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HESITANT FUZZY PARAOPEN AND HESITANT FUZZY MEAN OPEN SETS

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Abstract. The aim of this article is to introduce hesitant fuzzy paraopen and hesitant fuzzy mean open sets in hesitant fuzzy topological spaces. Moreover we investigate and extend some properties of hesitant fuzzy mean open with hesitant fuzzy paraopen, hesitant fuzzy minmimal open and maximal open sets in hesitant fuzzy topological spaces.

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1. Introduction

As an addendum to fuzzy sets[9], the notion hesitant fuzzy set introduced by Torra[8] in 2010. Deepak et. al. [2] introduced hesitant fuzzy topological space and extended the study to hesitant connectedness and compactness in hesitant fuzzy topological space. There are some certain fuzzy sets which are neither fuzzy minimal nor fuzzy maximal called as fuzzy mean open sets[6].Fuzzy mean open sets and fuzzy mean closed sets are complements to each other. The idea of hesitant fuzzy minimal and maximal open sets introduced by Swaminathan and Sivaraja [7].

Therefore from these notions we introduce and extend in this article some properties and connections of hesitant fuzzy mean open sets.

2. Preliminaries

Definition 2.1. [4] A hesitant fuzzy set h in X is a function $h : X \to P\,[0,1]$,the power set of [0,1].

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The hesitant fuzzy empty set h^0(resp. whole set h^1) is a hesitant fuzzy set in X are defined as follows: $h^0(x) = \phi$ (resp. $h^1(x) = [0,1]), \forall x \in X$. $HS(X)$ stands for collection of hesitant fuzzy set in X.

Definition 2.2. [3] If $h_1(x) \subset h_2(x), \forall x \in X$, for any two hesitant fuzzy set $h_1, h_2 \in HS(X)$, then h_1 is contained in h_2.

Definition 2.3. [3] Two hesitant fuzzy set h_1 and h_2 of X are said to be equal if $h_1 \subset h_2$ and $h_2 \subset h_1$.

Definition 2.4. [4] Let $h \in HS(X)$ for any nonempty set X. Then h^c is the complement of h which is hesitant fuzzy set in X such that $h^c(x) = [h(x)]^c = [0,1] \backslash h(x)$.

Definition 2.5. [5] Suppose that (X, τ) is a hesitant fuzzy topological space such that $x_\lambda \in H_p(X)$ and $N \in HS(\eta)$. Then the hesitant fuzzy neighbourhood N of x_λ is defined as if for an hestitant fuzzy set $U \in \tau$ such that $x_\lambda \in U \subset N$.

Definition 2.6. [5] Suppose that X is a nonempty set. A hesitant fuzzy topology τ of subsets X is said to be hesitant fuzzy topology on X if
(i) $h^0, h^1 \in \tau$.
(ii) $\bigcup_{i \in J} h_i \in \tau$ for each $(h_i)_{i \in J} \subset \tau$.
(iii) $h_1 \cap h_2 \in \tau$ for any $h_1,\ h_2 \in \tau$.

"The pair (X, τ) is called hesitant fuzzy topological space. The members of τ are called hesitant fuzzy open sets in X. A hesitant fuzzy set h in X with $h^c \in \tau$ is hesitant fuzzy closed set (in short hesitant fuzzy closed) in (X, τ)."

Definition 2.7. [7]*A proper nonzero hesitant fuzzy open set ξ of X is said to be*
(i)hesitant fuzzy minimal open set if ξ and h^0 are only hesitant fuzzy open sets contained in ξ.
(ii)hesitant fuzzy maximal open set if h^1 and ξ are only hesitant fuzzy open sets containing ξ.

Theorem 2.8. [7] If ξ is a hesitant fuzzy maximal open set then either $\xi \vee \upsilon = h^1$ or $\upsilon < \xi$, for any proper nonzero hesitant fuzzy open subset υ.

Theorem 2.9. [7] If ξ is a hesitant fuzzy minimal open set then either $\xi \wedge \upsilon = h^0$ or $\xi < \upsilon$, for any proper nonzero hesitant fuzzy open subset υ.

Definition 2.10. [7]*A proper nonzero hesitant fuzzy closed set η of X is said to be*
(i) hesitant fuzzy maximal closed set if any hesitant fuzzy closed set which contains η is h^1 or η.
(ii)hesitant fuzzy minimal closed set if any hesitant fuzzy closed set which is contained in η is h^0 or η.

