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Fisher Information and the Kullback-Leibler Distance in Concomitants of Generalized Order Statistics Under Iterated FGM family

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Abstract. We study the Fisher Information (FI) of $m-$generalized order statistics ($m-$GOSs) and their concomitants about the shape-parameter vector of the Iterated Farlie-Gumbel-Morgenstern (IFGM) bivariate distribution. We carry out a computational study and show how the FI matrix (FIM) helps in finding information contained in singly or multiply censored bivariate samples from the IFGM. We also run numerical computations about the FIM for the sub-models of order statistics (OSs) and sequential order statistics (SOSs). We evaluate FI about the mean and the shape-parameter of exponential and power distributions, respectively. Finally, we investigate the Kullback-Leibler distance in concomitants of $m-$GOSs.

1. Introduction

Suppose that we have a random variable (RV) X, which has an absolutely continuous distribution function (DF) $F(x;\theta)$ and a probability density function (PDF) $f(x;\theta)$, where θ is an unknown parameter (θ may be a single or vector valued parameter), $\theta \in \Theta$ and Θ is the parameter space. Under certain regularity conditions (cf. [2]), the FI about the real parameter θ contained in X is defined by $I_\theta(X) = \mathrm{E}\left(\frac{\partial \log f(X;\theta)}{\partial\theta}\right)^2 = -\mathrm{E}\left(\frac{\partial^2 \log f(X;\theta)}{\partial\theta^2}\right)$. Several authors have studied FI contained in OSs and record values about the unknown parameter of the given DF $F(x;\theta)$. Among those authors are Tukey [27], Mehrotra et al. [23], Park [25], Zheng and Gastwirth [28], Abo-Eleneen and Nagaraja [2], Ahmadi and Arghami [3], Hofmann and Ngaraja [18] , Hofmann [17], and Barakat et al. [12]. The FI plays a valuable role in statistical inference through the Cramer-Rao inequality. The present paper is devoted to study the FI contained in $m-$GOSs and their concomitants about the shape-parameter vector of the IFGM type bivariate distribution. The model of GOSs was suggested by Kamps [20] as a unified model for ordered RVs, which includes, among others, the following sub-models: OSs, SOSs, record values, $k-$record values, Pfeifer's records and progressive type II censored OSs. The subclass $m-$GOSs of GOSs contains many important models of ordered RVs such as OSs, SOSs, lower record values, $k-$records, and type

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II censored OSs. Let $n \in \mathbb{N}$, $m > -1, k > 0$ and $\gamma_i = k + (n-i)(m+1),\ i = 1, 2, ..., , n$, be parameters. Then the RVs $X_{1,n,m,k} \le X_{2,n,m,k} \le ... \le X_{n,n,m,k}$ are said to be $m-$GOSs based on an arbitrary continuous DF F with PDF f and the survival function $\overline{F} = 1 - F$, if their joint PDF is of the form

$$f_{1,2,...,n}(x_1, x_2, ...x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\overline{F}^m(x_i)f(x_i)\right)\overline{F}^{k-1}(x_n)f(x_n),$$

$F^{-1}(1) \ge x_n \ge ... \ge x_1 \ge F^{-1}(0)$. The marginal PDF of rth $m-$GOS, $X_{r,n,m,k}$, $1 \le r \le n$, is given by (cf. [20, 8])

$$f_{r,n,m,k}(x) = \frac{C_{r-1}}{(r-1)!}\overline{F}^{\gamma_r-1}(x)f(x)g_m^{r-1}(F(x)), \tag{1.1}$$

where $C_{r-1} = \prod_{i=1}^{r}\gamma_i, r = 1, 2, ..., n, g_m(x) = h_m(x) - h_m(0), x \in [0, 1)$ and $h_m(x) = -\frac{(1-x)^{m+1}}{(m+1)}$, if $m \neq -1$, while $h_{-1}(x) = -\log(1-x)$. For more details about the subject of the GOSs and its applications, see [21, 10, 11, 5, 6]. The concept of concomitants of OSs, or record values, arises when we have two random samples and we sort the members of one of them (e.g., the first sample) according to corresponding values of the second random sample. Specifically, in any data collection, several characteristics may be recorded, where some of them are often considered as primary and others can be observed from the primary data automatically. The latter ones are called concomitants. Concomitants of OSs and record values can arise in several applications. The most striking application of concomitants of OSs and record values arises in selection procedures, where items or subjects may be chosen on the basis of their X characteristic, and an associated characteristic Y that is hard to measure or can be observed only later may be of interest. For more details, see [16, 4, 26]. The concept of concomitants can also be easily extended to the model of GOSs. Kamps [20] derived and studied the distribution of concomitants of the Pfeifer's record values. Also, Bairamov and Eryilmaz [7] considered the concomitants for the model of progressive type II censoring. Generally speaking, the study of concomitants of any model of ordered RVs based on the random vector (X, Y) is strongly related to the bivariate DF that governs the random vector (X, Y). Several authors have considered the concomitants of $m-$GOSs for different bivariate models, see, for example [1, 13, 14, 15]. One of the most efficient model of the frequently used bivariate models is the IFGM DF. The initiation of this type dates back to Huang and Kotz [19], when they used successive iterations in the original FGM distribution to increase the correlation between components. They showed that just one single iteration can result in tripling the covariance for certain marginals. In this paper, we consider the bivariate FGM with a single iteration, which is defined by

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)\left[1 + \lambda\bar{F}_X(x)\bar{F}_Y(y) + \omega F_X(x)F_Y(y)\bar{F}_X(x)\bar{F}_Y(y)\right], \tag{1.2}$$

denoted by IFGM(λ, ω). The corresponding PDF is given by

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)\left[1 + \lambda C_1(x, y) + \omega C_2(x, y)\right], \tag{1.3}$$

where $C_1(x, y) = (1 - 2F_X(x))(1 - 2F_Y(y))$, $C_2(x, y) = F_X(x)F_Y(y)(2 - 3F_X(x))(2 - 3F_Y(y))$ and $f_X(x)$ and $f_Y(y)$ are the PDFs of the RVs X and Y, respectively. When the two marginals $F_X(x)$ and $F_Y(y)$ are continuous, they showed that the natural parameter space Θ (the admissible set of the parameters λ and ω that makes $F_{X,Y}(x,y)$ is a DF) is convex, where $\Theta=\{(\lambda,\omega):-1\le\lambda\le1;\lambda+\omega\ge-1;\omega\le\frac{3-\lambda+\sqrt{9-6\lambda-3\lambda^2}}{2}\}$. Barakat et al. [13] revisited the family IFGM(λ,ω) and showed that the maximum correlation is higher than previously known. They studied some distributional properties of concomitants of OSs for the family IFGM(λ,ω). Moreover, in that paper the authors gave several applications of this model in reliability theory and showed that, the utilization of the IFGM distribution instead of FGM distribution for studying these applications gives more accurate results. It is worth mentioning that the IFGM model has the same efficiency as the well-known Huang-Kotz FGM model (see [1]), but it is more tractable and flexible.

In this paper, we investigate the properties of the FI about the vector $(\lambda,\omega)\in\Theta$ (defined in the model (1.2)-(1.3)) contained in $(X_{r,n,m,k},Y_{[r,n,m,k]})$, i.e.,

$$mbox\mathbf{I}_{(\lambda,\omega)}(X_{r,n,m,k},Y_{[r,n,m,k]})=\begin{pmatrix} I_\lambda(X_{r,n,m,k},Y_{[r,n,m,k]}) & I_{\lambda,\omega}(X_{r,n,m,k},Y_{[r,n,m,k]}) \\ I_{\lambda,\omega}(X_{r,n,m,k},Y_{[r,n,m,k]}) & I_\omega(X_{r,n,m,k},Y_{[r,n,m,k]}) \end{pmatrix}, \quad (\lambda,\omega)\in\Theta.$$

Moreover, we evaluate the FI about the mean and the shape-parameter of the exponential and power distributions, respectively.

