac{3}{5}, \frac{9}{5}, \frac{11}{5}, \frac{13}{5}, \frac{1}{13}, \frac{1}{15}, \frac{3}{15}, \frac{2}{15}, \frac{1}{45}, \frac{1}{45}$, $\frac{1}{3}, \frac{1}{5}, \frac{3}{5}, \frac{9}{5}, \frac{11}{5}, \frac{13}{5}, \frac{1}{13}, \frac{1}{15}, \frac{3}{15}, \frac{2}{15}, \frac{1}{45}, \frac{1}{45}$, $\frac{1}{3}, \frac{1}{5}, \frac{3}{5}, \frac{9}{5}, \frac{11}{5}, \frac{13}{5}, \frac{1}{13}, \frac{1}{15}, \frac{3}{15}, \frac{2}{15}, \frac{1}{45}, \frac{1}{45}$.

In this paper, we use some theta-function identities involving parameters $h_{5,n}$ and $h_{3,n}$ defined in (1.1) and (1.2) below to show how to evaluate $R(e^{-2\pi\sqrt{5}})$ and $S(e^{-\pi\sqrt{5}})$ for $n = \frac{1}{5}, \frac{1}{5}$, where $m$ is any positive integer. Furthermore, we establish some explicit evaluations of them.
For $|q| < 1$, Ramanujan’s theta-functions $\varphi$ is defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$  

Let $k$ and $n$ be any positive real numbers. Define $h_{k,n}$ and $h'_{k,n}$ by

(1.1)  
$$h_{k,n} = \frac{\varphi(q)}{k^{1/4} \varphi(q^k)}, \text{ where } q = e^{-\pi \sqrt{n/k}},$$  

and

(1.2)  
$$h'_{k,n} = \frac{\varphi(-q)}{k^{1/4} \varphi(-q^k)}, \text{ where } q = e^{-2\pi \sqrt{n/k}}.$$  

We use the following formulas in [13, Theorem 3.5] to evaluate $R(e^{-2\pi \sqrt{n/5}})$ and $S(e^{-\pi \sqrt{n/5}})$ in terms of $h'_{5,n}$ and $h_{5,n}$, respectively.

(1.3)  
$$R^5(e^{-2\pi \sqrt{n/5}}) = \frac{1}{11 - R^5(e^{-2\pi \sqrt{n/5}})} = 5h_{5,n}^4 \left(\frac{\sqrt{5} h_{5,n}^2 - 5}{\sqrt{5} h_{5,n}^2 - 1}\right),$$  

and

(1.4)  
$$S^5(e^{-\pi \sqrt{n/5}}) = \frac{1}{11 - S^5(e^{-\pi \sqrt{n/5}})} = 5h_{5,n}^4 \left(\frac{5 - \sqrt{5} h_{5,n}^2}{\sqrt{5} h_{5,n}^2 - 1}\right).$$  

From (1.3) and (1.4), we have

(1.5)  
$$R^5(e^{-2\pi \sqrt{n/5}}) = \sqrt{a^2 + 1 - a}, \text{ where } 2a = 11 + 5h_{5,n}^4 \left(\frac{\sqrt{5} h_{5,n}^2 - 5}{\sqrt{5} h_{5,n}^2 - 1}\right),$$  

and

(1.6)  
$$S^5(e^{-\pi \sqrt{n/5}}) = \sqrt{b^2 + 1 - b}, \text{ where } 2b = -11 + 5h_{5,n}^4 \left(\frac{5 - \sqrt{5} h_{5,n}^2}{\sqrt{5} h_{5,n}^2 - 1}\right).$$  

Consequently, in order to find the values of $R(e^{-2\pi \sqrt{n/5}})$ and $S(e^{-\pi \sqrt{n/5}})$, it suffices to compute $h'_{5,n}$ and $h_{5,n}$, respectively. Thus, in this paper, we use theta-function identities involving $h'_{5,n}$ and $h_{5,n}$ to find some new explicit values of the Rogers-Ramanujan continued fraction.

2. Evaluations of $h'_{5,n}$ and $h_{5,n}$

In this section, we compute the values of $h'_{5,n}$ and $h_{5,n}$ for some positive real numbers $n$ to evaluate the Rogers-Ramanujan continued fraction. We begin by recalling the known values of $h'_{5,1}$ and $h'_{5,5}$ in [17] which will play key roles in evaluating the Rogers-Ramanujan continued fraction.
Lemma 2.1 ([17, Theorems 4.16 and 5.6(v)]). We have

(i) \( h_{5,1}^2 = \frac{\sqrt{5} + 1 - \sqrt{2}}{\sqrt{5} - 1} \),

(ii) \( h_{5,5}^2 = \frac{1}{4}(5 + \sqrt{5})(\sqrt{5} - 1)^2 \).