Theorem 2.11. [7] If η is a hesitant fuzzy maximal closed set then either $\eta \vee \omega = h^1$ or $\omega < \eta$, for any proper nonzero hesitant fuzzy closed set ω.

Theorem 2.12. [7] If η is a hesitant fuzzy minimal closed set then either $\eta \wedge \omega = h^0$ or $\eta < \omega$, for any proper nonzero hesitant fuzzy closed set ω.

3. Hesitant Fuzzy Paraopen Open and Closed Sets

Definition 3.1. *A hesitant fuzzy open set υ of a hesitant fuzzy topological space X is said to be a hesitant fuzzy paraopen set if it is neither hesitant fuzzy minimal open nor hesitant fuzzy maximal open set. The family of all hesitant fuzzy paraopen sets in a hesitant fuzzy topological space X is denoted by $HFPaO(X)$.*

Definition 3.2. *A hesitant fuzzy closed set ξ of a hesitant fuzzy topological space X is said to be a hesitant fuzzy paraclosed set iff its complement $h^1 - \xi$ is hesitant fuzzy paraopen set. The family of all hesitant fuzzy paraclosed Sets in a hesitant fuzzy topological space X is denoted by $HFPaC(X)$.*

Remark 3.1. *The following example proves that the converse is need not to be true of the statement: Every hesitant fuzzy paraopen set is a hesitant fuzzy open set and every hesitant fuzzy paraclosed set is a hesitant fuzzy closed set .*

Example 3.3. *Suppose that $X = \{a, b, c\}$ and $\tau = \{h^0, \xi_1, \xi_2, \xi_3, \xi_4, h^1\}$ where*
$\xi_1(a) = \{0.2, 0.3, 0.4\}$,$\xi_1(b) = [0, 0.4]$,$\xi_1(c) = \{0.3, 0.5\}$
$\xi_2(a) = \{0.2\}$,$\xi_2(b) = [0, 0.4]$,$\xi_2(c) = \{0.3, 0.5\}$
$\xi_3(a) = \{0.2, 0.3, 0.4\}$,$\xi_3(b) = [0, 0.4]$,$\xi_3(c) = \{0.3, 0.5, 0.7\}$ *and*
$\xi_4(a) = \{0.2\}$,$\xi_4(b) = [0, 0.4]$,$\xi_4(c) = \{0.3, 0.5, 0.7\}$
Hence (X, τ) is hesitant fuzzy topological space. Here the Hesitant fuzzy sets $\{\xi_1, \xi_4\}$ are hesitant fuzzy paraopen sets. But hesitant fuzzy minimal open set $\{\xi_2\}$ and hesitant fuzzy maximal open set $\{\xi_3\}$ are hesitant fuzzy open but not hesitant fuzzy paraopen sets. Similarly hesitant fuzzy maximal closed set $\{\xi_2^c\}$ and hesitant fuzzy minimal closed set $\{\xi_3^c\}$ are hesitant fuzzy closed sets but not hesitant fuzzy paraclosed set.

Remark 3.2. *Union and intersection of hesitant fuzzy paraopen (resp. hesitant fuzzy paraclosed) sets need not be a hesitant fuzzy paraopen (resp. hesitant fuzzy paraclosed) set which is shown by the following example.*

Example 3.4. *In Example 3.3, hesitant fuzzy sets $\{\xi_1, \xi_4\}$ are hesitant fuzzy paraopen sets but $\xi_1 \vee \xi_4 = \xi_3$ and $\xi_1 \wedge \xi_4 = \xi_2$ which are not hesitant fuzzy paraopen sets.*
Similarly for the hesitant fuzzy paraclosed sets $\{\xi_1^c, \xi_4^c\}$ but $\xi_1^c \vee \xi_4^c = \xi_2^c$ and $\xi_1^c \wedge \xi_4^c = \xi_3^c$ which are not hesitant fuzzy paraclosed sets.

Theorem 3.5. Let X be a hesitant fuzzy topological space and ξ be a nonempty proper hesitant fuzzy paraopen subset of X then there exists a hesitant fuzzy minimal open set υ with $\upsilon < \xi$.

Proof. By the definition of hesitant fuzzy minimal open set, we can write $\upsilon < \xi$. □

Theorem 3.6. Let X be a hesitant fuzzy topological space and ξ be a nonempty proper hesitant fuzzy paraopen subset of X then there exists a hesitant fuzzy maximal open set σ with $\xi < \sigma$.

