Commun. Korean Math. Soc. **37** (2022), No. 3, pp. 813–824
https://doi.org/10.4134/CKMS.c200471
pISSN: 1225-1763 / eISSN: 2234-3024

SOLITONS OF KÄHLERIAN NORDEN SPACE-TIME MANIFOLDS

Praveena Manjappa Mundalamane, Bagewadi Channabasappa Shanthappa, and Mallannara Siddalingappa Siddesha

Abstract. We study solitons of Kählerian Norden space-time manifolds and Bochner curvature tensor in almost pseudo symmetric Kählerian space-time manifolds. It is shown that the steady, expanding or shrinking solitons depend on different relations of energy density/isotropic pressure, the cosmological constant, and gravitational constant.

1. Introduction

Relativistic fluid models play a very important role in different branches of astrophysics, plasma physics, nuclear physics and cosmology. The space-time of general relativity and cosmology is studied using a four-dimensional pseudo-Riemannian manifold with Lorentzian metric, (M^4, g), where g is considered as a perfect fluid space-time. See [21] for an introduction to these topics.

The application of pseudo-Riemannian geometry in relativity is discussed by Neill [18] and by Kaigorodov [15] on the structure of space-time in 1983. In [7], Chaki and Roy explored that a general relativistic space-time with covariant constant energy-momentum tensor is Ricci symmetric. Many differential geometers have been extended the study to the curvature structure of space-time with special properties [11, 22, 23].

Definition 1.1. A non-flat Riemannian manifold M of dimension $n > 2$ is said to be an almost pseudo symmetric manifold [10] if its curvature tensor R satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z,U,W) = {} & [A(X)+B(X)]R(Y,Z,U,W) \\ & + A(Y)R(X,Z,U,W) + A(Z)R(Y,X,U,W) \\ & + A(U)R(Y,Z,X,W) + A(W)R(Y,Z,U,X),\end{aligned}$$

Received December 18, 2020; Accepted November 16, 2021.
2020 *Mathematics Subject Classification.* Primary 52Q15, 35C08, 81T20.
Key words and phrases. Kählerian Norden space-time manifolds, solitons, almost pseudo symmetric manifold.

©2022 Korean Mathematical Society

where A, B are two non zero 1-forms defined by

$$g(X,\rho) = A(X), \quad g(X,Q) = B(X),$$

$X, Y, Z, W \in \chi(M)$; ρ and Q are called the associated vector fields corresponding to the 1-forms A and B, respectively.

On the other side, the Ricci flow introduced by Hamilton [13] is an excellent mathematical model for simplifying the structure of the manifold. It is given by a partial differential equation connecting Ricci and metric tensors. Ricci solitons are self-similar solutions of this partial differential equation. Thus Ricci, Einstein and conformal Ricci flows are intrinsic geometric flows on a pseudo-Riemannian manifold, whose fixed points are solitons. Ricci solitons, Einstein solitons, and conformal Ricci solitons, which generate self-similar solutions of the Ricci flow, Einstein flow, and conformal Ricci flow are given by:

$$\frac{\partial g}{\partial t} = -2S, \ \frac{\partial g}{\partial t} = -2(S - \frac{r}{2}g), \ \frac{\partial g}{\partial t} = -2(S + \frac{g}{n}) - \phi g, \text{ and } r = -1.$$

In the papers (see [4, 16, 17]) solitons are studied to a great extent within the background of Riemannian geometry. In [8], Cho and Kimura defined an η-Ricci soliton as

$$L_\xi g + 2S + 2ag + 2b\eta \otimes \eta = 0, \tag{1.1}$$

where g is a pseudo-Riemannian metric, S is the Ricci curvature, ξ is a vector field, η is a 1-form and a and b are real constants. The data (g, ξ, a, b) in (1.1) is said to be an η-Ricci soliton in M^4; and in particular, if $b = 0$, (g, ξ, a) is a Ricci soliton [13] and it is called shrinking, steady or expanding according as a is negative, zero or positive, respectively [9].

Writing explicitly the Lie derivative $(L_\xi g)(X, Y)$ we get

$$(L_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \tag{1.2}$$

and from (1.1) we obtain:

$$S(X,Y) = -ag(X,Y) - b\eta(X)\eta(Y) - \frac{1}{2}[g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)] \tag{1.3}$$

for any $X, Y \in \chi(M^4)$.