We now need a couple of theta-function identities: one shows a relation between \( h_{5,n} \) and \( h_{5,n/4} \) and the other shows a relation between \( h_{5,n/4} \) and \( h_{5,n} \).

Lemma 2.2. For any positive real number \( n \), we have

\[
(2.1) \quad \frac{h_{5,n/4}^2}{h_{5,n}^2} + \frac{h_{5,n}^2}{h_{5,n/4}^2} + 4 = \sqrt{5} \left( h_{5,n}^2 + \frac{1}{h_{5,n}^2} \right).
\]

Proof. Let \( P = \frac{\varphi(-q)}{\varphi(-q^5)} \) and \( Q = \frac{\varphi(-q^2)}{\varphi(-q^{10})} \). Then \( \frac{P^2}{Q^2} + \frac{Q^2}{P^2} + 4 = Q^2 + \frac{5}{Q^2} \) by [2, Theorem 2.17]. Rewrite \( P \) and \( Q \) in terms of \( h_{5,n/4}^2 \) and \( h_{5,n}^2 \) to complete the proof. \( \square \)

Lemma 2.3 ([8, Corollary 3.4]). For every positive real number \( n \), we have

\[
(2.2) \quad \frac{h_{5,n}^2}{h_{5,n/4}^2} + \frac{h_{5,n/4}^2}{h_{5,n}^2} + 4 = \sqrt{5} \left( h_{5,n}^2 + \frac{1}{h_{5,n}^2} \right).
\]

Note also that (2.2) follows from a modular equation \( \frac{P^2}{Q^2} + \frac{Q^2}{P^2} + 4 = Q^2 + \frac{5}{Q^2} \) in [8, Theorem 3.3], where \( P = \frac{\varphi(q)}{\varphi(q^5)} \) and \( Q = \frac{\varphi(-q^2)}{\varphi(-q^{10})} \).

We are in position to evaluate \( h_{5,n}^2 \) for \( n = \frac{1}{4}, \frac{1}{16}, \) and \( \frac{1}{64} \).

Theorem 2.4. We have

(i) \( h_{5,1/4}^2 = 2 + \sqrt{5} - 2\sqrt{2 + \sqrt{5}} \),

(ii) \( h_{5,1/16}^2 = \frac{3 + 2\sqrt{5} - 2\sqrt{2 + 5\sqrt{5}} - 2\sqrt{11 + 7\sqrt{5} - 4\sqrt{22 + 10\sqrt{5}}} + \sqrt{5}}{-2 + \sqrt{5}} \),

(iii) \( h_{5,1/64}^2 = \frac{1}{\sqrt{5}} \left( 2c^2 - 2c + 1 - 2\sqrt{(c^2 + 1)(c^2 - 2c)} \right) \), where

\[
c = \frac{5 + 6\sqrt{5} - 5\sqrt{2 + 5\sqrt{5}} - 5\sqrt{11 + 7\sqrt{5} - 4\sqrt{22 + 10\sqrt{5}}}}{5 - 2\sqrt{5}}.
\]

Proof. For (i), let \( n = 1 \) in (2.1) and put \( h_{5,1/4}^2 = x \) and the value of \( h_{5,1}^2 \) in Lemma 2.1(i) to deduce that
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\[
x^2 - \left(3 + \sqrt{5} - 2\sqrt{2 + \sqrt{5}}\right)x + 2 + \sqrt{3} - 2\sqrt{2 + \sqrt{5}} = 0.
\]

Solving the last equation for \( x \) and using \( x < 1 \), we complete the proof.

For (ii), let \( n = \frac{1}{4} \) in (2.1). Putting \( h_{5,1/16}^2 = x \) and \( h_{5,1/4}^2 = 2 + \sqrt{5} - 2\sqrt{2 + \sqrt{5}} \) obtained in (i), we find that

\[
\left(2 + \sqrt{5} + 2\sqrt{2 + \sqrt{5}}\right)x^2 - (6 + 4\sqrt{5})x + 2 + \sqrt{3} + 2\sqrt{2 + \sqrt{5}} = 0.
\]

Solve the last equation for \( x \) with the help of Mathematica and use \( x < 1 \) to complete the proof.