Proof. By the definition of hesitant fuzzy maximal open set, we can write $\xi < \sigma$. □

Theorem 3.7. Suppose that X is a hesitant fuzzy topological space,then
(i) $\xi \wedge \upsilon = h^0$ or $\upsilon < \xi$ for any hesitant fuzzy paraopen ξ and a hesitant fuzzy minimal open set υ.
(ii) $\xi \vee \vartheta = h^1$ or $\xi < \vartheta$ for any hesitant fuzzy paraopen ξ and a hesitant fuzzy maximal open set ϑ.
(iii)Intersection of hesitant fuzzy paraopen sets is either hesitant fuzzy paraopen or hesitant fuzzy minimal open set.

Proof. (i) For any hesitant fuzzy paraopen set ξ and a hesitant fuzzy minimal open open set υ in X. Then $\xi \wedge \upsilon = h^0$ or $\xi \wedge \upsilon \neq h^0$. If$\xi \wedge \upsilon = h^0$, then we need not to prove anything. Assume $\xi \wedge \upsilon \neq h^0$. Then we write $\xi \wedge \upsilon$ is a hesitant fuzzy open set and $\xi \wedge \upsilon < \upsilon$. Hence $\upsilon < \xi$.

(ii) For any hesitant fuzzy paraopen set ξ and a hesitant fuzzy maximal open set γ in X. Then $\xi \vee \gamma = h^1$ or $\xi \vee \upsilon \neq h^1$.If $\xi \vee \gamma = h^1$, then we need not to prove anything. Assume $\xi \vee \gamma \neq h^1$. Then we write $\xi \vee \gamma$ is a hesitant fuzzy open set and $\gamma < \xi \vee \gamma$. Hence γ is a hesitant fuzzy maximal open set, $\xi \vee \gamma = \gamma$ which implies $\xi < \gamma$.

(iii)Let ξ and η be a hesitant fuzzy paraopen sets in X. If $\xi \wedge \eta$ is a hesitant fuzzy paraopen set then we need not to prove anything. Suppose $\xi \wedge \eta$ is not a hesitant fuzzy paraopen set.By definition, $\xi \wedge \eta$ is a hesitant fuzzy minimal open or hesitant fuzzy maximal open set. If $\xi \wedge \eta$ is a hesitant fuzzy minimal open set, then we need not to prove anything. Suppose $\xi \wedge \eta$ is a hesitant fuzzy maximal open set. Now $\xi \wedge \eta < \xi$ and $\xi \wedge \eta < \eta$ which contradicts the fact that ξ and η are hesitant fuzzy paraopen sets. Therefore $\xi \wedge \eta$ is not a hesitant fuzzy maximal open set. That is $\xi \wedge \eta$ is a hesitant fuzzy minimal open set. □

Theorem 3.8. Let X be a hesitant fuzzy topological space. A subset ϑ of X is hesitant fuzzy paraclosed iff it is neither hesitant fuzzy maximal closed nor hesitant fuzzy minimal closed set.

Proof. By the definition of hesitant fuzzy maximal closed set and the fact that the complement of hesitant fuzzy minimal open set is hesitant fuzzy maximal closed set and the complement of hesitant fuzzy maximal open set is hesitant fuzzy minimal closed set. □

Theorem 3.9. Let X be a hesitant fuzzy topological space and ϑ be a nonempty hesitant fuzzy paraclosed subset of X. Then there exists a hesitant fuzzy minimal closed set σ with $\sigma < \vartheta$.

Proof. clearly by hesitant fuzzy minimal closed set definition , it follows that $\sigma < \vartheta$. □

Theorem 3.10. Suppose that ϑ is a nonempty hesitant fuzzy paraclosed subset of hesitant fuzzy topological space X then there exists a hesitant fuzzy maximal closed set ρ such that $\vartheta < \rho$.

Proof. Clearly by hesitant fuzzy maximal closed set definition,it follows that $\vartheta < \rho$. □

Theorem 3.11. Suppose that X is a hesitant fuzzy topological space then
(i) $\zeta \wedge \omega = h^0$ or $\omega < \zeta$ for any hesitant fuzzy paraclosed set ζ and hesitant fuzzy minimal closed set ω.
(ii) $\zeta \vee \gamma = h^1$ or $\zeta < \gamma$ for any hesitant fuzzy paraclosed set ζ and hesitant fuzzy maximal closed set γ .
(iii) Intersection of hesitant fuzzy paraclosed sets is either hesitant fuzzy paraclosed or hesitant fuzzy minimal closed set.

