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INFINITELY MANY HOMOCLINIC SOLUTIONS FOR DAMPED VIBRATION SYSTEMS WITH LOCALLY DEFINED POTENTIALS

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Abstract. In this paper, we are concerned with the existence of infinitely many fast homoclinic solutions for the following damped vibration system

(1) $$\ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) + \nabla W(t,u(t)) = 0,\ \forall t \in \mathbb{R},$$

where $q \in C(\mathbb{R},\mathbb{R})$, $L \in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric and positive definite matix-valued function and $W \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$. The novelty of this paper is that, assuming that L is bounded from below unnecessarily coercive at infinity, and W is only locally defined near the origin with respect to the second variable, we show that (1) possesses infinitely many homoclinic solutions via a variant symmetric mountain pass theorem.

1. Introduction

We are interested in the existence of infinitely many homoclinic solutions for a class of damped vibration systems

$(\mathcal{DV})$ $$\ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) + \nabla W(t,u(t)) = 0,\ \forall t \in \mathbb{R},$$

where $q : \mathbb{R} \to \mathbb{R}$ is a continuous function, $L \in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric and positive definite matix-valued function and $W : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a continuous function, differentiable with respect to the second variable with continuous derivative $\frac{\partial W}{\partial x}(t,x) = \nabla W(t,x)$.

As usual, we say that a solution u of $(\mathcal{DV})$ is classical homoclinic (to 0) if $u \in C^2(\mathbb{R},\mathbb{R}^N)$ such that $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $t \to \mp\infty$. If $u \neq 0$, u is called nontrivial.

When $q = 0$, $(\mathcal{DV})$ is just the following second order Hamiltonian system:

$(\mathcal{HS})$ $$\ddot{u}(t) - L(t)u(t) + \nabla W(t,u(t)) = 0,\ \forall t \in \mathbb{R}.$$

Homoclinic orbits of Hamiltonian systems are very important in the study of gaz dynamics, fluid mechanics, relativistic mechanics, and nuclear physics. The

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homoclinic orbits are important in study of the behavior of dynamical systems which have been researched from Poincaré [12].

In the last three decades, the existence and multiplicity of homoclinic solutions of $(\mathcal{HS})$ have been intensively studied by many mathematicians with variational methods. Assuming that $L(t)$ and $W(t,x)$ are independent of t or periodic in t, many authors have studied the existence and multiplicity of homoclinic solutions for $(\mathcal{HS})$, see for instance [7,10,11,13,19,27] and the references therein. In this case, the existence of homoclinic solutions can be obtained by taking the limit of periodic solutions of approximating problems. If $L(t)$ and $W(t,x)$ are neither autonomous nor periodic in t, compactness arguments derived from Sobolev embedding theorem are not available for the study of $(\mathcal{HS})$, see [1,2,5,8,14–18,22–26] and the references cited therein.

When $q \neq 0$, i.e., the nonperiodic system $(\mathcal{DV})$ has been considered only by a few authors, see [3,4,6,20,21,28]. In all theses papers, $W(t,x)$ was always required to satisfy some kinds of growth conditions at infinity with respect to x, such as superquadratic, asymptotic quadratic or quadratic growth. Besides, the function L is required to satisfy one of the following conditions:
(1.1) There exists an $l \in C(\mathbb{R}, \mathbb{R}_+^*)$ such that $l(t) \to +\infty$ as $|t| \to \infty$ and

$$L(t)x \cdot x \geq l(t)\,|x|^2\,,\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

Here and in the following, " $\cdot$ " denotes the usual inner product of $\mathbb{R}^N$ and $|\cdot|$ is the induced norm.
(1.2) There are constants $0 < \tau_1 < \tau_2 < +\infty$ such that

$$\tau_1\,|x|^2 \leq L(t)x \cdot x \leq \tau_2\,|x|^2\,,\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

(1.3) (i) There exists an $l \in C(\mathbb{R}, \mathbb{R})$ such that

$$\inf_{t \in \mathbb{R}} l(t) > 0 \quad \text{and} \quad L(t)x \cdot x \geq l(t)\,|x|^2\,,\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$

(ii) There exists a constant $r_0 > 0$ such that

$$\lim_{|s| \to \infty} meas\Big(\,\{t \in]s - r_0, s + r_0[\, : L(t) < bI_N\}\,\Big) = 0,\ \forall b > 0,$$

where $meas_Q$ denotes the Lebesgue's measure on $\mathbb{R}$ with density $e^{Q(t)}$, $Q(t) = \int_0^t q(s)ds$. The conditions (1.1), (1.2) and (1.3) guarantee the compactness of the Sobolev embedding.