In information theory, the relative entropy is a measure of the distance between the PDFs $f_X(x)$ and $g_Y(y)$, see Kullback-Leibler [22]. This information measure is also known as the Kullback-Leibler distance (K-L distance). Since, in the context of concomitants theory, we often encounter the situation that the highest $X-$scores may be chosen and we wish to know something about the concomitant Y-scores. For example, the X's might refer to a characteristic in a parent and the Y's to the same characteristic in the offspring. The K-L distance can tell us how much information is lost when we approximate the DF of $Y_{[r,n,m,k]}$ by the DF of $Y_{r,n,m,k}$. In Section 4, we investigate the K-L distance from $Y_{[r,n,m,k]}$ to $Y_{r,n,m,k}$. The K-L distance is defined by

$$K(f_X(x),g_Y(y))=\int_{-\infty}^{\infty}f_X(x)\log\frac{f_X(x)}{g_Y(x)}dx. \tag{1.4}$$

2. FIM for (λ,ω) in $(X_{r,n,m,k},Y_{[r,n,m,k]})$

Since the conditional PDF of $Y_{[r,n,m,k]}$ given $X_{r,n,m,k}=x$ is $f_{Y_{[r,n,m,k]}|X_{r,n,m,k}}(y|x)=f_{Y|X}(y|x)$, then the joint PDF of $(X_{r,n,m,k},Y_{[r,n,m,k]})$ is given by

$$f_{X_{r,n,m,k},Y_{[r,n,m,k]}}(x,y;\lambda,\omega)=\frac{C_{r-1}}{(r-1)!}f_{X,Y}(x,y;\lambda,\omega)\overline{F}_X^{\gamma_r-1}(x)g_m^{r-1}(F_X(x)). \tag{2.1}$$

On the other hand, with $F_X(x)=x,\ 0\le x\le1$, and $F_Y(y)=y,\ 0\le y\le1$, one obtains the copula form of the IFGM(λ,ω). This copula (dependence function) is IFGM(λ,ω) with uniform marginals (i.e., free of any unknown parameters), cf. [24]. Therefore, in order to determine the FIM, $\mathbf{I}_{(\lambda,\omega)}(X_{r,n,m,k},Y_{[r,n,m,k]})$, we deal with the copula of (1.2), i.e., when X and $Y\sim U(0,1)$. The following theorem determines this FIM.

Theorem 2.1. *Let $m\neq-1$. Furthermore, let X and $Y\sim U(0,1)$ with the joint PDF (1.3). Then, for any $1\le r\le n$ and $(\lambda,\omega)\in\Theta\cap\Omega$, where $\Omega=\{(\lambda,\omega):\ \mid\lambda C_1(x,y)+\omega C_2(x,y)\mid<1,\forall 0\le x,y\le1\}$ (see Remark 2.1), the FIM about the parameter-vector (λ,ω) is given by*

$$\boldsymbol{I}_{(\lambda,\omega)}(X_{r,n,m,k},Y_{[r,n,m,k]})$$

$$= \begin{pmatrix} I_\lambda(X_{r,n,m,k}, Y_{[r,n,m,k]}) & I_{\lambda,\omega}(X_{r,n,m,k}, Y_{[r,n,m,k]}) \\ I_{\lambda,\omega}(X_{r,n,m,k}, Y_{[r,n,m,k]}) & I_\omega(X_{r,n,m,k}, Y_{[r,n,m,k]}) \end{pmatrix} = \frac{C_{r-1}}{(m+1)^r(r-1)!}$$

(2.2)

$$\times \begin{pmatrix} \sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^i\lambda^j\omega^{i-j}\binom{i}{j}\Phi(j+2,i-j) & \sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^i\lambda^j\omega^{i-j}\binom{i}{j}\Phi(j+1,i-j+1) \\ \sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^i\lambda^j\omega^{i-j}\binom{i}{j}\Phi(j+1,i-j+1) & \sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^i\lambda^j\omega^{i-j}\binom{i}{j}\Phi(j,i-j+2) \end{pmatrix},$$

where

$$\Phi(j+2,i-j) = \left(\sum_{h=0}^{j+2}\sum_{l=0}^{i-j}\sum_{t=0}^{i-j+l+h}(-1)^{h+l+t}(2)^{h+i-j-l}(3)^l\binom{j+2}{h}\binom{i-j}{l}\binom{i-j+l+h}{t}\right)$$

(2.3)

$$\times\beta(r,\frac{\gamma_r+t}{m+1})\left(\sum_{s=0}^{j+2}\sum_{p=0}^{i-j}(-1)^{s+p}(2)^{s+i-j-p}(3)^p\binom{j+2}{s}\binom{i-j}{p}(p+s+i-j+1)^{-1}\right).$$

Proof. From (1.3) and (2.1) we get

$$\log f_{X_{r,n,m,k},Y_{[r,n,m,k]}}(x,y;\lambda,\omega) = \log\frac{C_{r-1}}{(r-1)!} + \log(1+\lambda C_1(x,y)+\omega C_2(x,y))$$

$$+(\gamma_r-1)\log(1-x)+(r-1)\log\left(\frac{1-(1-x)^{m+1}}{m+1}\right),$$

where $C_1(x,y) = (1-2x)(1-2y)$ and $C_2(x,y) = xy(2-3x)(2-3y)$, $0 \le x,y \le 1$. Consider the matrix

(2.4)

$$\begin{pmatrix} I_{11}(x,y) & I_{12}(x,y) \\ I_{21}(x,y) & I_{22}(x,y) \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 \log f_{X_{r,n,m,k},Y_{[r,n,m,k]}}(x,y;\lambda,\omega)}{\partial\lambda^2} & \frac{\partial^2 \log f_{X_{r,n,m,k},Y_{[r,n,m,k]}}(x,y;\lambda,\omega)}{\partial\lambda\partial\omega} \\ \frac{\partial^2 \log f_{X_{r,n,m,k},Y_{[r,n,m,k]}}(x,y;\lambda,\omega)}{\partial\lambda\partial\omega} & \frac{\partial^2 \log f_{X_{r,n,m,k},Y_{[r,n,m,k]}}(x,y;\lambda,\omega)}{\partial\omega^2} \end{pmatrix}.$$

The four elements of the matrix (2.4) can easily be determined from the relations

$$\frac{\partial^2 \log f_{X_{r,n,m,k},Y_{[r,n,m,k]}}(x,y;\lambda,\omega)}{\partial\lambda^2} = \frac{-C_1^2(x,y)}{(1+\lambda C_1(x,y)+\omega C_2(x,y))^2}, \tag{2.5}$$

$$\frac{\partial^2 \log f_{X_{r,n,m,k},Y_{[r,n,m,k]}}(x,y;\lambda,\omega)}{\partial\lambda\partial\omega} = \frac{-C_1(x,y)C_2(x,y)}{(1+\lambda C_1(x,y)+\omega C_2(x,y))^2} \tag{2.6}$$

and

$$\frac{\partial^2 \log f_{X_{r,n,m,k},Y_{[r,n,m,k]}}(x,y;\lambda,\omega)}{\partial\omega^2} = \frac{-C_2^2(x,y)}{(1+\lambda C_1(x,y)+\omega C_2(x,y))^2}. \tag{2.7}$$