Proof. (i) Suppose that ζ is a hesitant fuzzy paraclosed and ω is a hesitant fuzzy minimal closed set in X. Then $(h^1 - \zeta)$ is hesitant fuzzy paraopen and $(h^1 - \omega)$ is hesitant fuzzy maximal open set in X. Then by Theorem 3.7(ii) we have $(h^1 - \zeta) \vee (h^1 - \omega) = X$ or $(h^1 - \zeta) < (h^1 - \omega)$ implies that $h^1 - (\zeta \wedge \omega) = h^1$ or $\omega < \zeta$. Hence, $\zeta \wedge \omega = h^0$ or $\omega < \zeta$.

(ii) Suppose that ζ is a hesitant fuzzy paraclosed and γ is a hesitant fuzzy maximal closed set in X. Then $(h^1 - \zeta)$ is hesitant fuzzy paraopen and $(h^1 - \gamma)$ is hesitant fuzzy minimal open sets in X. Then by Theorem 3.7(i) we have $(h^1 - \zeta) \wedge (h^1 - \gamma) = h^0$ or $h^1 - \gamma < h^1 - \zeta$ implies that $h^1 - (\zeta \vee \gamma) = h^0$ or $\zeta < \gamma$. Hence, $\zeta \vee \gamma = h^1$ or $\zeta < \gamma$.

(iii)Suppose that ζ and η is a hesitant fuzzy paraclosed sets in X. If $\zeta \wedge \eta$ is a hesitant fuzzy paraclosed set then we need not to prove anything. Suppose $\zeta \wedge \eta$ is not a hesitant fuzzy paraclosed set. Then clearly, $\zeta \wedge \eta$ is a hesitant fuzzy minimal closed or hesitant fuzzy maximal closed set. Suppose $\zeta \wedge \eta$ is a hesitant fuzzy minimal closed set then we need not to prove anything. Suppose $\zeta \wedge \eta$ is a hesitant fuzzy maximal closed set. Now $\zeta < \zeta \wedge \eta$ and $\eta < \zeta \wedge \eta$ contradicts the fact that ζ and η are hesitant fuzzy paraclosed sets. Hence, $\zeta \wedge \eta$ is not a hesitant fuzzy maximal closed set. That is $\zeta \wedge \eta$ is a hesitant fuzzy minimal closed set. □

4. Hesitant Fuzzy Mean Open and Closed Sets

In this section we introduce hesitant fuzzy mean open and hesitant fuzzy mean closed sets.

Definition 4.1. If there exist two distinct nonempty proper hesitant fuzzy open sets $\omega_1, \omega_2 (\neq \beta)$ with $\omega_1 < \beta < \omega_2$ for any hesitant fuzzy open set $\beta \in X$ then β is hesitant fuzzy mean open in X.

Remark 4.1. Union and intersection of hesitant fuzzy mean open sets may not be hesitant fuzzy mean open sets which depicted in next example.

Example 4.2. Let $X = \{a, b, c\}$ and $\tau = \{h^0, \xi_1, \xi_2, \xi_3, \xi_4, h^1\}$ where
$\xi_1(a) = \{0.2, 0.3\}$,$\xi_2(b) = \{0.3, 0.4\}$,$\xi_3(c) = \{0.4, 0.5\}$
$\xi_2(a) = \{0.2, 0.3, 0.4\}$,$\xi_2(b) = \{0.3, 0.4\}$,$\xi_3(c) = \{0.4, 0.5\}$
$\xi_3(a) = \{0.2, 0.3, 0.4\}$,$\xi_2(b) = \{0.3, 0.4\}$,$\xi_3(c) = \{0.4, 0.5\}$ and

$\xi_4(a) = \{0.2, 0.3, 0.4\}$,$\xi_2(b) = \{0.3, 0.4, 0.5\}$,$\xi_3(c) = \{0.4, 0.5, 0.6\}$.
Hence (X, τ) is hesitant fuzzy topological space. Obviously ξ_2 and ξ_3 are hesitant fuzzy mean open sets. Now the union and intersection $\xi_2 \vee \xi_3 = \xi_4$ and $\xi_2 \wedge \xi_3 = \xi_1$ are not hesitant fuzzy mean open sets.

Definition 4.3. If there exist two distinct proper hesitant fuzzy closed sets $\zeta_1, \zeta_2 (\neq \varrho)$ with $\zeta_1 < \varrho < \zeta_2$ for any hesitant fuzzy closed set $\varrho \in X$ then ϱ is hesitant fuzzy mean closed .