Blaga defined an η-Einstein soliton which is defined as [2]

$$L_\xi g + 2S + (2a - r)g + 2b\eta \otimes \eta = 0, \tag{1.4}$$

where g, S, r, ξ, η, a and b have the meaning are already stated. The data (g, ξ, a, b) in (1.4) is said to be an η-Einstein soliton in M^4; and in particular, if $b = 0$, (g, ξ, a) is an Einstein soliton [5]. Using (1.2) in (1.4) we get

$$S(X,Y) = -(a - \frac{r}{2})g(X,Y) - b\eta(X)\eta(Y) - \frac{1}{2}[g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)] \tag{1.5}$$

for any $X, Y \in \chi(M^4)$.

In [24] Siddique studied a conformal η-Ricci soliton which is defined as

$$L_\xi g + 2S + [2a - (\phi + \frac{2}{n})]g + 2b\eta \otimes \eta = 0, \tag{1.6}$$

where ϕ is a time dependent scalar field, n is dimension of the manifold and g, S, ξ, η, a, b have the meaning are already stated. The data (g, ξ, a, b) in (1.6) is said to be a conformal η-Ricci soliton in M^4; and in particular, if $b = 0$, (g, ξ, a) is a conformal Ricci soliton [1]. Using (1.2) in (1.6) we get

$$\begin{aligned} S(X,Y) = & -[a - \frac{1}{2}(\phi + \frac{1}{2})]g(X,Y) - b\eta(X)\eta(Y) \\ & - \frac{1}{2}[g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)] \end{aligned} \tag{1.7}$$

for any $X, Y \in \chi(M^4)$.

In [19, 20] Praveena and Bagewadi studied Ricci solitons and obtained results in almost pseudo symmetric Kähler manifolds. In [25, 26], Venkatesha and Kumara studied conformally flat quasi-Einstein space-times with applications in general relativity and Ricci soliton structure in a perfect fluid space-time whose time like velocity vector field ξ is torse-forming. Blaga [3] investigated that if perfect fluid space-time (M^4, g) is either an η-Ricci soliton or an η-Einstein soliton with $\xi = grad(f)$, where f is a scalar function, then Laplacian of f can be expressed as a linear combination of the cosmological constant, isotropic pressure, energy density and gravitational constant. Siddiqi and Siddiqui [24] explored if perfect fluid space-time (M^4, g) is a conformal η-Ricci soliton with $\xi = grad(f)$, where f is a scalar function, then Laplacian of f can be expressed as a linear combination of the cosmological constant, isotropic pressure, energy density, gravitational constant and time-dependent scalar field. Recently, Praveena and Bagewadi [21] studied solitons of an almost pseudo symmetric Kählerian space-time manifold with different curvature tensors. Motivated by the above studies in this paper we study solitons (an η-Ricci soliton, an η-Einstein soliton and a conformal-η-Ricci soliton) of Kählerian Norden space-time manifolds and Bochner curvature tensor in almost pseudo symmetric Kählerian Norden space-time manifolds.

2. Basic properties of Kählerian Norden space-time manifold

A four-dimensional manifold has general relativistic perfect fluid space-time such type of manifold is said to be a Kählerian Norden space-time manifold if it admits Norden metric which satisfies

$$J^2(Z) = -Z, \;\; g(JZ, JY) = -\,g(Z,Y), \;\text{ and }\; (\nabla_Z J)(Y) = 0,$$

where J is a $(1,1)$ tensor and g is a pseudo-Riemannian metric. We know that in a Kähler Norden manifold the Riemannian curvature tensor R and Ricci tensor S satisfy [12]

$$R(JX, JY, Z, W) = -R(X, Y, Z, W), \tag{2.1}$$

$$S(JZ,JY) = -S(J,Y), \quad S(Z,JY) - S(JZ,Y) = 0, \tag{2.2}$$

$$g(JZ,JY) = -S(J,Y), \quad g(Z,JY) - g(JZ,Y) = 0. \tag{2.3}$$

Let $E_1, E_2, \ldots, E_n$ be an orthonarmal frame in a Kähler Norden manifold. Then $g(E_i, E_i) = \dim(M)$ and $g(JE_i, E_i) = 0$.

The scalar curvature r and the $*$-scalar curvature r^* which are defined as the trace $S(E_i, E_i)$ and $S(JE_i, E_i)$, respectively.

We define the following for almost pseudo symmetric and Bochner curvature tensor in a Kähler Norden manifold. It is similar to Definition 1.1.