The proof of (iii) is similar to that of (ii). \( \square \)

By letting \( n = \frac{1}{128} \) in (2.1) and using the value \( h_{5,1/64}^2 \) in Theorem 2.4(iii), we can evaluate \( h_{5,1/128}^2 \). Thus we can evaluate \( h_{5,1/4m}^2 \) for every positive integer \( m \).

We now evaluate \( h_{5,5}^2 \) for \( n = \frac{5}{4}, \frac{5}{16}, \text{ and } \frac{5}{64} \).

**Theorem 2.5.** We have

(i) \( h_{5,1/4}^2 = \frac{3 + 3\sqrt{2} + 2\sqrt{5} + \sqrt{10} - 4\sqrt{1 + \sqrt{5}} - 2\sqrt{2 + \sqrt{5}}}{5 - 2\sqrt{5}} \),

(ii) \( h_{5,1/16}^2 = \sqrt{5} \left(10c^2 - 14c + 5 - 2\sqrt{(c - 1)(5c^2 - 6c + 2)}\right) \), where \( c = \frac{4 - 2\sqrt{2 + 5\sqrt{5}} - \sqrt{11 + 7\sqrt{5} - 4\sqrt{22 + 10\sqrt{5}}}}{5 - 2\sqrt{5}} \),

(iii) \( h_{5,1/64}^2 = \frac{1}{\sqrt{5}} \left(2c^2 + 6c + 5 + 2\sqrt{(c^2 + 2c)(c^2 + 4c + 5)}\right) \), where \( c = \frac{(d + 1)(d + 2) - \sqrt{(d^2 + 1)(d^2 - 2d)} \text{ and}}{5 + 6\sqrt{5} - 5\sqrt{2 + 5\sqrt{5}} - 5\sqrt{11 + 7\sqrt{5} - 4\sqrt{22 + 10\sqrt{5}}}} \).

**Proof.** For (i), let \( n = \frac{1}{4} \) in (2.2) and put \( h_{5,1/4}^2 = x \) and \( h_{5,1/4}^2 = 2 + \sqrt{5} - 2\sqrt{2 + \sqrt{5}} \) in Theorem 2.4(i) to deduce that

\[
x^2 - \left(32 + 14\sqrt{5} - 4\sqrt{118 + 53\sqrt{5}}\right)x + \left(2 + \sqrt{5} - 2\sqrt{2 + \sqrt{5}}\right)^2 = 0.
\]

Solving the last equation for \( x \) with the help of Mathematica and using \( x > 1 \), we complete the proof.

The proofs of (ii) and (iii) are similar to that of (ii). \( \square \)

We now evaluate \( h_{5,n}^2 \) for \( n = \frac{5}{4}, \frac{5}{16}, \text{ and } \frac{5}{64} \).
Theorem 2.6. We have

(i) \( h_{5,5/4}^2 = 60 - 40\sqrt{5} + 27\sqrt{5} - 18\sqrt{5^3}, \)

(ii) \( h_{5,5/16}^2 = \frac{-95 + \sqrt{5} + 2\sqrt{5^3} + 4\sqrt{10 \left( 76 - 4\sqrt{5} - 15\sqrt{5} - 4\sqrt{5^3} \right)}}{19 - 68\sqrt{5} - 45\sqrt{5} + 6\sqrt{5^3}} , \)

(iii) \( h_{5,5/64}^2 = \frac{\sqrt{5}}{2} \left( 5c^2 - 4c + 1 - \sqrt{(c - 1)(5c - 1)(5c^2 - 2c + 1)} \right) , \) where

\[ c = \frac{1 + 2\sqrt{5} - 19\sqrt{5} + 4\sqrt{2 \left( 76 - 4\sqrt{5} - 15\sqrt{5} - 4\sqrt{5^3} \right)}}{19 - 68\sqrt{5} - 45\sqrt{5} + 6\sqrt{5^3}} . \]

Proof. For (i), let \( n = 5 \) in (2.1). Putting \( h_{5,5/4}^2 = x \) and \( h_{5,5}^2 = \frac{1}{2} (\sqrt{5} - 1) \sqrt{5 + \sqrt{5}} \) in Lemma 2.1(ii), we find that

\[ x^2 - (65 - 40\sqrt{5} + 25\sqrt{5} - 18\sqrt{5^3})x + 5(2 + \sqrt{5})(3 - 2\sqrt{5}) = 0 . \]

Solving the last equation for \( x \) and using \( x < \frac{1}{2} \), we complete the proof with the help of Mathematica.