Theorem 4.4. *Both hesitant fuzzy mean open sets and hesitant fuzzy mean closed sets of a hesitant fuzzy topological space are complement to each other.*

Proof. Suppose that β is a hesitant fuzzy mean open set in X.Obviously, hesitant fuzzy open sets $\omega_1 \neq h^0, \beta$ and $\omega_2 \neq \beta, h^1$ with $\omega_1 < \beta < \omega_2$ and $h^1 - \omega_2 < h^1 - \beta < h^1 - \omega_1$.As $h^1 - \omega_2 \neq h^0, h^1 - \beta$ and $h^1 - \omega_1 \neq h^1 - \beta, h^1$; hence $h^1 - \beta$ is a hesitant fuzzy mean closed set.

Conversely, Suppose that β is a hesitant fuzzy mean open set with $h^1 - \beta$ is a hesitant fuzzy mean closed set. Now we have hesitant fuzzy closed sets $\zeta_1 \neq h^0, h^1 - \beta$ and $\zeta_2 \neq h^1 - \beta, h^1$ such that $\zeta_1 < h^1 - \beta < \zeta_2$. It means that $h^1 - \zeta_2 < \beta < h^1 - \zeta_1$. Since $h^1 - \zeta_2 \neq h^0, \beta$ and $h^1 - \zeta_1 \neq \beta, h^1$ and so β is a hesitant fuzzy mean open set.

□

Theorem 4.5. *A proper hesitant fuzzy open set is hesitant fuzzy paraopen set iff it is a hesitant fuzzy mean open set.*

Proof. Suppose that G is a proper hesitant fuzzy paraopen set, then we have $\zeta_1 < G < \zeta_2$ for any 2 proper hesitant fuzzy open sets ζ_1, ζ_2 we conclude that it is a hesitant fuzzy mean open set.

Conversely, Suppose that β is a hesitant fuzzy mean open set in h^1. Then there exist proper hesitant fuzzy open sets $\xi_1, \xi_2 \neq \beta$ with $\xi_1 < \beta < \xi_2$. As $\xi_1 \neq h^0, \beta$ and $\xi_2 \neq h^1, \beta$,β is neither a hesitant fuzzy minimal open nor a hesitant fuzzy maximal open set. Hence $\beta \neq h^0, h^1$ and β is a proper hesitant fuzzy paraopen set. □

Theorem 4.6. *A proper hesitant fuzzy open is hesitant fuzzy paraclosed set iff it is a hesitant fuzzy mean closed set.*

Proof. For any nonempty proper hesitant fuzzy paraclosed set κ, then we have $\zeta_1 < \kappa < \zeta_2$ for any 2 proper hesitant fuzzy closed sets ζ_1, ζ_2 we conclude that κ is a hesitant fuzzy mean closed set.

Conversely, Suppose that κ is hesitant fuzzy mean closed in X. Then there exist proper hesitant fuzzy closed sets $\varrho_1, \varrho_2 \neq \kappa$ with $\varrho_1 < \kappa < \varrho_2$. As $\varrho_1 \neq h^0, \kappa$ and $\varrho_2 \neq h^1, \kappa$, κ is neither a hesitant fuzzy minimal closed nor a hesitant fuzzy maximal closed set. Therefore $\kappa \neq h^0, h^1$, κ is a proper hesitant fuzzy paraclosed set.

□

Theorem 4.7. (**[6]**) *(i) If ξ is a hesitant fuzzy minimal open and ζ is a hesitant fuzzy open sets in X, then $\xi \wedge \zeta = h^0$ or $\xi < \zeta$.*
(ii)If ξ and φ are hesitant fuzzy minimal open sets, then $\xi \wedge \zeta = h^0$ or $\xi = \zeta$.

Theorem 4.8. (**[6]**) *(i) If ξ is a hesitant fuzzy maximal open and ζ is a hesitant fuzzy open sets in X, then $\xi \vee \zeta = h^1$ or $\zeta < \xi$.*
(ii)If ξ and φ are hesitant fuzzy maximal open set, then $\xi \vee \varphi = h^1$ or $\xi = \varphi$.