In the present paper, we will study the existence of infinity many homoclinic solutions for $(\mathcal{DV})$ in the case where $W(t,x)$ is still only locally defined near the origin with respect to x and L is bounded from below and unnecessary coercive. More precisely, we make the following assumptions:
(L) There exists a constant $l_0 > 0$ such that

$$l(t) = \min_{|\xi|=1} L(t)\xi \cdot \xi \geq l_0,\ \forall t \in \mathbb{R}.$$

There exists a constant $\delta > 0$ such that $W \in C(\mathbb{R} \times B_\delta(0), \mathbb{R})$ is continuously differentiable in the second variable with continuous derivative, where $B_\delta(0)$ is the open ball in $\mathbb{R}^N$ centered at 0 with radius δ, and satisfies
(W_1) $W(t,x)$ is even in x and $W(t,0) = 0, \forall t \in \mathbb{R}$;
(W_2) There exist constants $\nu \in]1,2[$, $\beta_1 \in [1,2]$, $\beta_2 \in [1, \frac{2}{2-\nu}]$ and nonnegative functions $a \in L_Q^{\beta_1}(\mathbb{R}, \mathbb{R}^+)$, $b \in L_Q^{\beta_2}(\mathbb{R}, \mathbb{R}^+)$ such that

$$|\nabla W(t,x)| \leq a(t) + b(t)\,|x|^{\nu-1}\,,\ \forall (t,x) \in \mathbb{R} \times B_\delta(0),$$

where L_Q^s will be defined in Section 2;

$$(W_3) \qquad \lim_{|x|\to 0} \frac{W(t,x)}{|x|^2} = +\infty,\ \text{uniformly in } t \in \mathbb{R}.$$

Our main result reads as follows.

Theorem 1.1. *Suppose that (L) and (W_1)-(W_3) are satisfied. Then the damped vibration system $(\mathcal{DV})$ possesses a sequence of homoclinic solutions (u_k) such that*

$$\max_{t\in\mathbb{R}} |u_k(t)| \to 0 \ \text{as}\ k \to \infty.$$

2. Preliminaries

In order to introduce the concept of homoclinic solutions for $(\mathcal{DV})$ conveniently, we firstly describe some properties of the weighted Sobolev space E on which certain variational functional associated with $(\mathcal{DV})$ is defined and the homoclinic solutions of $(\mathcal{DV})$ are the critical points of such functional. We shall use $L_Q^2(\mathbb{R})$ to denote the Hilbert space of measurable functions from $\mathbb{R}$ into $\mathbb{R}^N$ under the inner product

$$\langle u, v\rangle_{L_Q^2} = \int_{\mathbb{R}} e^{Q(t)} u(t)\cdot v(t)dt$$

and the induced norm

$$\|u\|_{L_Q^2} = \Big(\int_{\mathbb{R}} e^{Q(t)}\,|u(t)|^2\,dt\Big)^{\frac{1}{2}}.$$

Similarly, $L_Q^p(\mathbb{R})$ $(2 < p < \infty)$ denotes the Banach space of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norm

$$\|u\|_{L_Q^p} = \Big(\int_{\mathbb{R}} e^{Q(t)}\,|u(t)|^p\,dt\Big)^{\frac{1}{p}}$$

and $L_Q^\infty(\mathbb{R})$ denotes the Banach space of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norm

$$\|u\|_{L_Q^\infty} = \operatorname{ess\,sup}\Big\{e^{\frac{Q(t)}{2}}\,|u(t)| : t \in \mathbb{R}\Big\}.$$

Consider the Hilbert space

$$E = \Big\{u \in H_Q^1(\mathbb{R}) : \int_{\mathbb{R}} e^{Q(t)} L(t)u(t)\cdot u(t)dt < \infty\Big\}$$

equipped with the following inner product

$$\langle u, v\rangle = \int_{\mathbb{R}} e^{Q(t)}\Big[\dot{u}(t)\cdot\dot{v}(t) + L(t)u(t)\cdot v(t)\Big]dt$$

and the induced norm $\|u\| = \langle u, u\rangle^{\frac{1}{2}}$. Here $H^1_Q(\mathbb{R})$ denotes the Sobolev space

$$H^1_Q(\mathbb{R}) = \left\{u \in L^2_Q(\mathbb{R}) : \dot{u} \in L^2_Q(\mathbb{R})\right\}.$$