For (ii) and (iii), repeat the same argument as in the proof of (i).

By letting \( n = \frac{5}{64} \) in (2.1) and using the value \( h_{5,5/64}^2 \) in Theorem 2.6(iii), we can evaluate \( h_{5,5/256}^2 \). Thus we can evaluate \( h_{5,5/4m}^2 \) for every positive integer \( m \).

We end this section by evaluating \( h_{5,n}^2 \) for \( n = \frac{5}{4}, \frac{5}{16}, \) and \( \frac{5}{64} \).

Theorem 2.7. We have

(i) \( h_{5,5/4}^2 \)

\[ = \frac{10 + 80\sqrt{5} - 49\sqrt{5} - 6\sqrt{5^3} - 2\sqrt{5 \left( 363 + 710\sqrt{5} + 281\sqrt{5} - 614\sqrt{5^3} \right)}}{-161 + 72\sqrt{5}} , \]

(ii) \( h_{5,5/16}^2 = \frac{\sqrt{5}}{2} \left( 5c^2 - 4c + 1 + \sqrt{(c - 1)(5c - 1)(5c^2 - 2c + 1)} \right) , \) where

\[ c = \frac{1 + 2\sqrt{5} - 19\sqrt{5} + 4\sqrt{2 \left( 76 - 4\sqrt{5} - 15\sqrt{5} - 4\sqrt{5^3} \right)}}{19 - 68\sqrt{5} - 45\sqrt{5} + 6\sqrt{5^3}} . \]

(iii) \( h_{5,5/64}^2 = \frac{\sqrt{5}}{2} \left( 5c^2 - 4c + 1 + \sqrt{(c - 1)(5c - 1)(5c^2 - 2c + 1)} \right) , \) where

\[ c = \frac{1}{2} \left( 5d^2 - 4d + 1 + \sqrt{(d - 1)(5d - 1)(5d^2 - 2d + 1)} \right) \) and

\[ d = \frac{1 + 2\sqrt{5} - 19\sqrt{5} + 4\sqrt{2 \left( 76 - 4\sqrt{5} - 15\sqrt{5} - 4\sqrt{5^3} \right)}}{19 - 68\sqrt{5} - 45\sqrt{5} + 6\sqrt{5^3}} . \]
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Proof. The results follow directly from (2.2) and Theorem 2.6(ii) and (iv) with the help of Mathematica. \qed

3. Evaluations of \( R(q) \) and \( S(q) \)

We first evaluate \( R(e^{-2\pi \sqrt{n}}) \) for \( n = \frac{1}{5^4 m} \), where \( m \) is any positive integer. We show the cases when \( m = 1, 2, \) and 3.

Theorem 3.1. We have

(i) \( R^5(e^{-\pi/\sqrt{5}}) \)
\[
= -\frac{1}{4} \left( 147 + 65\sqrt{5} - 25\sqrt{58 + 26\sqrt{5}} \right) + \frac{\sqrt{5}^3 \left( 5 - 4\sqrt{5} + \sqrt{2 + 5\sqrt{5}} \right)}{11 - 5\sqrt{5}},
\]

(ii) \( R^5(e^{-\pi/2\sqrt{5}}) \) = \( \frac{1}{2} (-a - 11 + \sqrt{a^2 + 22a + 225}) \), where
\[
a = \frac{10(11 - \sqrt{2} - 3\sqrt{10})^2 \left(30 + 31\sqrt{5} - 5\sqrt{142 + 113\sqrt{5}} \right)}{11(2 - \sqrt{5})^2(14 - \sqrt{2} - 2\sqrt{5} - 3\sqrt{10})},
\]

(iii) \( R^5(e^{-\pi/4\sqrt{5}}) \) = \( \frac{1}{2} (-a - 11 + \sqrt{a^2 + 22a + 125}) \), where
\[
a = \frac{(c - 2)(2c + 1) \left(2c^2 - 2c + 1 - 2\sqrt{(c^2 + 1)(c^2 - 2c)} \right)^2}{c^2 - 2c - \sqrt{(c^2 + 1)(c^2 - 2c)}}, \\
c = \frac{5 + 6\sqrt{5} - 5\sqrt{2 + 5\sqrt{5} - 5\sqrt{11 + 7\sqrt{5} - 4\sqrt{22 + 10\sqrt{5}}}}}{5 - 2\sqrt{5}}.
\]