Definition 2.1. A Kähler Norden manifold is called an almost pseudo Bochner symmetric manifold if its Bochner curvature tensor D of type (0,4) is not zero and satisfies the condition

$$\begin{aligned}(\nabla_X D)(Y,Z,U,V) = {} & [A(X) + B(X)]D(Y,Z,U,V) \\ & + A(Y)D(X,Z,U,V) + A(Z)D(Y,X,U,V) \\ & + A(U)D(Y,Z,X,V) + A(V)D(Y,Z,U,X),\end{aligned} \tag{2.4}$$

where A, B are 1-forms (not simultaneously zero) and D is given by [27],

$$\begin{aligned}D(X,Y,Z,U) = R(X,Y,Z,U) - \frac{1}{2n+4}[& g(Y,Z)S(X,U) - S(X,Z)g(Y,U) \\ & + g(JY,Z)S(JX,U) - S(JX,Z)g(JY,U) \\ & + S(Y,Z)g(X,U) - g(X,Z)S(Y,U) \\ & + S(JY,Z)g(JX,U) - g(JX,Z)S(JY,U) \\ & - 2S(Y,JX)g(JZ,U) - 2S(JZ,U)g(JX,Y)] \\ & + \frac{r}{(2n+2)(2n+4)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U) \\ & + g(JY,Z)g(JX,U) - g(JX,Z)g(JY,U) \\ & - 2g(JX,Y)g(JZ,U)].\end{aligned} \tag{2.5}$$

Put $X = Y = E_i$ in the above equation we get

$$K(Y,Z) = \frac{n}{2n+4}[S(Y,Z) - \frac{r}{2(n+1)}g(Y,Z) - \frac{r^*}{n}g(JY,Z)]. \tag{2.6}$$

We know that the Einstein equation with cosmological constant for the perfect fluid space-time is given by

$$S(X,Y) = -(\lambda - \frac{r}{2} - kp)g(X,Y) + k(\sigma + p)\eta(X)\eta(Y), \tag{2.7}$$

for any $X, Y \in \chi(M^4)$, where p is the isotropic pressure, σ is the energy-density, λ is the cosmological constant, k is the gravitational constant, g is the metric tensor of Minkowski space-time [14], ξ is the velocity vector of the fluid and η is associated 1-form $\eta(\xi) = -1$, S is the Ricci tensor and r is the scalar curvature of g.

Here the Ricci tensor S is a functional combination of g and $\eta \otimes \eta$ and is called quasi-Einstein [6]. Quasi-Einstein manifolds arose during the study of exact solutions of Einstein field equations.

Consider $\{E_i\}_{1\leq i\leq 4}$ an orthonormal frame field, i.e., $g(E_i, E_j) = \varepsilon_{ij}\delta_{ij}$, $i, j \in \{1, 2, 3, 4\}$ with $\varepsilon_{11} = -1$, $\varepsilon_{ii} = -1$, $i \in \{2, 3, 4\}$, $\varepsilon_{ij} = 0$, $i, j \in \{1, 2, 3, 4\}$, $i \neq j$. Let $\xi = \Sigma_{i=1}^{4}\xi^i E_i$. Then

$$-1 = g(\xi, \xi) = \Sigma_{1\leq i,j\leq 4}\xi^i\xi^j g(E_i, E_i) = \sum_{i=1}^{4}\varepsilon_{ii}(\xi^i)^2, \tag{2.8}$$

and

$$\eta(E_i) = g(E_i, \xi) = \sum_{j=1}^{4}\xi^j g(E_i, E_j) = \varepsilon_{ii}\xi^j. \tag{2.9}$$

Contracting (2.7) and taking into account that $g(\xi, \xi) = -1$, we get:

$$r = 4\lambda + k(\sigma - 3p). \tag{2.10}$$

Substitute above value in equation (2.7), we get:

$$S(X, Y) = (\lambda + \frac{k(\sigma - p)}{2})g(X, Y) + k(\sigma + p)\eta(X)\eta(Y) \tag{2.11}$$

for any $X, Y \in \chi(M^4)$.

Example 2.1. A radiation fluid has constant scalar curvature equal to 4λ.