Evidently, E is continuously embedded into $H^1_Q(\mathbb{R})$. Hence E is continuously embedded in $L^p_Q(\mathbb{R})$ for all $p \in [2, \infty]$ and compactly embedded in $L^p_{Q,loc}(\mathbb{R})$ for all $p \in [2, \infty]$, where $L^p_{Q,loc}(\mathbb{R})$ denotes the space of measurable functions u from $\mathbb{R}$ into $\mathbb{R}^N$ such that for all compact $K \subset \mathbb{R}$, $\int_K e^{Q(t)}\,|u(t)|^p\,dt < \infty$. Consequently, for all $p \in [2, \infty]$, there exists a constant $\eta_p > O$ such that

$$\|u\|_{L^p_Q} \leq \eta_p\,\|u\|\,,\ \forall u \in E. \tag{2.1}$$

To prove our main result via critical point theory, we shall use the following symmetric mountain pass theorem developed by Kajikiya [9]. We will first recall the notion of genus.

Let E be a Banach space and let A be a subset of E. A is said to be symmetric if $u \in A$ implies $-u \in A$. For a closed symmetric set A which does not contain the origin, we define the genus $\gamma(A)$ of A by the smallest integer k for which there exists an odd continuous mapping from $\mathbb{R}$ to $\mathbb{R}^k \setminus \{0\}$. If such a k does not exist, we define $\gamma(A) = +\infty$. Moreover, we set $\gamma(\phi) = 0$. Let

$$\Gamma_k = \{A \subset E : A \text{ is a close symmetric subset},\ 0 \notin A,\ \gamma(A) \geq k\}.$$

The properties of genus used in the proof of our main result are summarized as follows.

Lemma 2.1 ([9, Proposition 7.5]). *Let A and B be closed symmetric subsets of E that do not contain the origin. Then the following hold.*

(i) *If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*

(ii) *The N-dimensional sphere S^N has a genus of $N+1$ by the Borsuk-Ulam theorem.*

Lemma 2.2 ([9, Theorem 1]). *Let E be an infinite-dimensional Banach space and let $\Phi \in C^1(E, \mathbb{R})$ be an even functional with $\Phi(0) = 0$. Suppose that Φ satisfies*

(1) *Φ is bounded from below and satisfies the (PS)-condition*;

(2) *For each $k \in \mathbb{N}$, there exists $A_k \subset \Gamma_k$ such that*

$$\sup_{u \in A_k} \Phi(u) < 0.$$

Then (a) *or* (b) *below holds.*

(a) *There exists a critical point sequence (u_k) such that $\Phi(u_k) < 0$ and $\lim_{k\to\infty} u_k = 0$;*

(b) *There exist two critical point sequences* (u_k) *and* (v_k) *such that* $\Phi(u_k) = 0$, $u_k \neq 0$ *and* $\lim_{k\to\infty} u_k = 0$, $\Phi(v_k) < 0$, $\lim_{k\to\infty} \Phi(v_k) = 0$ *and* (v_k) *converges to a non-zero limit.*

3. Proof of Theorem 1.1

In order to prove our main result via critical point theory, we need to modify $W(t,x)$ for x outside a neighborhood of the origin to get $\widetilde{W}(t,x)$ as follows. Choose a constant $r \in]0, \frac{\delta}{2}[$ and define a cut-off function $\chi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ such that $\chi(s) = 1$ for $0 \leq s \leq r$, $\chi(s) = 0$ for $s \geq 2r$ and $-\frac{2}{r} \geq \chi'(s) < 0$ for $r < s < 2r$. Let

$$\widetilde{W}(t,x) = \chi(|x|)W(t,x),\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N. \tag{3.1}$$