Proof. The results follow directly from (1.5) and Theorem 2.4 with the help of Mathematica. \qed

See [16, Corollary 4.6(iv)] and [14, Theorem 3.13] for different proofs of Theorem 3.1(i). We next evaluate \( S(e^{-\pi \sqrt{m}}) \) for \( n = \frac{1}{5^4 m} \), where \( m \) is any positive integer. We show the cases when \( m = 1, 2, \) and 3.

Theorem 3.2. We have

(i) \( S^5(e^{-\pi/2\sqrt{5}}) \) = \( \frac{11}{2} + \frac{b(4b - 5)^2}{2(b - 1)} + \sqrt{1 + \left(\frac{11}{2} + \frac{b(4b - 5)^2}{2(b - 1)}\right)^2} \), where
\[
b = \frac{5 \left(2 - 3\sqrt{2} - 4\sqrt{5} - \sqrt{10 + 4\sqrt{1 + \sqrt{5} + 2\sqrt{2 + 5\sqrt{5}}}} \right)}{4(5 - 2\sqrt{5})},
\]

(ii) \( S^5(e^{-\pi/4\sqrt{5}}) \) = \( \frac{1}{2} \left(25b + 11 + 5\sqrt{25b^2 + 22b + 5} \right), \) where
The results follow directly from (1.5) and Theorem 2.6. We have
\[ b = \frac{5(2c^2 + c) \left(10c^2 + 6c + 1 + 2\sqrt{(5c^2 + 2c)(5c^2 + 4c + 1)}\right)^2}{5c^2 + 2c + \sqrt{(5c^2 + 2c)(5c^2 + 4c + 1)}} \]
and
\[ c = \frac{-1 + 2\sqrt{5} - \sqrt{2 + 5\sqrt{5} - \sqrt{11 + 7\sqrt{5} - 4\sqrt{22} + 10\sqrt{5}}}}{5 - 2\sqrt{5}}. \]

(iii) \(S^5(e^{-\pi/8\sqrt{5}}) = \frac{1}{2} \left(b + 11 + \sqrt{b^2 + 22b + 125}\right),\) where
\[ b = \frac{1}{4} \left(2c + 1\right)^4 \left(-c^2 + 2c + \sqrt{(c^2 + 1)(c^2 - 2c)}\right)^2, \]
\[ c = d^2 - d - \sqrt{(d^2 + 1)(d^2 - 2d)}, \text{ and} \]
\[ d = \frac{5 + 6\sqrt{5} - 5\sqrt{2} + 5\sqrt{5} - 5\sqrt{11 + 7\sqrt{5} - 4\sqrt{22} + 10\sqrt{5}}}{5 - 2\sqrt{5}}. \]

Proof. The results are immediate consequences of (1.6) and Theorem 2.5 with the help of Mathematica.

See [14, Theorem 3.13] for an alternative proof for Theorem 3.2(i). We now evaluate \(R(e^{-2\pi\sqrt{n}})\) for \(n = \frac{1}{4}\pi m\), where \(m\) is any positive integer. We show the cases when \(m = 1, 2,\) and 3.

Theorem 3.3. We have
(i) \(R^3(e^{-\pi}) = \frac{1}{2} \left(-25a - 11 + 5\sqrt{25a^2 + 22a + 5}\right),\) where
\[ a = \frac{5 \left(3 + 7\sqrt{5} - 2\sqrt{40 + 21\sqrt{5}}\right)}{123 - 55\sqrt{5}}. \]
(ii) \(R^3(e^{-\pi/2}) = -\frac{11}{2} - \frac{125a^2(a - 1)}{2(5a - 1)} + \sqrt{1 + \left(\frac{11}{2} + \frac{125a^2(a - 1)}{2(5a - 1)}\right)^2},\) where
\[ a = \frac{1 + 2\sqrt{5} - 19\sqrt{5} + 4\sqrt{2(76 - 4\sqrt{5} - 15\sqrt{5} - 4\sqrt{53})}}{19 - 68\sqrt{5} - 45\sqrt{5} + 6\sqrt{53}}. \]
(iii) \(R^3(e^{-\pi/4}) = \frac{1}{4} (-25a - 22 + 5\sqrt{25a^2 + 44a + 20}),\) where
\[ a = \frac{5c(c - 1) \left(5c^2 - 4c + 1 - \sqrt{(c - 1)(5c^2 - 2c + 1)\left(5c^2 - 2c + 1\right)}\right)^2}{5c^2 - 6c + 1 - \sqrt{(c - 1)(5c^2 - 2c + 1)(5c^2 - 2c + 1)}} \]
and
\[ c = \frac{1 + 2\sqrt{5} - 19\sqrt{5} + 4\sqrt{2(76 - 4\sqrt{5} - 15\sqrt{5} - 4\sqrt{53})}}{19 - 68\sqrt{5} - 45\sqrt{5} + 6\sqrt{53}}. \]