3. Solitons of Kählerian Norden space-time manifold

Now replacing X and Y by JX and JY, respectively, in (2.11) and using equations (2.2) and (2.3) we get

$$-S(X, Y) = -(\lambda + \frac{k(\sigma - p)}{2})g(X, Y) + k(\sigma + p)\eta(JX)\eta(JY). \tag{3.1}$$

Adding (3.1) from (2.11), we have

$$k(\sigma + p)[\eta(JX)\eta(JY) + \eta(X)\eta(Y)].$$

Put $Y = \xi$ in above equation we obtained

$$k(\sigma + p)\eta(x) = 0.$$

Since $k \neq 0$ and $\eta(X) \neq 0$, we have

$$\sigma = -p.$$

Now using this in (3.1) we get

$$S(X, Y) = (\lambda + \sigma k)g(X, Y). \tag{3.2}$$

From (3.2) and (1.3) we get

$$(\lambda+\sigma k)\ g(X, Y) = -ag(X, Y) - b\eta(X)\eta(Y) - \frac{1}{2}[g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi)]. \tag{3.3}$$

Multiplying (3.3) by ε_{ii} and summing over i for $X = Y := E_i$ and then using equations (2.8) and (2.9), we get:

$$4a - b = -4(\lambda + \sigma k) - div\xi. \tag{3.4}$$

Writing $X = Y = \xi$ in (3.3), we obtain:

$$a - b = -(\lambda + \sigma k). \tag{3.5}$$

Solving (3.4) and (3.5) we have

$$a = -(\lambda + \sigma k) - \frac{div\xi}{3} \quad \text{and} \quad b = \frac{-div\xi}{3}.$$

If $b = 0$, then we obtain the Ricci soliton with $a = -(\lambda + \sigma k)$ which is steady if $\sigma = -\frac{\lambda}{k}$, expanding if $\sigma < -\frac{\lambda}{k}$ and shrinking if $\sigma > -\frac{\lambda}{k}$.

Theorem 3.1. *Let (g, ξ, a, b) be an η-Ricci soliton in a Kählerian Norden space-time manifold with $p = -\sigma$ and $b = 0$. Then it is steady if $\sigma = -\frac{\lambda}{k}$, expanding if $\sigma < -\frac{\lambda}{k}$ and shrinking if $\sigma > -\frac{\lambda}{k}$.*

From (3.2) and (1.5) we get

$$(\lambda + \sigma k)\, g(Y, Z) = -(a - \frac{r}{2})g(Y, Z) - b\eta(Y)\eta(Z) - \frac{1}{2}[g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi)],$$

then we write the above equation

$$[(\lambda + \sigma k) + (a - \frac{r}{2})]g(Y, Z) + b\eta(Y)\eta(Z) + \frac{1}{2}[g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi)] = 0. \tag{3.6}$$

Multiplying (3.6) by ε_{ii} and contracting over Y and Z and then using equations (2.8), (2.10) and (2.9), we get:

$$4a - b = 4\lambda + 4k\sigma - div\xi. \tag{3.7}$$

Writing $Y = Z = \xi$ in (3.6) and using (2.10) we obtain:

$$a - b = \lambda + k\sigma, \tag{3.8}$$

solving (3.7) and (3.8) we get

$$a = \lambda + k\sigma - \frac{div\xi}{3} \quad \text{and} \quad b = -\frac{div\xi}{3}.$$

If $b = 0$, then we obtain Einstein soliton with $a = \lambda + k\sigma$ which is steady if $\sigma = -\frac{\lambda}{k}$, expanding if $\sigma > -\frac{\lambda}{k}$ and shrinking if $\sigma < -\frac{\lambda}{k}$.

Theorem 3.2. *Let (g, ξ, a, b) be an η-Einstein soliton in a Kählerian Norden space-time manifold with $p = -\sigma$ and $b = 0$. Then it is steady if $\sigma = -\frac{\lambda}{k}$, expanding if $\sigma > -\frac{\lambda}{k}$ and shrinking if $\sigma < -\frac{\lambda}{k}$.*

From (3.2) and (1.7) we get

$$\begin{aligned}(\lambda + k\sigma)\, g(Y, Z) = & -[a - \frac{1}{2}(\phi + \frac{1}{2})]g(Y, Z) - b\eta(Y)\eta(Z) \\ & - \frac{1}{2}[g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi)],\end{aligned}$$

then we write the above equation

$$[(\lambda + k\sigma) + (a - \frac{1}{2}(\phi + \frac{1}{2}))]g(Y, Z) + b\eta(Y)\eta(Z)$$

$$+ \frac{1}{2}[g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi)] = 0. \tag{3.9}$$