Then, by using (2.1), (2.4), (2.5), (2.6) and (2.7), the FIM about parameter-vector (λ,ω) can be expressed by

$$\mathbf{I}_{(\lambda,\omega)}(X_{r,n,m,k}, Y_{[r,n,m,k]}) = \begin{pmatrix} I_\lambda(X_{r,n,m,k}, Y_{[r,n,m,k]}) & I_{\lambda,\omega}(X_{r,n,m,k}, Y_{[r,n,m,k]}) \\ I_{\lambda,\omega}(X_{r,n,m,k}, Y_{[r,n,m,k]}) & I_\omega(X_{r,n,m,k}, Y_{[r,n,m,k]}) \end{pmatrix}$$

$$= -\mathrm{E}\begin{pmatrix} I_{11}(X_{r,n,m,k}, Y_{[r,n,m,k]}) & I_{12}(X_{r,n,m,k}, Y_{[r,n,m,k]}) \\ I_{21}(X_{r,n,m,k}, Y_{[r,n,m,k]}) & I_{22}(X_{r,n,m,k}, Y_{[r,n,m,k]}) \end{pmatrix}$$

$$= -\frac{C_{r-1}}{(r-1)!}\int_0^1\int_0^1 \begin{pmatrix} \frac{\partial^2 \log f_{X_{r,n,m,k},Y_{[r,n,m,k]}}(x,y;\lambda,\omega)}{\partial\lambda^2} & \frac{\partial^2 \log f_{X_{r,n,m,k},Y_{[r,n,m,k]}}(x,y;\lambda,\omega)}{\partial\lambda\partial\omega} \\ \frac{\partial^2 \log f_{X_{r,n,m,k},Y_{[r,n,m,k]}}(x,y;\lambda,\omega)}{\partial\lambda\partial\omega} & \frac{\partial^2 \log f_{X_{r,n,m,k},Y_{[r,n,m,k]}}(x,y;\lambda,\omega)}{\partial\omega^2} \end{pmatrix}$$

$$\times(1+\lambda C_1(x,y)+\omega C_2(x,y))(1-x)^{\gamma_r-1}\left(\frac{1-(1-x)^{m+1}}{m+1}\right)^{r-1} dxdy$$

$$= \frac{C_{r-1}}{(r-1)!}\int_0^1\int_0^1 \begin{pmatrix} \frac{C_1^2(x,y)}{1+\lambda C_1(x,y)+\omega C_2(x,y)} & \frac{C_1(x,y)C_2(x,y)}{1+\lambda C_1(x,y)+\omega C_2(x,y)} \\ \frac{C_1(x,y)C_2(x,y)}{1+\lambda C_1(x,y)+\omega C_2(x,y)} & \frac{C_2^2(x,y)}{1+\lambda C_1(x,y)+\omega C_2(x,y)} \end{pmatrix}$$

$$(1-x)^{\gamma_r-1}\left(\frac{1-(1-x)^{m+1}}{m+1}\right)^{r-1} dxdy. \tag{2.8}$$

The factor $(1+\lambda C_1(x,y)+\omega C_2(x,y))^{-1}$ in the denominator of the integrand in each element of the matrix (2.8) can be expanded as $\sum_{i=0}^{\infty}(-1)^i(\lambda C_1(x,y)+\omega C_2(x,y))^i$ provided $(\lambda,\omega)\in\Omega$. Moreover, this infinite expansion is uniformly convergent for $\mid \lambda C_1(x,y) + \omega C_2(x,y) \mid < 1$. Thus, we get

$$I_\lambda(X_{r,n,m,k}, Y_{[r,n,m,k]}) = \frac{C_{r-1}}{(r-1)!}\sum_{i=0}^{\infty}(-1)^i\int_0^1\int_0^1 C_1^2(x,y)(\lambda C_1(x,y)+\omega C_2(x,y))^i$$

(2.9)

$$\times(1-x)^{\gamma_r-1}\left(\frac{1-(1-x)^{m+1}}{m+1}\right)^{r-1} dxdy = \frac{C_{r-1}}{(r-1)!}\sum_{i=0}^{\infty}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\lambda^j\omega^{i-j}J_1J_2,$$

where

$$J_1 = \int_0^1 (1-2x)^{j+2}x^{i-j}(2-3x)^{i-j}(1-x)^{\gamma_r-1}\left(\frac{1-(1-x)^{m+1}}{m+1}\right)^{r-1} dx$$

$$= \sum_{h=0}^{j+2}\sum_{l=0}^{i-j}(-1)^{l+h}(2)^{h+i-j-l}(3)^l\binom{j+2}{h}\binom{i-j}{l}$$

$$\times\int_0^1 x^{h+l+i-j}(1-x)^{\gamma_r-1}\left(\frac{1-(1-x)^{m+1}}{m+1}\right)^{r-1} dx$$

$$= \sum_{h=0}^{j+2}\sum_{l=0}^{i-j}\sum_{t=0}^{i-j+l+h}(-1)^{l+h+t}(2)^{h+i-j-l}(3)^l\binom{j+2}{h}\binom{i-j}{l}\binom{i-j+l+h}{t}$$

$$\times\int_0^1 (1-x)^{\gamma_r+t-1}\left(\frac{1-(1-x)^{m+1}}{m+1}\right)^{r-1} dx$$

and

$$J_2 = \int_0^1 (1-2y)^{j+2}y^{i-j}(2-3y)^{i-j}dy$$

$$= \sum_{s=0}^{j+2}\sum_{p=0}^{i-j}(-1)^{s+p}(2)^{s+i-j-p}(3)^p\binom{j+2}{s}\binom{i-j}{p}\int_0^1 y^{s+p+i-j}dy$$

$$= \sum_{s=0}^{j+2}\sum_{p=0}^{i-j}(-1)^{s+p}(2)^{s+i-j-p}(3)^p \binom{j+2}{s}\binom{i-j}{p}(s+p+i-j+1)^{-1}. \tag{2.10}$$

Now, by making the transformation $u=\frac{1-(1-x)^{m+1}}{m+1}$ in J_1, we get

$$J_1 = \sum_{h=0}^{j+2}\sum_{l=0}^{i-j}\sum_{t=0}^{i-j+l+h}(-1)^{l+h+t}(2)^{h+i-j-l}(3)^l \binom{j+2}{h}\binom{i-j}{l}\binom{i-j+l+h}{t}$$

$$\times\int_0^{\frac{1}{m+1}} u^{r-1}(1-(m+1)u)^{\frac{\gamma_r+t}{m+1}-1}du$$

$$=\frac{1}{(m+1)^r}\sum_{h=0}^{j+2}\sum_{l=0}^{i-j}\sum_{t=0}^{i-j+l+h}(-1)^{l+h+t}(2)^{h+i-j-l}(3)^l \binom{j+2}{h}\binom{i-j}{l}\binom{i-j+l+h}{t}$$

$$\times\beta(r,\frac{\gamma_r+t}{m+1}). \tag{2.11}$$

Combining (2.9), (2.10), and (2.11), we get $I_\lambda(X_{r,n,m,k},Y_{[r,n,m,k]})$. The other elements of the matrix (2.2) (i.e., $I_{\lambda,\omega}(X_{r,n,m,k},Y_{[r,n,m,k]})$ and $I_\omega(X_{r,n,m,k},Y_{[r,n,m,k]})$) can be obtained by the same procedure. The theorem is proved. □