Combining (W_1), (W_2) and the definition of χ, we obtain

$$\left|\widetilde{W}(t,x)\right| \leq a(t)\,|x| + b(t)\,|x|^{\nu}\,,\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N, \tag{3.2}$$

and

$$\left|\nabla\widetilde{W}(t,x)\right| \leq 5\Big(a(t) + b(t)\,|x|^{\nu-1}\Big),\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N. \tag{3.3}$$

Now, we introduce the following modified system:

$$\ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) = \nabla\widetilde{W}(t,u(t)),\ t \in \mathbb{R} \tag{$\widetilde{\mathcal{DV}}$}$$

and define the variational functional Φ associated with $(\widetilde{\mathcal{DV}})$ by

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}\Big[\,|\dot{u}(t)|^2 + L(t)u(t)\cdot u(t)\Big]dt - \int_{\mathbb{R}} e^{Q(t)}\widetilde{W}(t,u(t))dt\\ &= \frac{1}{2}\,\|u\|^2 - \varphi(u),\end{aligned} \tag{3.4}$$

where $\varphi(u) = \int_{\mathbb{R}} e^{Q(t)}\widetilde{W}(t,u(t))dt$.

Lemma 3.1. *Assume that* (L), (W_1) *and* (W_2) *are satisfied. Then* $\varphi \in C^1(E,\mathbb{R})$ *and* $\varphi' : E \to E'$ *is compact, and hence* $\Phi \in C^1(E,\mathbb{R})$*. Moreover*

$$\varphi'(u)v = \int_{\mathbb{R}} e^{Q(t)}\nabla\widetilde{W}(t,u(t))\cdot v(t)dt, \tag{3.5}$$

$$\Phi'(u)v = \langle u,v\rangle - \int_{\mathbb{R}} e^{Q(t)}\nabla\widetilde{W}(t,u(t))\cdot v(t)dt \tag{3.6}$$

for all $u,v \in E$*, and nontrivial critical points of* Φ *on* E *are homoclinic solutions of* $(\mathcal{DV})$*.*

Proof. In the following, we will note

$$\overline{\beta}_1 = \frac{\beta_1}{\beta_1 - 1},\ \overline{\beta}_2 = \frac{\nu\beta_2}{\beta_2 - 1},\ (\overline{\beta}_1 = \infty, \overline{\beta}_2 = \infty,\ \text{if } \beta_1 = 1\ \text{ or }\ \beta_2 = 1). \tag{3.7}$$

It is easy to see that $\overline{\beta}_1, \overline{\beta}_2 \in [2, \infty]$. By (2.1), (3.2) and Hölder's inequality, we have for $u \in E$

$$\begin{aligned}\int_{\mathbb{R}} e^{Q(t)} \left|\widetilde{W}(t,u(t))\right| dt &\le \int_{\mathbb{R}} a(t)\,|u(t)|\,dt + \int_{\mathbb{R}} b(t)\,|u(t)|^{\nu}\,dt \\ &\le \|a\|_{L_Q^{\beta_1}} \|u\|_{L_Q^{\overline{\beta}_1}} + \|b\|_{L_Q^{\beta_2}} \|u\|^{\nu}_{L_Q^{\overline{\beta}_2}} \\ &\le \eta_{\overline{\beta}_1} \|a\|_{L_Q^{\beta_1}} \|u\| + \eta^{\nu}_{\overline{\beta}_2} \|b\|_{L_Q^{\beta_2}} \|u\|^{\nu} < \infty,\end{aligned} \tag{3.8}$$

which implies that φ and Φ are both well defined. Now, we prove that $\varphi \in C^1(E, \mathbb{R})$ and $\varphi' : E \to E'$ is compact. By (3.3), for any $u, v \in E$ and $s \in [0, 1]$, there holds

$$\begin{aligned}\left|\nabla\widetilde{W}(t,u+sv)v\right| &\le 5\Big[a(t) + b(t)\,|u+sv|^{\nu-1}\Big]\,|v| \\ &\le 5\Big[a(t) + b(t)\Big(\,|u|^{\nu-1} + |v|^{\nu-1}\Big)\Big]\,|v| \\ &\le 5\Big[a(t) + b(t)\Big(\,|u|^{\nu-1}\,|v| + |v|^{\nu}\Big)\Big]\,|v|\,.\end{aligned}$$