Multiplying (3.9) by ε_{ii} and contracting over Y and Z and then using equations (2.8) and (2.9), we get:

$$4a - b = -4(\lambda + k\sigma) + 2\phi + 1 - div\xi. \tag{3.10}$$

Writing $Y = Z = \xi$ in (3.9) then we obtain:

$$a - b = -(\lambda + k\sigma) + \frac{1}{2}(\phi + \frac{1}{2}), \tag{3.11}$$

solving (3.10) and (3.11) we get

$$a = -(\lambda + k\sigma) + \frac{1}{2}\phi + \frac{1}{4} - \frac{div\xi}{3} \quad \text{and} \quad b = -\frac{div\xi}{3}.$$

If $b = 0$, then we obtain an Einstein soliton with $a = -(\lambda + k\sigma) + \frac{1}{2}\phi + \frac{1}{4}$ which is steady if $\sigma = -\frac{\lambda}{k} + \frac{\phi}{2k} + \frac{1}{4k}$, expanding if $\sigma > -\frac{\lambda}{k} + \frac{\phi}{2k} + \frac{1}{4k}$ and shrinking if $\sigma < -\frac{\lambda}{k} + \frac{\phi}{2k} + \frac{1}{4k}$.

Theorem 3.3. *Let (g, ξ, a, b) be an η-Einstein soliton in a Kählerian Norden space-time manifold with $p = -\sigma$ and $b = 0$. Then it is steady if $\sigma = -\frac{\lambda}{k} + \frac{\phi}{2k} + \frac{1}{4k}$, expanding if $\sigma > -\frac{\lambda}{k} + \frac{\phi}{2k} + \frac{1}{4k}$ and shrinking if $\sigma < -\frac{\lambda}{k} + \frac{\phi}{2k} + \frac{1}{4k}$.*

4. Almost pseudo Bochner symmetric Kählerian Norden space-time manifold

In this section we study solitons of an almost pseudo symmetric Bochner curvature tensor in a Kählerian Norden space-time manifold. Using (2.1), (2.2), and (2.3) in (2.5) we have

$$D(JY, JZ, U, V) = -D(Y, Z, U, V). \tag{4.1}$$

Taking the covariant derivative of (4.1), we get

$$(\nabla_X D)(JY, JZ, U, V) = -(\nabla_X D)(Y, Z, U, V). \tag{4.2}$$

Using (2.4) in (4.2), we get

$$-A(Y)D(X, Z, U, V) - D(Z)R(Y, X, U, V) = A(JY)D(X, JZ, U, V)$$

$$A(JZ)D(JY, X, U, V). \tag{4.3}$$

Setting $Z = U = E_i$ in (4.3) then we have

$$-A(Y)K(X, V) + A(D(Y, X)V) = A(JY)K(JX, V) - A(D(JY, X)JV).$$

Again putting $Y = \rho = E_i$ in the above equation we get

$$K(X, V) = 0. \tag{4.4}$$

Using equation (4.4) in (2.6) we obtain

$$S(Y,Z) = \frac{r}{10}g(Y,Z) + \frac{r^*}{4}g(JY,Z). \tag{4.5}$$

From (4.5) and (1.3) we get

$$\begin{aligned} \frac{r}{10}g(Y,Z) + \frac{r^*}{4}g(JY,Z) = & -ag(Y,Z) - b\eta(Y)\eta(Z) \\ & -\frac{1}{2}[g(\nabla_Y\xi, Z) + g(Y, \nabla_Z\xi)]. \end{aligned} \tag{4.6}$$

Multiplying (4.6) by ε_{ii} and summing over i for $Y = Z := E_i$ then using equations (2.8), (2.10) and (2.9), we get:

$$4a - b = \frac{-2[4\lambda + k(\sigma - 3p)]}{5} - div\xi. \tag{4.7}$$

Put $X = Y = \xi$ in (4.6) then we obtain:

$$a - b = \frac{-[4\lambda + k(\sigma - 3p)]}{10}. \tag{4.8}$$

Solving (4.7) and (4.8) we get

$$a = \frac{-[4\lambda + k(\sigma - 3p)]}{10} - \frac{div\xi}{3} \quad \text{and} \quad b = \frac{-div\xi}{3}.$$

If $b = 0$, then we obtain Ricci soliton $a = \frac{-[4\lambda+k(\sigma-3p)]}{10}$ which is steady if $p = \frac{4}{3}(\frac{\lambda}{k}) + \frac{\sigma}{3}$, expanding if $p > \frac{4}{3}(\frac{\lambda}{k}) + \frac{\sigma}{3}$ and shrinking if $p < \frac{4}{3}(\frac{\lambda}{k}) + \frac{\sigma}{3}$.