Proof. The results follow directly from (1.5) and Theorem 2.6. We used Mathematica to verify (i)–(iii).
See [16, Corollary 3.8(ii)] and [15, Theorem 2.1] for different proofs for Theorem 3.3(i). See also [16, Corollary 3.10(ii)] for a different proof for Theorem 3.3(ii).

We end this section by evaluating \( S(e^{-\pi/4}) \) for \( n = \frac{1}{4}m \), where \( m \) is any positive integer. We show the cases when \( m = 1, 2, \) and \( 3. \)

**Theorem 3.4.** We have

(i) \( S^5(e^{-\pi/2}) = \frac{11}{2} + \frac{b^2(b - 5)}{2(b - 1)} + \sqrt{1 + \left( \frac{11}{2} + \frac{b^2(b - 5)}{2(b - 1)} \right)^2} \), where

\[
b = \frac{5(49 + 6\sqrt{5} - 2\sqrt{5} - 16\sqrt{5}^3 + 2\sqrt{363} + 710\sqrt{5} - 281\sqrt{5}^3)}{161 - 72\sqrt{5}^3},
\]

(ii) \( S^5(e^{-\pi/4}) = \frac{1}{2} \left( 25b + 11 + 5\sqrt{25b^2 + 22b + 5} \right) \), where

\[
b = \frac{5(c^2 - c) \left( 5c^2 - 4c + 1 + \sqrt{(c - 1)(5c - 1)(5c^2 - 2c + 1)} \right)^2}{2 \left( 5c^2 - 6c + 1 + \sqrt{(c - 1)(5c - 1)(5c^2 - 2c + 1)} \right)}, \text{ and }
\]

\[
c = \frac{1 + 2\sqrt{5} - 19\sqrt{5} + 4\sqrt{2(76 - 4\sqrt{5} - 15\sqrt{5} - 4\sqrt{5}^3)}}{19 - 68\sqrt{5} - 45\sqrt{5} + 6\sqrt{5}^3},
\]

(iii) \( S^5(e^{-\pi/8}) = \frac{1}{2} \left( 25b + 11 + 5\sqrt{25b^2 + 22b + 5} \right) \), where

\[
b = \frac{5(c^2 - c) \left( 5c^2 - 4c + 1 + \sqrt{(c - 1)(5c - 1)(5c^2 - 2c + 1)} \right)^2}{2 \left( 5c^2 - 6c + 1 + \sqrt{(c - 1)(5c - 1)(5c^2 - 2c + 1)} \right)}, \text{ and }
\]

\[
c = \frac{1}{2} \left( 5d^2 - 4d + 1 - \sqrt{(d - 1)(5d - 1)(5d^2 - 2d + 1)} \right), \text{ and }
\]

\[
d = \frac{1 + 2\sqrt{5} - 19\sqrt{5} + 4\sqrt{2(76 - 4\sqrt{5} - 15\sqrt{5} - 4\sqrt{5}^3)}}{19 - 68\sqrt{5} - 45\sqrt{5} + 6\sqrt{5}^3}.
\]

**Proof.** The results follow from (1.6) and Theorem 2.7. We used Mathematica to verify (i)–(iii).

**References**

14. N. Saikia: Some new explicit values of the parameters s_n and t_n connected with the Rogers-Ramanujan continued fraction and applications. Afr. Mat. 25 (2014), 961-973.

aPh.D: Department of Mathematics and Computer Science, Korea Science Academy of KAIST, Busan 47162, Korea
Email address: jhyi100@ksea.kaist.ac.kr

bProfessor: Department of Mathematics Education, Busan National University of Education, Busan 47503, Korea
Email address: daehyunpaek@gmail.com