Hence, by the Mean Value Theorem and Lebesgue's Dominated Convergence Theorem, we get for all $u, v \in E$

$$\begin{aligned}\lim_{s\to 0}\frac{\varphi(u+sv)-\varphi(u)}{s} &= \lim_{s\to 0}\int_{\mathbb{R}} e^{Q(t)} \int_0^1 \nabla\widetilde{W}(t,u+rsv)v\,dr\,dt \\ &= \int_{\mathbb{R}} e^{Q(t)} \nabla\widetilde{W}(t,u)v\,dt = \mathcal{L}(u)v.\end{aligned}$$

Moreover, it follows from (2.1), (3.3) and Hölder's inequality that

$$\begin{aligned}|\mathcal{L}(u)v| &\le \int_{\mathbb{R}} e^{Q(t)} \left|\nabla\widetilde{W}(t,u)\right| |v|\,dt \\ &\le 5\Big[\int_{\mathbb{R}} e^{Q(t)} a(t)\,|v|\,dt + \int_{\mathbb{R}} e^{Q(t)} a(t)\,|u|^{\nu-1}\,|v|\,dt \\ &\le 5\Big[\,\|a\|_{L_Q^{\beta_1}} \|v\|_{L_Q^{\overline{\beta}_1}} + \|b\|_{L_Q^{\beta_2}} \|u\|^{\nu-1}_{L_Q^{\overline{\beta}_2}} \|v\|_{L_Q^{\overline{\beta}_2}}\,\Big] \\ &\le 5\Big[\eta_{\overline{\beta}_1} \|a\|_{L_Q^{\beta_1}} + \eta^{\nu}_{\overline{\beta}_2} \|b\|_{L_Q^{\beta_2}} \|u\|^{\nu-1}\Big]\,\|v\|\,,\ \forall v \in E,\end{aligned} \tag{3.9}$$

which means that $\mathcal{L}(u)$ is bounded. This means that φ is Gâteaux-differentiable on E and its Gâteaux-derivative at u is $\mathcal{L}(u)$. Let $u_n \rightharpoonup u$ in E as $n \to \infty$, then (u_n) is bounded in E and

$$u_n \to u \text{ in } L^{\infty}_{Q,loc}(\mathbb{R}) \text{ as } n \to \infty. \tag{3.10}$$

Therefore, there exists a constant $c_1 > 0$ such that

$$\|u_n\|^{\nu-1} + \|u\|^{\nu-1} \le c_1,\ \forall n \in \mathbb{N}. \tag{3.11}$$

By (W_2), for any $\epsilon > 0$, there exists $R_\epsilon > 0$ such that

$$\Big(\int_{|t|\ge R_\epsilon} e^{Q(t)} (a(t))^{\beta_1} dt\Big)^{\frac{1}{\beta_1}} \le \frac{\epsilon}{40\eta_{\overline{\beta}_1}}, \tag{3.12}$$

$$\Big(\int_{|t|\geq R_\epsilon} e^{Q(t)}(b(t))^{\beta_2}dt\Big)^{\frac{1}{\beta_2}} \leq \frac{\epsilon}{20c_1\eta^{\nu}_{\overline{\beta}_2}}. \tag{3.13}$$

Combining (3.3) with (3.11)-(3.13), the Hölder's inequality implies

$$\begin{aligned}
&\int_{|t|\geq R_\epsilon} e^{Q(t)}\left|\nabla\widetilde{W}(t,u_n)-\nabla\widetilde{W}(t,u)\right||v|\,dt\\
&\leq 5\int_{|t|\geq R_\epsilon} e^{Q(t)}\Big[2a(t)+b(t)(|u_n|^{\nu-1}+|u|^{\nu-1})\Big]|v|\,dt\\
&\leq 10\Big(\int_{|t|\geq R_\epsilon} e^{Q(t)}(a(t))^{\beta_1}dt\Big)^{\frac{1}{\beta_1}}\|v\|_{L_Q^{\overline{\beta}_1}}\\
&\quad +5\Big(\int_{|t|\geq R_\epsilon} e^{Q(t)}(b(t))^{\beta_2}dt\Big)^{\frac{1}{\beta_2}}\Big(\|u_n\|^{\nu-1}_{L_Q^{\overline{\beta}_2}}+\|u\|^{\nu-1}_{L_Q^{\overline{\beta}_2}}\Big)\|v\|_{L_Q^{\overline{\beta}_2}}\\
&\leq 10\eta_{\overline{\beta}_1}\Big(\int_{|t|\geq R_\epsilon} e^{Q(t)}(a(t))^{\beta_1}dt\Big)^{\frac{1}{\beta_1}}\\
&\quad +5\eta^{\nu}_{\overline{\beta}_2}\Big(\int_{|t|\geq R_\epsilon} e^{Q(t)}(b(t))^{\beta_2}dt\Big)^{\frac{1}{\beta_2}}\Big(\|u_n\|^{\nu-1}+\|u\|^{\nu-1}\Big)\\
&\leq \frac{\epsilon}{4}+\frac{\epsilon}{4}=\frac{\epsilon}{2},\ \forall n\in\mathbb{N},\ \text{and}\ \|v\|=1.
\end{aligned} \tag{3.14}$$