Theorem 4.1. *Let (g, ξ, a, b) be an η-Ricci soliton in an almost pseudo Bochner symmetric Kählerian Norden space-time manifold with $b = 0$. Then it is steady if $p = \frac{4}{3}(\frac{\lambda}{k}) + \frac{\sigma}{3}$, expanding if $p > \frac{4}{3}(\frac{\lambda}{k}) + \frac{\sigma}{3}$ and shrinking if $p < \frac{4}{3}(\frac{\lambda}{k}) + \frac{\sigma}{3}$.*

Remark 4.2. For radiation fluid $\sigma = 3p$ we have $a = -\frac{2\lambda}{5} - \frac{div\xi}{3}$ and $b = -\frac{div\xi}{3}$.

From (4.5) and (1.5) we get

$$\begin{aligned} \frac{r}{10}g(Y,Z) + \frac{r^*}{4}g(JY,Z) = & -(a - \frac{r}{2})g(Y,Z) - b\eta(Y)\eta(Z) \\ & -\frac{1}{2}[g(\nabla_Y\xi, Z) + g(Y, \nabla_Z\xi)]. \end{aligned} \tag{4.9}$$

Multiplying (4.9) by ε_{ii} and contracting over Y and Z then using equations (2.8), (2.10) and (2.9), we get:

$$4a - b = \frac{8[4\lambda + k(\sigma - 3p)]}{5} - div\xi. \tag{4.10}$$

Setting $Y = Z = \xi$ in (4.9) and using (2.10) we obtain:

$$a - b = \frac{2[4\lambda + k(\sigma - 3p)]}{5}, \tag{4.11}$$

solving (4.10) and (4.11) we get

$$a = \frac{2[4\lambda + k(\sigma - 3p)]}{5} - \frac{div\xi}{3} \quad \text{and} \quad b = -\frac{div\xi}{3}.$$

If $b = 0$, then we obtain a Einstein soliton with $a = \frac{2[4\lambda + k(\sigma - 3p)]}{5}$ which is steady if $p = \frac{4}{k}(\frac{\lambda}{3}) + \frac{\sigma}{3}$, expanding if $p > \frac{4}{k}(\frac{\lambda}{3}) + \frac{\sigma}{3}$ and shrinking if $p < \frac{4}{k}(\frac{\lambda}{3}) + \frac{\sigma}{3}$.

Theorem 4.3. *Let (g, ξ, a, b) be an η-einstein soliton in an almost pseudo Bochner symmetric Kählerian Norden space-time manifold with $b = 0$. Then it is steady if $p = \frac{4}{3}(\frac{\lambda}{k}) + \frac{\sigma}{3}$, expanding if $p > \frac{4}{3}(\frac{\lambda}{k}) + \frac{\sigma}{3}$ and shrinking if $p < \frac{4}{3}(\frac{\lambda}{k}) + \frac{\sigma}{3}$.*

Remark 4.4. For radiation fluid $\sigma = 3p$ we have $a = \frac{8\lambda}{5} - \frac{div\xi}{3}$ and $b = -\frac{div\xi}{3}$.

From (4.5) and (1.7) we get

$$\frac{r}{10}g(Y,Z) + \frac{r^*}{4}g(JY,Z) = -(a - \frac{r}{2})g(Y,Z) - b\eta(Y)\eta(Z) - \frac{1}{2}[g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi)]. \tag{4.12}$$

Multiplying (4.12) by ε_{ii} and contracting over Y and Z then using equations (2.8), (2.10) and (2.9), we get:

$$4a - b = \frac{-[4\lambda + k(\sigma - 3p)]}{5} + 2\phi + 1 - div\xi. \tag{4.13}$$

Setting $Y = Z = \xi$ in (4.12) and using (2.10) we obtain:

$$a - b = \frac{-[4\lambda + k(\sigma - 3p)]}{10} + \frac{\phi}{2} + \frac{1}{4}, \tag{4.14}$$

solving (4.13) and (4.14) we get

$$a = \frac{-[4\lambda + k(\sigma - 3p)]}{10} + \frac{\phi}{2} + \frac{1}{4} - \frac{div\xi}{3} \quad \text{and} \quad b = -\frac{div\xi}{3}.$$

If $b = 0$, then we obtain a conformal Ricci soliton with $a = \frac{-[4\lambda + k(\sigma - 3p)]}{10} + \frac{\phi}{2} + \frac{1}{4}$ which is steady if $p = \frac{4}{k}(\frac{\lambda}{3}) + \frac{\sigma}{3} - \frac{5\phi}{3} - \frac{5}{6}$, expanding if $p > \frac{4}{k}(\frac{\lambda}{3}) + \frac{\sigma}{3} - \frac{5\phi}{3} - \frac{5}{6}$ and shrinking if $p < \frac{4}{k}(\frac{\lambda}{3}) + \frac{\sigma}{3} - \frac{5\phi}{3} - \frac{5}{6}$.