Remark 2.2. Since $(\lambda=0,\omega=0)\in\Theta\cap\Omega$, then the set $\Theta\cap\Omega\neq\phi$, where ϕ is the empty set. On the other hand, $(\lambda=0,\omega=0)\in\Omega$, then $\{(\lambda,\omega):\mid \lambda C_1(x,y)+\omega C_2(x,y)\mid>1,\forall 0\le x,y\le 1\}=\phi$. Thus, $\Omega\cup\Omega^\star=\mathcal{U}$ (while $\Omega\cap\Omega^\star\neq\phi$), where $\mathcal{U}$ is the universal set and $\Omega^\star=\{(\lambda,\omega):\mid \lambda C_1 x,y)+\omega C_2(x,y)\mid<1,\ 0\le x\le x_0,\ 0\le y\le y_0$, and $\mid \lambda C_1(x,y)+\omega C_2 x,y)\mid\ge 1, x>x_0, y>y_0$, for some $0<x_0,y_0<1\}$. In order to check $(\lambda_0,\omega_0)\in\Omega$, for any $(\lambda_0,\omega_0)\in\Theta$, draw the function $\mathcal{F}(x,y;\lambda_0,\omega_0)=\mid\lambda_0C_1(x,y)+\omega_0C_2(x,y)\mid, 0\le x,y\le 1$, as 3D diagram $(x,y,\mathcal{F})$, by using Mathematica 12. If the curve (surface) of $\mathcal{F}$ falls entirely within the cube $\kappa=\{(x,y,z):-1\le x,y,z\le +1\}$, then $(\lambda_0,\omega_0)\in\Omega$, otherwise $(\lambda_0,\omega_0)\notin\Omega$. Note that $\Omega\cup\Omega^\star=\mathcal{U}$, means that there are only the following possibilities:

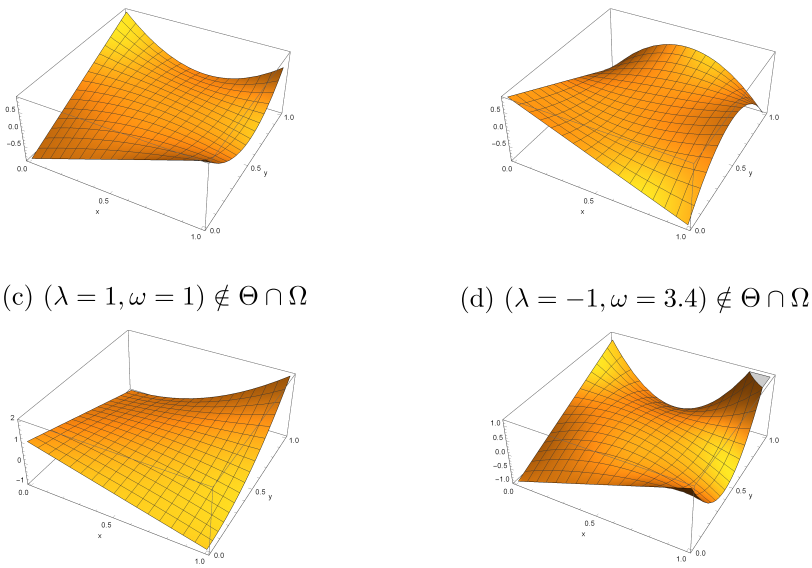
(1) The curve of $\mathcal{F}$ falls entirely within the cube κ, represented by the set Ω;

(2) A portion of that curve falls within κ and the other portion is outside the cube κ, represented by the set $\Omega^\star$.

Figure 1 (Parts a,b,c, and d) shows that how can we apply this check for some values of $(\lambda,\omega)\in\Theta$. It is worth noting that $(\lambda,\omega)\notin\Omega$, only for boundary values (or close to them) of λ and ω, such as $\lambda=-1,+1$ and $\omega=-2,3+\sqrt{3}$.

2.1 Discussion

Table 1 displays the FIM $\mathbf{I}_{(\lambda,\omega)}(X_{r,n,m,k},Y_{[r,n,m,k]})$ for the models of OSs and SOSs (i.e., $\mathbf{I}_{(\lambda,\omega)}(X_{r,n,0,1},\ Y_{[r,n,0,1]})$ and $\mathbf{I}_{(\lambda,\omega)}(X_{r,n,1,1},Y_{[r,n,1,1]})$) as a function of $n, r\le\frac{n+1}{2}$, λ and ω, for $n=1,2,...,5,10$, $\lambda=-0.99$ and different values for ω, for which $(\lambda,\omega)\in\Theta\cap\Omega$. The entries were computed by using the FIM (2.2), the relation (2.3) and MATHEMATICA Ver. 12. The infinite series was cut off after 11 terms and this gives a satisfactory accuracy. Table 1 is constructed as a matrix. Namely, every entry in the two parts (i.e. $(m,k)=(0,1)$ and $(m,k)=(1,1)$) of Table 1 has the form $a\mid b\mid c$, where $a=I_\lambda(X_{r,n,m,k},Y_{[r,n,m,k]})$,



(a) $(\lambda = -0.9, \omega = 1.4) \in \Theta \cap \Omega$ (b) $(\lambda = 0.9, \omega = -1) \in \Theta \cap \Omega$

(c) $(\lambda = 1, \omega = 1) \notin \Theta \cap \Omega$ (d) $(\lambda = -1, \omega = 3.4) \notin \Theta \cap \Omega$

Figure 2: 3D Diagrams for checking the belonging relationship $(\lambda, \omega) \in \Theta \cap \Omega$

$b = I_{\lambda,\omega}(X_{r,n,m,k}, Y_{[r,n,m,k]})$ and $c = I_{\omega}(X_{r,n,m,k}, Y_{[r,n,m,k]})$, respectively. The first raw in each of the two parts of Table 1 represents the FIM $\mathbf{I}_{(\lambda,\omega)}(X, Y)$ (with elements $I_{\lambda}(X, Y)$, $I_{\lambda,\omega}(X, Y)$ and $I_{\omega}(X, Y)$) in a single pair. Since the FIM $\mathbf{I}_{(\lambda,\omega)}(X, Y)$ in a random sample of size n is $n\mathbf{I}_{(\lambda,\omega)}(X, Y)$, Table 1 enables us to compute the proportion of the sample FIM $\mathbf{I}_{(\lambda,\omega)}(X_{r,n,m,k}, Y_{[r,n,m,k]})$ contained in a single pair. For example, in $\mathbf{I}_{(\lambda,\omega)}(X_{r,n,0,1}, Y_{[r,n,0,1]})$ (i.e., Part 1 of Table 1), when $n = 10$, the FI about λ (i.e., the element $I_{\lambda}(X_{r,n,0,1}, Y_{[r,n,0,1]})$) in the extreme pair ranges from 23.6% ($\simeq$ 24%) to 32% of the total FI as $\lambda = -0.99$. In contrast, the FI in the central pair no more than 2% of what is available in the complete sample in all cases $\lambda = -0.99$. Moreover, the FI about the vector (λ, ω) (i.e., the element $I_{\lambda,\omega}(X_{r,n,0,1}, Y_{[r,n,0,1]})$) in the extreme pair (actually the second lower extreme, i.e., $r = 2$) ranges from 3.5% ($\simeq$ 4%) to 11% of the total FI as $\lambda = -0.99$. In contrast, the FI in the extreme pair is no more than 2% of what is available in the complete sample in all cases $\lambda = -0.99$. Finally, the FI about ω (i.e., the element $I_{\omega}(X_{r,n,0,1}, Y_{[r,n,0,1]})$) in the extreme pair is no more than 3% of what is available in the complete sample in all cases $\lambda = -0.99$. On the other hand, the FI in the central pair ($r = 4$) ranges from 2% to 7.5% ($\simeq$ 8%) of the total FI as $\lambda = -0.99$.