For the R_ϵ given above, by (2.1), (3.10) and the continuity of $\nabla\widetilde{W}$, there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$ and $\|v\|=1$

$$\begin{aligned}
&\int_{-R_\epsilon}^{R_\epsilon} e^{Q(t)}\left|\nabla\widetilde{W}(t,u_n)-\nabla\widetilde{W}(t,u)\right||v|\,dt\\
&\leq \eta_\infty\int_{-R_\epsilon}^{R_\epsilon} e^{Q(t)}\left|\nabla\widetilde{W}(t,u_n)-\nabla\widetilde{W}(t,u)\right|dt<\frac{\epsilon}{2}.
\end{aligned} \tag{3.15}$$

Combining (3.14) with (3.15), we get

$$\begin{aligned}
\|\mathcal{L}(u_n)-\mathcal{L}(u)\|_{E'} &= \sup_{\|v\|=1}|(\mathcal{L}(u_n)-\mathcal{L}(u))v|\\
&= \sup_{\|v\|=1}\left|\int_{\mathbb{R}} e^{Q(t)}(\nabla\widetilde{W}(t,u_n)-\nabla\widetilde{W}(t,u))\cdot v dt\right|\\
&\leq \sup_{\|v\|=1}\int_{-R_\epsilon}^{R_\epsilon} e^{Q(t)}\left|\nabla\widetilde{W}(t,u_n)-\nabla\widetilde{W}(t,u)\right||v|\,dt\\
&\quad+\sup_{\|v\|=1}\int_{|t|\geq R_\epsilon} e^{Q(t)}\left|\nabla\widetilde{W}(t,u_n)-\nabla\widetilde{W}(t,u)\right||v|\,dt\\
&< \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon \text{ for all } n\geq n_0.
\end{aligned}$$

This implies that $\mathcal{L}$ is continuous. Thus $\varphi\in C^1(E,\mathbb{R})$ and (3.5) holds with $\varphi'=\mathcal{L}$. This together with the reflexivity of the Hilbert space E implies that

φ' is compact. In addition, due to the form of Φ, we see that $\Phi \in C^1(E, \mathbb{R})$ and (3.6) also holds. The proof of Lemma 3.1 is completed. $\square$

Lemma 3.2. *Assume that* (L)*,* (W_1) *and* (W_2) *hold. Then* Φ *is bounded from below and satisfies the* (PS)*-condition.*

Proof. Firstly, we prove that Φ is bounded from below. By (3.8), it follows

$$\begin{aligned} \Phi(u) &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} e^{Q(t)} \left|\nabla \widetilde{W}(t,u)\right| dt \\ &\geq \frac{1}{2}\|u\|^2 - \eta_{\overline{\beta}_1} \|a\|_{L_Q^{\beta_1}} \|u\| - \eta_{\overline{\beta}_2}^{\nu} \|b\|_{L_Q^{\beta_2}} \|u\|^{\nu}. \end{aligned} \tag{3.16}$$