Theorem 4.5. *Let (g, ξ, a, b) be an conformal η-Ricci soliton in an almost pseudo Bochner symmetric Kählerian Norden space-time manifold with $b = 0$. Then it is steady if $p = \frac{-[4\lambda + k(\sigma - 3p)]}{10} + \frac{\phi}{2} + \frac{1}{4}$, expanding if $p > \frac{-[4\lambda + k(\sigma - 3p)]}{10} + \frac{\phi}{2} + \frac{1}{4}$ and shrinking if $p < \frac{-[4\lambda + k(\sigma - 3p)]}{10} + \frac{\phi}{2} + \frac{1}{4}$.*

Remark 4.6. For radiation fluid $\sigma = 3p$ we have $a = \frac{-2\lambda}{5} + \frac{\phi}{2} + \frac{1}{4} - \frac{div\xi}{3}$ and $b = -\frac{div\xi}{3}$.

5. Conclusion

It is proved in the theorems of Section 3 solitons of Kählerian Norden space-time manifolds depend on energy density σ, cosmological constant λ and gravitational constant k because energy density is expressed as a linear combination of cosmological constant and gravitational constant. Since a content matter of the fluid is not the pure and perfect fluid is dust which looks isotropic or stars in its rest frame. In modern cosmology, it is considered as a candidate for dark energy, the cause of the acceleration of the expansion of the universe. We also proved in the theorems of Section 4 that solitons of almost pseudo-Bochner symmetric Kählerian Norden space-time manifolds depend on isotropic pressure p, cosmological constant λ, energy density σ and gravitational constant k because isotropic pressure is expressed as a linear combination of cosmological constant, energy density and gravitational constant. Since the content matter of the fluid is viscous and perfect hence it is known as a viscous fluid and is further distinguished as Newtonian fluids if the viscosity of the fluid is constant for a different rate of shear stress concerning time. The matter content of the universe is assumed to perform like a perfect fluid in standard cosmological models. It is concluded that the evolution of the universe depends on Ricci, Einstein, and conformal Ricci solitons.

References

[1] N. Basu and A. Bhattacharyya, *Conformal Ricci soliton in Kenmotsu manifold*, Glob. J. Adv. Res. Class. Mod. Geom. **4** (2015), no. 1, 15–21.

[2] A. M. Blaga, *On gradient η-Einstein solitons*, Kragujevac J. Math. **42** (2018), no. 2, 229–237. https://doi.org/10.5937/kgjmath1802229b

[3] A. M. Blaga, *Solitons and geometrical structures in a perfect fluid spacetime*, Rocky Mountain J. Math. **50** (2020), no. 1, 41–53. https://doi.org/10.1216/rmj.2020.50.41

[4] A. M. Blaga, A. Ishan, and S. Deshmukh, *A note on solitons with generalized geodesic vector field*, Symmetry **13** (2021), no. 7, 1104. https://doi.org/10.3390/sym13071104

[5] G. Catino and L. Mazzieri, *Gradient Einstein solitons*, Nonlinear Anal. **132** (2016), 66–94. https://doi.org/10.1016/j.na.2015.10.021

[6] M. C. Chaki and R. K. Maity, *On quasi Einstein manifolds*, Publ. Math. Debrecen **57** (2000), no. 3-4, 297–306.

[7] M. C. Chaki and S. Ray, *Space-times with covariant-constant energy-momentum tensor*, Internat. J. Theoret. Phys. **35** (1996), no. 5, 1027–1032. https://doi.org/10.1007/BF02302387

[8] J. T. Cho and M. Kimura, *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J. (2) **61** (2009), no. 2, 205–212. https://doi.org/10.2748/tmj/1245849443

[9] B. Chow, P. Lu, and L. Ni, *Hamilton's Ricci Flow*, Graduate Studies in Mathematics, 77, American Mathematical Society, Providence, RI, 2006. https://doi.org/10.1090/gsm/077

[10] U. C. De and A. K. Gazi, *On almost pseudo symmetric manifolds*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **51** (2008), 53–68 (2009).