Also, as example in $\mathbf{I}_{(\lambda,\omega)}(X_{r,n,1,1}, Y_{[r,n,1,1]})$ (i.e., Part 2 of Table 1), when $n = 10$, the FI about λ (i.e., the element $I_{\lambda}(X_{r,n,1,1}, Y_{[r,n,1,1]})$) in the extreme pair ranges from 32% to 41% of the total FI, as $\lambda = -0.99$. In contrast, the FI in the central pair ranges from 5% to 7% of what is available in the complete sample in all cases $\lambda = -0.99$. Moreover, the FI about the vector (λ, ω) (i.e., the element $I_{\lambda,\omega}(X_{r,n,1,1}, Y_{[r,n,1,1]})$) in the extreme pair (actually the third lower extreme, i.e., $r = 3$) ranges from 4% to 9% of the total FI, as $\lambda = -0.99$. In contrast, the FI in the extreme pair is no more than 4% of what is available in the complete sample in all cases $\lambda = -0.99$. Finally, the FI about ω (i.e., the element $I_\omega(X_{r,n,1,1}, Y_{[r,n,1,1]})$) in the extreme pair is no more than 1% of what is available in the complete sample in all cases $\lambda = -0.99$. On the other hand, the FI in the central pair ranges from 2% to 6% of the total FI, as $\lambda = -0.99$.

Another important usage of Table 1 is that it can readily be used to obtain the FI contained in singly or multiply censored bivariate samples from the IFGM(λ, ω) distribution. One just adds up the FI in individual pairs that constitute the censored sample. For example, in $\mathbf{I}_{(\lambda,\omega)}(X_{r,n,0,1}, Y_{[r,n,0,1]})$ (i.e., the element $I_\lambda(X_{r,n,0,1}, Y_{[r,n,0,1]})$), when $n = 10$, the FI about λ in the Type II censored sample consisting of the bottom (or the top) three pairs ranges from 45% to 58% (as ω varies over $0, 0.4, 0.8, 1.2, 1.4$) when $\lambda = -0.99$. Another example, in $\mathbf{I}_{(\lambda,\omega)}(X_{r,n,1,1}, Y_{[r,n,1,1]})$ (i.e., the element $I_\lambda(X_{r,n,0,1}, Y_{[r,n,0,1]})$), when $n = 10$, the FI about λ in the Type II censored sample consisting of the bottom (or the top) three pairs ranges from 70% to 86% (as ω varies over $0, 0.4, 0.8, 1.2, 1.4$) when $\lambda = -0.99$. Note that, for the preceding ratios (when $n = 10$), we have in general FI $(\lambda, \omega_1) <$ FI (λ, ω_2), if $\omega_1 < \omega_2$, where FI(λ, ω) denotes to the FI about λ in the Type II censored sample consisting of the bottom three pairs at ω.

From Table 1, the following properties can be extracted for the models of OSs and SOSs:

1. In general, for $(m, k) = (0, 1)$, or $(m, k) = (1, 1)$, we have

$$I_\lambda(X_{r,n,m,k}, Y_{[r,n,m,k]}) > I_{\lambda,\omega}(X_{r,n,m,k}, Y_{[r,n,m,k]}) > I_\omega(X_{r,n,m,k}, Y_{[r,n,m,k]}).$$

2. For $(m, k) = (0, 1)$, or $(m, k) = (1, 1)$, $I_\lambda(X_{r,n,m,k}, Y_{[r,n,m,k]})$ increases with increasing the difference between r and n, for $r \le \frac{n+1}{2}$. For any n, the greatest value of $I_\lambda(X_{r,n,m,k}, Y_{[r,n,m,k]})$ is always obtained at the lower extreme.

3. $I_{\lambda,\omega}(X_{r,n,0,1}, Y_{[r,n,0,1]})$ increases with increasing the difference between $r > 1$ and n, for $r \le \frac{n+1}{2}$. For any n, the greatest value of $I_{\lambda,\omega}(X_{r,n,0,1}, Y_{[r,n,0,1]})$ is obtained at the second lower extreme. On the other hand, the greatest value of $I_{\lambda,\omega}(X_{r,n,1,1}, Y_{[r,n,1,1]})$ is attained frequently at $r = 3$, or $r = 4$.

4. For $(m, k) = (0, 1)$, or $(m, k) = (1, 1)$, $I_\omega(X_{r,n,m,k}, Y_{[r,n,m,k]})$ decreases with increasing the difference between r and n, for $r < \frac{n+1}{2}$.

5. In general, we have $I_\lambda(X_{r,n,1,1}, Y_{[r,n,1,1]}) > I_\lambda(X_{r,n,0,1}, Y_{[r,n,0,1]})$.

Table 1: FIM for $(X_{r,n,0,1}, Y_{[r,n,0,1]})$ and $(X_{r,n,1,1}, Y_{[r,n,1,1]})$ about the parameter-vector $(\lambda = -0.99, \omega)$