Theorem 4.9. (**[6]**) *If ζ_1 and ζ_2 are hesitant fuzzy maximal open set and hesitant fuzzy minimal open set respectively of an hesitant fuzzy topological space X, then either $\zeta_2 < \zeta_1$ or the space is HFD.*

Theorem 4.10. *Suppose that X contains a hesitant fuzzy maximal open set ξ_2, a hesitant fuzzy minimal open set $\xi_1 \neq \xi_2$ and a proper hesitant fuzzy open set $\zeta \neq \xi_1, \xi_2$.Then any one of the succeeding is true:*
(i) $\xi_1 < \zeta < \xi_2$ implies ζ is a hesitant fuzzy mean open set .
(ii)$\xi_1 < h^1 - \zeta < \xi_2$.
(iii) $\xi_1 < \zeta, \xi_1 \vee \zeta = h^1$ and $\xi_2 \wedge \zeta \neq h^0$.
(iv)$\zeta < \xi_2, \xi_1 \wedge \xi_2 = h^0$ and $\xi_1 \vee \xi_2 \neq h^1$.

Proof. By theorem 4.9 , we can write $\xi_1 < \xi_2$. Since ξ_1 is a hesitant fuzzy minimal open set and ξ_2 is a hesitant fuzzy maximal open set, we have $\xi_1 < \zeta$ or $\xi_1 \wedge \zeta = h^0$ and $\zeta < \xi_2$ or $\xi_2 \vee \zeta = h^1$.The feasible combinations are (i) $\xi_1 < \zeta < \xi_2$, (ii) $\xi_1 \wedge \zeta = h^0$ and $\xi_2 \vee \zeta = h^1$, (iii) $\xi_1 < \zeta$ and $\xi_2 \vee \zeta = h^1$, (iv) $\xi_1 \wedge \zeta = h^0$ and $\zeta < \xi_2$. $\xi_1 \wedge \zeta = h^0$ and $\xi_2 \vee \zeta = h^1$ imply that $\xi_1 < h^1 - \zeta < \xi_2$. If $\xi_1 < \zeta$ and $\xi_2 \vee \zeta = h^1$, then $h^0 \neq \xi_1 < \xi_2 \wedge \zeta$ since $\xi_1 < \xi_2$.If $\xi_1 \wedge \zeta = h^0$ and $\zeta < \xi_2$, then $\xi_1 \vee \zeta < \xi_2 \neq h^1$ since $\xi_1 < \xi_2$.

By assuming (i),(ii) as true, then clearly $\xi_1 < \zeta \vee (h^1 - \zeta) < \xi_2$ and $\xi_1 < \zeta \wedge (h^1 - \zeta) < \xi_2$. $\xi_1 < \zeta \vee (h^1 - \zeta) < \xi_2$ gives $\xi_1 < h^1 < \xi_2$ and then we get $\xi_2 = h^1$, which is false. Also $\xi_1 < \zeta \wedge (h^1 - \zeta) < \xi_2$ gives $\xi_1 < h^0 < \xi_2$ and then we get $\xi_1 = h^0$, again which is false.

By assuming (i),(iii) as true,we have $\zeta < \xi_2$ and $\xi_2 \vee \zeta = h^1 \implies \xi_2 = h^1$, which is false.

By assuming (i),(iv) as true,we have $\xi_1 < \zeta$ and $\xi_1 \wedge \zeta = h^0 \implies \xi_1 = h^0$, which is false.

By assuming (ii),(iii) as true,we have $\xi_1 < h^1 - \zeta$ and $\xi_1 < \zeta \implies \xi_1 < (h^1 - \zeta) \wedge \zeta = h^0$ and thus $\xi_1 = h^0$, which is false.

By assuming (ii),(iv) as true, we have $h^1 - \zeta < \xi_2$ and $\zeta < \xi_2 \implies (h^1 - \zeta) \vee \zeta = h^1 < \xi_2$ and thus $\xi_2 = h^1$, which is false.

By assuming (iii),(iv) as true,we have $\xi_1 < \zeta < \xi_2, \xi_2 \vee \zeta = h^1$ and $\xi_1 \wedge \zeta = h^0$. $\zeta < \xi_2$ and $\xi_2 \vee \zeta = h^1$ give $\xi_2 = h^1$, which is false. $\xi_1 < \zeta$ and $\xi_1 \wedge \zeta = h^0$ give $\xi_1 = h^0$, again which is false. □

Theorem 4.11. *Let X contain a distinct hesitant fuzzy minimal closed set ϱ_1 and hesitant fuzzy maximal closed set ϱ_2 and a proper hesitant fuzzy closed set $\upsilon \neq \varrho_1, \varrho_2$.Then any one of the succeeding is true:*

(i) upsilon is hesitant fuzzy mean closed set with $\varrho_1 < \upsilon < \varrho_2$.
(ii) $\varrho_1 < h^1 - \upsilon < \varrho_2$.
(iii) $\upsilon < \varrho_2, \varrho_1 \wedge \upsilon = h^0$ *and* $\varrho_1 \vee \upsilon \neq h^1$.
(iv)$\varrho_1 < \upsilon, \varrho_2 \vee \upsilon = h^1$ *and* $\varrho_2 \wedge \upsilon \neq h^0$.