[11] U. C. De and G. C. Ghosh, *On weakly Ricci symmetric spacetime manifolds*, Rad. Mat. **13** (2004), no. 1, 93–101.

[12] G. T. Ganchev and A. V. Borisov, *Note on the almost complex manifolds with a Norden metric*, C. R. Acad. Bulgare Sci. **39** (1986), no. 5, 31–34.

[13] R. S. Hamilton, *The Ricci flow on surfaces*, in Mathematics and general relativity (Santa Cruz, CA, 1986), 237–262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988. https://doi.org/10.1090/conm/071/954419

[14] K. M. Haradhan, *Minkowski geometry and space-time manifold in relativity*, J. Environmental Treatment Techniques **1** (2013), no. 2, 101–110.

[15] V. R. Kaĭgorodov, *Structure of the curvature of space-time*, in Problems in geometry, Vol. 14, 177–204, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1983.

[16] D. M. Naik and V. Venkatesha, *η-Ricci solitons and almost η-Ricci solitons on para-Sasakian manifolds*, Int. J. Geom. Methods Mod. Phys. **16** (2019), no. 9, 1950134, 18 pp. https://doi.org/10.1142/S0219887819501342

[17] D. M. Naik, V. Venkatesha, and H. A. Kumara, *Ricci solitons and certain related metrics on almost co-Kaehler manifolds*, Zh. Mat. Fiz. Anal. Geom. **16** (2020), no. 4, 402–417.

[18] B. O'Neill, *Semi-Riemannian Geometry*, Pure and Applied Mathematics, 103, Academic Press, Inc., New York, 1983.

[19] M. M. Praveena and C. S. Bagewadi, *On almost pseudo Bochner symmetric generalized complex space forms*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **32** (2016), no. 1, 149–159.

[20] M. M. Praveena and C. S. Bagewadi, *On almost pseudo symmetric Kähler manifolds*, Palest. J. Math. **6** (2017), Special Issue II, 272–278.

[21] M. M. Praveena, C. S. Bagewadi, and M. R. Krishnamurthy, *Solitons of Kählerian space-time manifolds*, Int. J. Geom. Methods Mod. Phys. **18** (2021), no. 2, Paper No. 2150021, 12 pp. https://doi.org/10.1142/S0219887821500213

[22] A. K. Raychaudhuri, S. Banerji, and A. Banerjee, *General Relativity, Astrophysics, and Cosmology*, Astronomy and Astrophysics Library, Springer-Verlag, New York, 1992.

[23] A. A. Shaikh, D. W. Yoon, and S. K. Hui, *On quasi-Einstein spacetimes*, Tsukuba J. Math. **33** (2009), no. 2, 305–326. https://doi.org/10.21099/tkbjm/1267209423

[24] Mohd. D. Siddiqi and S. A. Siddiqui, *Conformal Ricci soliton and geometrical structure in a perfect fluid spacetime*, Int. J. Geom. Methods Mod. Phys. **17** (2020), no. 6, 2050083, 18 pp. https://doi.org/10.1142/S0219887820500838

[25] Venkatesha and H. A. Kumara, *Ricci soliton and geometrical structure in a perfect fluid spacetime with torse-forming vector field*, Afr. Mat. **30** (2019), no. 5-6, 725–736. https://doi.org/10.1007/s13370-019-00679-y

[26] Venkatesha and H. A. Kumara, *A study of conformally flat quasi-Einstein spacetimes with applications in general relativity*, Kragujevac J. Math. **45** (2021), no. 3, 477–489. https://doi.org/10.46793/kgjmat2103.477v

[27] K. Yano and M. Kon, *Structures on Manifolds*, Series in Pure Mathematics, 3, World Scientific Publishing Co., Singapore, 1984.

Praveena Manjappa Mundalamane
Department of Mathematics
M. S. Ramaiah Institute of Technology (Affiliated to VTU)
Bangalore-560054, India
Email address: mmpraveenamaths@gmail.com

Bagewadi Channabasappa Shanthappa
Department of Mathematics
Kuvempu University
Shivamogga, Karnataka, India
Email address: prof_bagewadi@yahoo.co.in

Mallannara Siddalingappa Siddesha
Department of Mathematics
Jain University
Global Campus-562112
Email address: mssiddesha@gmail.com