		$\mathbf{I}_{(\lambda,\omega)}(X_{r,n,0,1}, Y_{[r,n,0,1]})$,		$\lambda = -0.99$		
n	**r**	$\omega = 0$	$\omega = 0.4$	$\omega = 0.8$	$\omega = 1.2$	$\omega = 1.4$
1	1	0.1950|0.0620|0.0450	0.1650|0.0391|0.0261	0.1540|0.0321|0.0210	0.1490|0.0287|0.0188	0.1470|0.0277|0.0182
2	1	0.1950|0.0210|0.0110	0.1900|0.0182|0.0095	0.1887|0.0175|0.0091	0.1860|0.0172|0.0091	0.1850|0.0172|0.0092
3	1	0.2430|0.0170|0.0070	0.2420|0.0168|0.0072	0.2400|0.0169|0.0073	0.2380|0.0170|0.0076	0.2380|0.0172|0.0078
3	2	0.0960|0.0270|0.0180	0.0870|0.0209|0.0141	0.0830|0.0187|0.0130	0.0810|0.0175|0.0123	0.0800|0.0171|0.0122
4	1	0.2930|0.0180|0.0060	0.2910|0.0174|0.0064	0.2890|0.0176|0.0065	0.2870|0.0178|0.0068	0.2870|0.0180|0.0070
4	2	0.0960|0.0160|0.0110	0.0940|0.0150|0.0096	0.0920|0.0147|0.0097	0.0915|0.0147|0.0098	0.0920|0.0148|0.0100
5	1	0.3370|0.0180|0.0050	0.334|0.0179|0.0057	0.3320|0.0180|0.0058	0.3310|0.0182|0.0061	0.3290|0.0184|0.0061
5	2	0.1190|0.0150|0.0100	0.1170|0.0157|0.0090	0.1160|0.0158|0.0093	0.1150|0.0161|0.0096	0.1160|0.0163|0.0098
5	3	0.0620|0.0160|0.0110	0.0600|0.0139|0.0106	0.0550|0.0130|0.0103	0.0540|0.0125|0.0103	0.0550|0.0124|0.0104
10	1	0.4850|0.0170|0.0030	0.4830|0.0166|0.0032	0.4790|0.0167|0.0033	0.4770|0.0167|0.0033	0.4760|0.0168|0.0034
10	2	0.2550|0.0220|0.0090	0.2530|0.0213|0.0070	0.2510|0.0215|0.0072	0.2490|0.0218|0.0074	0.2480|0.0220|0.0075
10	3	0.1310|0.0190|0.0090	0.1290|0.0191|0.0096	0.1280|0.0200|0.0098	0.1270|0.0200|0.0103	0.1260|0.0202|0.0110
10	4	0.0650|0.0140|0.0090	0.0650|0.0138|0.0102	0.0640|0.0141|0.0123	0.0610|0.0145|0.0133	0.0600|0.0148|0.0115
10	5	0.0360|0.0090|0.0090	0.0360|0.0086|0.0091	0.0358|0.0088|0.0095	0.0352|0.0090|0.0099	0.0340|0.0090|0.0095
		$\mathbf{I}_{(\lambda,\omega)}(X_{r,n,1,1}, Y_{[r,n,1,1]})$,		$\lambda = -0.99$		
n	**r**	$\omega = 0$	$\omega = 0.4$	$\omega = 0.8$	$\omega = 1.2$	$\omega = 1.4$
1	1	0.1950|0.0618|0.0442	0.1650|0.0391|0.0261	0.1540|0.0321|0.0210	0.1490|0.0287|0.0188	0.1470|0.0277|0.0182
2	1	0.2440|0.0169|0.0072	0.2420|0.0168|0.0072	0.2390|0.0168|0.0073	0.2380|0.0170|0.0075	0.2380|0.0172|0.0077
3	1	0.3370|0.0178|0.0056	0.3340|0.0179|0.0057	0.3320|0.0180|0.0058	0.3300|0.0182|0.0060	0.3290|0.0184|0.0062
3	2	0.1040|0.0158|0.0096	0.1030|0.0152|0.0094	0.1010|0.0151|0.0095	0.1010|0.0152|0.0097	0.1000|0.0153|0.0099
4	1	0.4080|0.0178|0.0044	0.4050|0.0178|0.0045	0.4020|0.0179|0.0046	0.4000|0.0181|0.0047	0.3990|0.0182|0.0048
4	2	0.1590|0.0179|0.0084	0.1580|0.0181|0.0086	0.1560|0.0183|0.0089	0.1560|0.0187|0.0093	0.1550|0.0189|0.0095
5	1	0.4630|0.0169|0.0035	0.4590|0.0170|0.0036	0.4570|0.0171|0.0037	0.4540|0.0172|0.0037	0.4530|0.0173|0.0038
5	2	0.2160|0.0203|0.0076	0.2140|0.0204|0.0078	0.2120|0.0207|0.0079	0.2110|0.0210|0.0083	0.2090|0.0212|0.0085
5	3	0.0880|0.0149|0.0094	0.0870|0.0151|0.0097	0.0870|0.0154|0.0100	0.0870|0.0158|0.0105	0.0860|0.0161|0.0108
10	1	0.6190|0.0125|0.0014	0.6160|0.0125|0.0014	0.6130|0.0125|0.0014	0.6090|0.0126|0.0014	0.6080|0.0126|0.0014
10	2	0.4120|0.0198|0.0037	0.4080|0.0199|0.0037	0.4050|0.0199|0.0038	0.4020|0.0110|0.0038	0.4030|0.0201|0.0039
10	3	0.2690|0.0229|0.0063	0.2670|0.0230|0.0064	0.2640|0.0232|0.0065	0.2620|0.0234|0.0067	0.2610|0.0236|0.0068
10	4	0.1710|0.0225|0.0087	0.1690|0.0227|0.0089	0.1680|0.0230|0.0091	0.1670|0.0235|0.0095	0.1660|0.0237|0.0097
10	5	0.1020|0.0193|0.0103	0.1010|0.0196|0.0106	0.1000|0.0200|0.0111	0.1002|0.0206|0.0115	0.1000|0.0209|0.0118

3. FI in $Y_{[r,n,m,k]}$ of $m-$GOSs

We begin this section by representing a result of Barakat and Husseiny [9] that gives an explicit form of the marginal PDF of the concomitant $Y_{[r,n,m,k]}$ of $m-$GOS for IFGM. This will enable us to derive and study the FI about mean and the shape parameter of the exponential and power distributions, respectively.

Lemma 3.1. ([9]) *Let $m \neq -1$, $X \sim F_X$ and $Y \sim F_Y$. Furthermore, let $V_1 \sim F_X^2$ (with PDF f_{V_1}) and $V_2 \sim F_X^3$ (with PDF f_{V_2}). Then,*

$$\begin{aligned} f_{[r,n,m,k]}(y) &= f_Y(y)\left[1+\lambda\mathfrak{D}_1(r,n,m,k)(1-2F_Y(y))+\omega\mathfrak{D}_2(r,n,m,k)(2F_Y(y)-3F_Y^2(y))\right] \\ &= f_Y(y)+\lambda\mathfrak{D}_1(r,n,m,k)(f_Y(y)-f_{V_1}(y))+\omega\mathfrak{D}_2(r,n,m,k)(f_{V_1}(y)-f_{V_2}(y)), \quad (3.1) \end{aligned}$$

where $\mathfrak{D}_1(r,n,m,k) = 2\prod_{i=1}^{r}\frac{\gamma_i}{\gamma_i+1} - 1$ *and* $\mathfrak{D}_2(r,n,m,k) = 4\prod_{i=1}^{r}\frac{\gamma_i}{\gamma_i+1} - 3\prod_{i=1}^{r}\frac{\gamma_i}{\gamma_i+2} - 1.$

Theorem 3.2. (FI in $Y_{[r,n,m,k]}$ about $\mathrm{E}(Y)$) *Suppose that $m \neq -1$, and Y has exponential*

DF with mean θ, then the FI about θ, contained in $Y_{[r,n,m,k]}$, is given by

$$(3.2)\qquad I_\theta(Y_{[r,n,m,k]}) = \frac{1}{\theta^2}\left(1 - 2\delta_1 - \frac{2\delta_3}{27} + \int_0^\infty \frac{\delta_1^2 e^{-w} + \delta_3^2 e^{-5w} - 2\delta_1\delta_3 e^{-3w}}{\delta_1 + \delta_2 e^{-w} + \delta_3 e^{-2w}} w^2 dw\right),$$

where $w = \frac{y}{\theta}$, $\delta_1 = 1 - (\lambda\mathcal{D}_1(r,n,m,k) + \omega\mathcal{D}_2(r,n,m,k))$, $\delta_2 = 2(\lambda\mathcal{D}_1(r,n,m,k) + 2\omega\mathcal{D}_2(r,n,m,k))$ and $\delta_3 = -3\omega\mathcal{D}_2(r,n,m,k)$.