Since $\nu < 2$, it follows that Φ is bounded from below. Next, we show that Φ satisfies the (PS)-condition. Let (u_n) be a (PS)-sequence, that is

$$|\Phi(u_n)| \leq M, \ \forall n \in \mathbb{N}, \ \Phi'(u_n) \to 0 \text{ as } n \to \infty \tag{3.17}$$

for some constant $M > 0$. By (3.16) and (3.17), it holds

$$M \geq \frac{1}{2}\|u_n\|^2 - \eta_{\overline{\beta}_1} \|a\|_{L_Q^{\beta_1}} \|u_n\| - \eta_{\overline{\beta}_2}^{\nu} \|b\|_{L_Q^{\beta_2}} \|u_n\|^{\nu}$$

which implies that (u_n) is bounded in E since $\nu < 2$. Hence, up to a subsequence if necessary, we can assume that

$$u_n \rightharpoonup u \text{ in } E \text{ as } n \to \infty \tag{3.18}$$

for some $u \in E$. By virtue of the Riez Representation Theorem, $\varphi : E \to E'$ and $\Phi' : E \to E'$ can be viewed as $\varphi : E \to E$ and $\Phi' : E \to E$, respectively. This together with (3.5) and (3.6) yields

$$u_n = \Phi'(u_n) + \varphi'(u_n), \ \forall n \in \mathbb{N}. \tag{3.19}$$

By Lemma 3.1, φ' is compact. Combining this with (3.17)-(3.19), the right side of (3.19) converges strongly in E and hence $u_n \to u$ in E as $n \to \infty$. Then Φ satisfies the (PS)-condition. The proof of Lemma 3.2 is completed. $\square$

Lemma 3.3. *Suppose that* (L) *and* (W_3) *hold. Then for each* $k \in \mathbb{N}$*, there exists an* $A_k \subset E$ *with genus* $\gamma(A_k) \geq k$ *such that* $\sup_{u \in A_k} \Phi(u) < 0$.

Proof. Let (e_n) be an orthonormal basis of E. Then for each $k \in \mathbb{N}$, let

$$E_k = \oplus_{m=1}^{k} span\{e_m\}.$$

Since E_k is finite dimensional, there exists a constant $\tau_k > 0$ such that

$$\|u\| \leq \tau_k \|u\|_{L_Q^2}, \ \forall u \in E_k. \tag{3.20}$$

By (W_3), there exists a constant $R_k > 0$ such that

$$\widetilde{W}(t,x) \geq \tau_k^2 |x|^2, \ \forall t \in \mathbb{R}, \ |x| \leq R_k. \tag{3.21}$$

Let $u \in E$ such that $\|u\| \leq \frac{R_k}{\eta_\infty}$. By (2.1), we know that $|u(t)| \leq R_k$ for all $t \in \mathbb{R}$, thus by (3.21), it holds

$$\widetilde{W}(t,u(t)) \geq \tau_k^2 |u(t)|^2, \ \forall t \in \mathbb{R}. \tag{3.22}$$

Therefore, by (3.20) and (3.22), for all $u \in E_k \setminus \{0\}$ with $0 < \|u\| = \frac{\min\{r, R_k\}}{\eta_\infty} = \rho_k$, we have

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} e^{Q(t)} \nabla \widetilde{W}(t,u)dt \\ &\le \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} e^{Q(t)} \tau_k^2 |u(t)|^2 dt \\ &\le \frac{1}{2}\|u\|^2 - \|u\|^2 \\ &= -\frac{1}{2}\rho_k^2,\end{aligned}$$

which implies

$$\{u \in E_k \setminus \{0\} : \|u\| = \rho_k\} \subset A_k = \left\{u \in E_k : \Phi(u) \le -\frac{1}{2}\rho_k^2\right\}. \tag{3.23}$$

Thus, by Lemma 2.1, (3.23) implies

$$\gamma(A_k) \ge \gamma\Big(\{u \in E_k \setminus \{0\} \,/\, \|u\| = \rho_k\} \Big) \ge k$$

hence, by the definition of Γ_k, we have $A_k \subset \Gamma_k$. Moreover, the definition of A_k implies

$$\sup_{u \in A_k} \Phi(u) \le -\frac{1}{2}\rho_k^2 < 0.$$

The proof of Lemma 3.3 is completed. □

Consequently, Φ possesses a sequence of nontrivial critical points (u_k) satisfying $u_k \to 0$ in E as $k \to \infty$. By virtue of Lemma 3.1, (u_k) is a sequence of homoclinic solutions of $(\widetilde{\mathcal{DV}})$. By (2.1), it follows that $\max_{t\in\mathbb{R}} |u_k(t)| \to 0$ as $k \to \infty$. Therefore, there exists a positive constant $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$, u_k is a homoclinic solution of $(\mathcal{DV})$. This ends the proof of Theorem 1.1.

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