Proof. Note that X contains a hesitant fuzzy maximal open set $h^1-\varrho_1$, a hesitant fuzzy minimal open set $h^1 - \varrho_2$ and a proper hesitant fuzzy open set $h^1 - \upsilon$ with $h^1 - \varrho_1 \neq h^1 - \varrho_2$ and $h^1 - \upsilon \neq h^1 - \varrho_1, h^1 - \varrho_2$. By Theorem 4.10, anyone of the following is true:

(i) $h^1 - \upsilon$ is hesitant fuzzy mean open set with $h^1 - \varrho_2 < h^1 - \upsilon < h^1 - \varrho_1$ which leads to $\varrho_1 < \upsilon < \varrho_2$ Theorem 4.4 , we see that υ is a hesitant fuzzy mean closed set.
(ii) $h^1 - \varrho_2 < h^1 - (h^1 - \upsilon) < h^1 - \varrho_1$, which implies $\varrho_1 < h^1 - \upsilon < \varrho_2$.
(iii) $h^1 - \varrho_2 < h^1 - \upsilon, (h^1 - \varrho_1) \vee (h^1 - \upsilon) = h^1$ and $(h^1 - \varrho_1) \wedge (h^1 - \upsilon) \neq h^0$ which implies $\upsilon < \varrho_2, \varrho_1 \wedge \upsilon = h^0$ and $\varrho_1 \vee \upsilon \neq h^1$.
(iv) $h^1 - \upsilon < h^1 - \varrho_1, (h^1 - \varrho_2) \wedge (h^1 - \upsilon) = h^0$ and $(h^1 - \varrho_2) \vee (h^1 - \upsilon) \neq h^1$ $\Longrightarrow$ $\varrho_1 < \upsilon, \varrho_2 \vee \upsilon = h^1$ and $\varrho_2 \wedge \upsilon \neq h^0$. □

Theorem 4.12. *Suppose that there exists distinct hesitant fuzzy maximal open sets and a hesitant fuzzy mean open sets in a hesitant fuzzy topological space then intersection of the two hesitant fuzzy maximal open is nonempty.*

Proof. Suppose that φ_1, φ_2 are distinct hesitant fuzzy maximal open and β is a hesitant fuzzy mean open set in a hesitant fuzzy topological space X.Clearly, $\varphi_1 \vee \varphi_2 = h^1$ by Theorem 4.8. As β is a hesitant fuzzy mean open set, it is neither hesitant fuzzy maximal open nor hesitant fuzzy minimal open which means that,$\beta \neq \varphi_1, \varphi_2$. Also $\beta \neq h^1$. By Theorem 4.8, we get $\beta \gneqq \varphi_1$ or $\beta \vee \varphi_1 = h^1$ and $\beta \gneqq \varphi_2$ or $\beta \vee \varphi_2 = h^1$. The feasible possibilities are (I) $\beta \gneqq \varphi_1$ and $\beta \gneqq \varphi_2$, (II) $\beta \gneqq \varphi_1$ and $\beta \vee \varphi_2 = h^1$, (III) $\beta \vee \varphi_1 = h^1$ and $\beta \gneqq \varphi_2$ and (IV) $\beta \vee \varphi_1 = h^1$ and $\beta \vee \varphi_2 = h^1$.
Case I: Obviously, $\varphi_1 \wedge \varphi_2 \neq h^0$ if $\beta \gneqq \varphi_1$ and $\beta \gneqq \varphi_2$.
Case II:If $\beta \wedge \varphi_2 \neq h^0$,then obviously $\varphi_1 \wedge \varphi_2 \neq h^0$. Now suppose $\beta \wedge \varphi_2 \neq h^0$. As $\beta \gneqq \varphi_1$, then there exists $x_\alpha \in \varphi_1$ such that $x_\alpha \notin \varphi_2$. Siince $\beta \vee \varphi_2 = h^1$, $x_\alpha \in \varphi_2$. So $\varphi_1 \wedge \varphi_2 \neq h^0$.
Case III: Similar to Case II.
Case IV: $\beta \vee \varphi_1 = h^1$ and $\beta \vee \varphi_2 = h^1$ imply that $\beta \vee (\varphi_1 \wedge \varphi_2) = h^1$ which in turn imply that $\beta = h^1$ if $\varphi_1 \wedge \varphi_2 = h^0$. As $\beta \neq h^1$, we have $\varphi_1 \wedge \varphi_2 \neq h^0$. □