Proof. Let $f_Y(y) = \frac{1}{\theta}\exp(\frac{-y}{\theta})$, $\theta > 0$, $y \geq 0$, by using (3.1) we get the PDF of the concomitant $Y_{[r,n,m,k]}$ based on the exponential distribution as

$$f_{[r,n,m,k]}(y;\theta) = \frac{1}{\theta}\exp(\frac{-y}{\theta})\Big[1 + \lambda\mathcal{D}_1(r,n,m,k)\left(1 - 2\left(1 - \exp(\frac{-y}{\theta})\right)\right)$$
$$+\omega\mathcal{D}_2(r,n,m,k)\left(2\left(1 - \exp(\frac{-y}{\theta})\right) - 3\left(1 - \exp(\frac{-y}{\theta})\right)^2\right)\Big].$$

This expression, after some algebra, can be written as

$$f_{Y_{[r,n,m,k]}}(y;\theta) = \frac{1}{\theta}\exp(\frac{-y}{\theta})\left(\delta_1 + \delta_2\exp(\frac{-y}{\theta}) + \delta_3\exp(\frac{-2y}{\theta})\right),$$

where $\delta_1 = 1-(\lambda\mathcal{D}_1(r,n,m,k)+\omega\mathcal{D}_2(r,n,m,k))$, $\delta_2 = 2(\lambda\mathcal{D}_1(r,n,m,k)+2\omega\mathcal{D}_2(r,n,m,k))$ and $\delta_3 = -3\omega\mathcal{D}_2(r,n,m,k)$. Therefore,

$$\frac{\partial\log(f_{Y_{[r,n,m,k]}}(y;\theta))}{\partial\theta} = \frac{1}{\theta}\left(\frac{y}{\theta} - 1 + \frac{y}{\theta}\frac{\delta_2\exp(\frac{-y}{\theta}) + 2\delta_3\exp(\frac{-2y}{\theta})}{\delta_1 + \delta_2\exp(\frac{-y}{\theta}) + \delta_3\exp(\frac{-2y}{\theta})}\right)$$

$$= \frac{1}{\theta}\left(\frac{2y}{\theta} - 1 - \frac{\delta_1\frac{y}{\theta}}{\delta_1 + \delta_2\exp(\frac{-y}{\theta}) + \delta_3\exp(\frac{-2y}{\theta})} + \frac{\delta_3\frac{y}{\theta}\exp(\frac{-2y}{\theta})}{\delta_1 + \delta_2\exp(\frac{-y}{\theta}) + \delta_3\exp(\frac{-2y}{\theta})}\right).$$

Thus,

$$\left(\frac{\partial\log f_{Y_{[r,n,m,k]}}(y;\theta)}{\partial\theta}\right)^2 = \frac{1}{\theta^2}\Bigg(4w^2 + 1 + \frac{\delta_1^2 w^2}{(\delta_1 + \delta_2 e^{-w} + \delta_3 e^{-2w})^2}$$
$$+\frac{\delta_3^2 w^2 e^{-4w}}{(\delta_1 + \delta_2 e^{-w} + \delta_3 e^{-2w})^2} - 4w - \frac{4\delta_1 w^2}{\delta_1 + \delta_2 e^{-w} + \delta_3 e^{-2w}}$$
$$+\frac{4\delta_3 w^2 e^{-2w}}{\delta_1 + \delta_2 e^{-w} + \delta_3 e^{-2w}} + \frac{2\delta_1 w}{\delta_1 + \delta_2 e^{-w} + \delta_3 e^{-2w}}$$
$$(3.3)\qquad -\frac{2\delta_3 w e^{-2w}}{\delta_1 + \delta_2 e^{-w} + \delta_3 e^{-2w}} - \frac{2\delta_1\delta_3 w^2 e^{-2w}}{(\delta_1 + \delta_2 e^{-w} + \delta_3 e^{-2w})^2}\Bigg),$$

where $w = \frac{y}{\theta}$. On the other hand, the PDF of the RV $W = \frac{Y_{[r,n,m,k]}}{\theta}$ is $f_W(w) = e^{-w}(\delta_1 + \delta_2 e^{-w} + \delta_3 e^{-2w})$. Therefore (3.3) yields

$$I_\theta(Y_{[r,n,m,k]}) = \int_0^\infty \left(\frac{\partial\log f_W(w)}{\partial\theta}\right)^2 f_W(w)dw = \frac{1}{\theta^2}\sum_{i=1}^{10} J_i,$$

where

$$J_1 = 4\int_0^\infty w^2 e^{-w}\left(\delta_1 + \delta_2 e^{-w} + \delta_3 e^{-2w}\right)dw = 4\left(2\delta_1 + \frac{\delta_2}{4} + \frac{2\delta_3}{27}\right),$$

$$J_2=\int_0^\infty e^{-w}\left(\delta_1+\delta_2e^{-w}+\delta_3e^{-2w}\right)dw=1,$$

$$J_3=\delta_1^2\int_0^\infty \frac{w^2e^{-w}}{\delta_1+\delta_2e^{-w}+\delta_3e^{-2w}}dw,\ J_4=\delta_3^2\int_0^\infty \frac{w^2e^{-5w}}{\delta_1+\delta_2e^{-w}+\delta_3e^{-2w}}dw,$$

$$J_5=-4\int_0^\infty we^{-w}\left(\delta_1+\delta_2e^{-w}+\delta_3e^{-2w}\right)dw=-4\left(\delta_1+\frac{\delta_2}{4}+\frac{\delta_3}{9}\right),$$

$$J_6=-4\delta_1\int_0^\infty w^2e^{-w}dw=-8\delta_1, J_7=4\delta_3\int_0^\infty w^2e^{-3w}dw=\frac{8\delta_3}{27},$$

$$J_8=2\delta_1\int_0^\infty we^{-w}dw=2\delta_1, J_9=-2\delta_3\int_0^\infty we^{-3w}dw=\frac{-2\delta_3}{9},$$

and

$$J_{10}=-2\delta_1\delta_3\int_0^\infty \frac{w^2e^{-3w}}{\delta_1+\delta_2e^{-w}+\delta_3e^{-2w}}dw.$$

Therefore, we get $I_\theta(Y_{[r,n,m,k]})=\frac{1}{\theta^2}\left(1-2\delta_1-\frac{2\delta_3}{27}+J_3+J_4+J_{10}\right).$ □

Theorem 3.3. (FI in $Y_{[r,n,m,k]}$ about the shape parameter of power distribution) *Suppose that* Y *has power DF* $F_Y(y)=y^\alpha,\alpha>0,0\le y\le 1$, *then the FI in* $Y_{[r,n,m,k]}$ *about* α *is given by*

$$I_\alpha(Y_{[r,n,m,k]})=\frac{1}{\alpha^2}\left(1-\frac{\delta_2^\star}{4}-\frac{8\delta_3^\star}{27}\right)+E\left(\frac{\partial\log\left(\delta_1^\star+\delta_2^\star Y_{[r,n,m,k]}^\alpha+\delta_3^\star Y_{[r,n,m,k]}^{2\alpha}\right)}{\partial\alpha}\right)^2,$$

where $\delta_1^\star=1+\lambda\mathcal{D}_1(r,n,m,k)$, $\delta_2^\star=2(\omega\mathcal{D}_2(r,n,m,k)-\lambda\mathcal{D}_1(r,n,m,k))$ *and* $\delta_3^\star=-3\omega\mathcal{D}_2(r,n,\ m,k)$.