Theorem 4.13. *Suppose that there exists distinct hesitant fuzzy minimal open sets and a hesitant fuzzy mean open set in a hesitant fuzzy topological space, then the union of the two hesitant fuzzy minimal open sets is not equal to* h^1.

Proof. Let φ_1, φ_2 be two distinct hesitant fuzzy minimal open sets and β be a hesitant fuzzy mean open set in a hesitant fuzzy topological space X. By Theorem 4.7, $\varphi_1 \vee \varphi_2 = h^0$. β being a hesitant fuzzy mean open set, it is neither hesitant fuzzy maximal open nor hesitant fuzzy minimal open which

means that,$\beta \neq \varphi_1, \varphi_2$. Also $\beta \neq h^0, h^1$. By Theorem 4.7, we get $\varphi_1 \gneqq \beta$ or $\beta \wedge \varphi_1 = h^0$ and $\varphi_2 \gneqq \beta$ or $\beta \wedge \varphi_2 = h^0$. The feasible possibilities are (I) $\varphi_1 \gneqq \beta$ and $\varphi_2 \gneqq \beta$, (II) $\varphi_1 \gneqq \beta$ and $\beta \wedge \varphi_2 = h^0$, (III) $\beta \wedge \varphi_1 = h^0$ and $\varphi \gneqq \beta$ and (IV) $\beta \wedge \varphi_1 = h^0$ and $\beta \wedge \varphi_2 = h^0$ as $\beta \neq h^1$.
Case I: Obviously $\varphi_1 \vee \varphi_2 \neq h^1$ if $\varphi_1 \gneqq \beta$ and $\varphi_2 \gneqq \beta$.
Case II:If $\beta \vee \varphi_2 \neq h^1$, then obviously $\varphi_1 \vee \varphi_2 \neq h^1$. Now suppose $\beta \vee \varphi_2 \neq h^1$. Since $\varphi_1 \gneqq \beta$, then there exists $x_\alpha \in \beta$ such that $x_\alpha \notin \varphi_1$. As $\beta \wedge \varphi_2 = h^0$, $x_\alpha \notin \varphi_2 . x_\alpha \notin \varphi_1, \varphi_2$ imply that $\varphi_1 \vee \varphi_2 \neq h^1$.
Case III: Similar to Case II.
Case IV: $\beta \wedge \varphi_1 = h^0$ and $\beta \wedge \varphi_2 = h^0$ imply that $\beta \wedge (\varphi_1 \vee \varphi_2) = h^0$ which in turn imply that $\beta = h^0$ if $\varphi_1 \vee \varphi_2 = h^1$. As $\beta \neq h^0$, we have $\varphi_1 \vee \varphi_2 \neq h^1$.

□

On combininig Theorems 4.12 and 4.13, we get Theorems 4.14 and 4.15 and the proofs of these two theorems can be proved similar to that of Theorems 4.12 and 4.13.

Theorem 4.14. *Suppose that there exists distinct hesitant fuzzy maximal closed sets and a hesitant fuzzy mean closed set in a hesitant fuzzy topological space, then the intersection of the two hesitant fuzzy maximal open is nonempty.*

Theorem 4.15. *Suppose that there exists distinct hesitant fuzzy minimal closed sets and a hesitant fuzzy mean closed set in a hesitant fuzzy topological space, then the intersection of the two hesitant fuzzy minimal closed sets is not equal to h^1.*

5. Discussion and Conclusion

The idea of hesitant fuzzy minimal and maximal open sets having important role in literature. Certain family between hesitant fuzzy maximal and minimal sets called as hesitant fuzzy mean open sets. Like hesitant fuzzy mean open sets, some family of sets known to be hesitant fuzzy paraopen sets. Even though hesitant fuzzy mean open and hesitant fuzzy paraopen sets seem to be same, there is a difference between them. Some properties examined by means of these sets in this paper. Therefore the hesitant fuzzy minimal, maximal, mean and paraopen open sets play dominant role and in further study these notions can be investigated via various kinds of open sets in the literature.

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