Proof. Let $f_Y(y)=\alpha y^{\alpha-1}$, $\alpha>0$, $0\le y\le 1$, by using (3.1) we get the PDF of $Y_{[r,n,m,k]}$ based on the power distribution as follow

$$f_{[r,n,m,k]}(y;\alpha)=\alpha y^{\alpha-1}\left[1+\lambda\mathcal{D}_1(r,n,m,k)(1-2y^\alpha)+\omega\mathcal{D}_2(r,n,m,k)(2y^\alpha-3y^{2\alpha})\right].$$

This expression, after some algebra, can be written as

$$f_{[r,n,m,k]}(y;\alpha)=\alpha y^{\alpha-1}\left(\delta_1^\star+\delta_2^\star y^\alpha+\delta_3^\star y^{2\alpha}\right),$$

where $\delta_1^\star=1+\lambda\mathcal{D}_1(r,n,m,k)$, $\delta_2^\star=2(\omega\mathcal{D}_2(r,n,m,k)-\lambda\mathcal{D}_1(r,n,m,k))$ and $\delta_3^\star=-3\omega\mathcal{D}_2(r,n,\ m,k)$. Therefore,

$$\frac{\partial\log f_{[r,n,m,k]}(y;\alpha)}{\partial\alpha}=\frac{1}{\alpha}+\log y+\frac{\delta_2^\star y^\alpha\log y+2\delta_3^\star y^{2\alpha}\log y}{\delta_1^\star+\delta_2^\star y^\alpha+\delta_3^\star y^{2\alpha}}.$$

Then,

$$\left(\frac{\partial\log f_{[r,n,m,k]}(y;\alpha)}{\partial\alpha}\right)^2=\frac{1}{\alpha^2}+(\log y)^2+\left(\frac{\delta_2^\star y^\alpha\log y+2\delta_3^\star y^{2\alpha}\log y}{\delta_1^\star+\delta_2^\star y^\alpha+\delta_3^\star y^{2\alpha}}\right)^2+\frac{2\log y}{\alpha}$$

$$+\frac{2}{\alpha}\left(\frac{\delta_2^\star y^\alpha\log y+2\delta_3^\star y^{2\alpha}\log y}{\delta_1^\star+\delta_2^\star y^\alpha+\delta_3^\star y^{2\alpha}}\right)+2\left(\frac{\delta_2^\star y^\alpha\log y+2\delta_3^\star y^{2\alpha}\log y}{\delta_1^\star+\delta_2^\star y^\alpha+\delta_3^\star y^{2\alpha}}\right)\log y. \tag{3.4}$$

Thus, (3.4) yields

$$I_\alpha(Y_{[r,n,m,k]}) = \int_0^1 \left(\frac{\partial \log f_{Y_{[r,n,m,k]}}(y;\alpha)}{\partial \alpha}\right)^2 f_{[r,n,m,k]}(y;\alpha) = \sum_{i=1}^{6} J_i,$$

where

$$J_1 = \frac{1}{\alpha^2}\int_0^1 \alpha y^{\alpha-1}\left(\delta_1^\star + \delta_2^\star y^\alpha + \delta_3^\star y^{2\alpha}\right) dy = \frac{1}{\alpha^2},$$

$$J_2 = \alpha\int_0^1 y^{\alpha-1}\left(\delta_1^\star + \delta_2^\star y^\alpha + \delta_3^\star y^{2\alpha}\right)(\log y)^2 dy = \frac{27(8\delta_1^\star + \delta_2^\star) + 8\delta_3^\star}{108\alpha^2},$$

$$J_3 = \alpha\int_0^1 y^{\alpha-1}\frac{\left(\delta_2^\star y^\alpha \log y + 2\delta_3^\star y^{2\alpha}\log y\right)^2}{\delta_1^\star + \delta_2^\star y^\alpha + \delta_3^\star y^{2\alpha}} dy,$$

$$J_4 = 2\int_0^1 y^{\alpha-1}(\delta_1^\star + \delta_2^\star y^\alpha + \delta_3^\star y^{2\alpha})\log y dy = -\frac{36\delta_1^\star + 9\delta_2^\star + 4\delta_3^\star}{18\alpha 2},$$

$$J_5 = 2\int_0^1 y^{\alpha-1}\left(\delta_2^\star y^\alpha \log y + 2\delta_3^\star y^{2\alpha}\log y\right) dy = -\frac{9\delta_2^\star + 8\delta_3^\star}{18\alpha^2},$$

and

$$J_6 = 2\alpha\int_0^1 y^{\alpha-1}\left(\delta_2^\star y^\alpha \log y + 2\delta_3^\star y^{2\alpha}\log y\right)\log y dy = \frac{27\delta_2^\star + 16\delta_3^\star}{54\alpha^2}.$$

Therefore, we get $I_\alpha(Y_{[r,n,m,k]}) = \frac{1}{\alpha^2}\left(1 - \frac{1}{4}\delta_2^\star - \frac{8}{27}\delta_3^\star\right) + J_3$. □

We compute the FI in $Y_{[r,n,0,1]}$ and $Y_{[r,n,1,1]}$ about $\mathrm{E}(Y) = \theta$ by using formula (3.2). Table 2 provides $I_\theta(Y_{[r,n,0,1]})$ and $I_\theta(Y_{[r,n,1,1]})$ values for $n = 5, 15$, $\lambda = 0.99$, $\omega = -1.8, -1.6$, $-1.4, -1.2, 0, 0.4, 0.8$, for $\theta = 1$. Moreover, Table 2 reveals that the greatest values of FI are almost obtained at the maximum OSs.

4. K-L distance

Upon relying on the definition in relation (1.4), we get the following theorem:

Theorem 4.1. *Let $m \neq -1$ and $Y_{[r,n,m,k]}$ is concomitants of rth $m-$GOS in IFGM family, then the K-L distance from $Y_{[r,n,m,k]}$ to $Y_{r,n,m,k}$ is given by*

$$K(Y_{[r,n,m,k]}, Y_{r,n,m,k}) = -H(Y_{[r,n,m,k]}) + I(Y_{[r,n,m,k]}, Y_{r,n,m,k})$$

$$= -\delta_{r,n,m,k} - \log\left(\tfrac{C_{r-1}}{(r-1)!}\right) - (\gamma_r - 1)(-1 + \frac{\lambda\mathfrak{D}_1(r,n,m,k)}{2} + \frac{\omega\mathfrak{D}_2(r,n,m,k)}{3})$$

(4.1)

$$-(r-1)\left(\lambda\mathfrak{D}_1(r,n,m,k)H_n^\star + \omega\mathfrak{D}_2(r,n,m,k)H_n^{\star\star} - H_n[\tfrac{1}{m+1}] - \log(1+m)\right),$$

Table 2: FI in $Y_{[r,n,0,1]}$ and $Y_{[r,n,1,1]}$ for θ at $\theta = 1$, where $\mathrm{E}(Y) = \theta$

$I_\theta(Y_{[r,n,0,1]})$		$\lambda = 0.99$					$I_\theta(Y_{[r,n,1,1]})$		$\lambda = 0.99$				
n	**r**	$\omega = -1.8$	$\omega = -1.6$	$\omega = -1.4$	$\omega = -1.2$	$\omega = 0$	**n**	**r**	$\omega = -1.8$	$\omega = -1.6$	$\omega = -1.4$	$\omega = -1.2$	$\omega = 0$
5	1	0.7694	0.7706	0.7730	0.7766	0.827	5	1	0.7564	0.7623	0.7692	0.7770	0.8497
5	2	0.9014	0.8928	0.8853	0.8791	0.870	5	2	0.8002	0.7966	0.7943	0.7936	0.8275
5	3	1.045	1.038	1.032	1.026	1.0000	5	3	0.9205	0.9096	0.9000	0.8919	0.8756
5	4	1.162	1.165	1.168	1.172	1.196	5	4	1.058	1.050	1.044	1.037	1.008
5	5	1.312	1.310	1.314	1.322	1.454	5	5	1.227	1.231	1.238	1.245	1.323
15	1	0.7787	0.7866	0.7952	0.8044	0.8789	15	1	0.8293	0.8369	0.8450	0.8534	0.9149
15	2	0.7529	0.7566	0.7614	0.7676	0.8376	15	2	0.7699	0.7777	0.7862	0.7956	0.8737
15	3	0.7836	0.7810	0.7800	0.7807	0.8261	15	3	0.7509	0.7562	0.7627	0.7704	0.8475
15	4	0.8385	0.8301	0.8235	0.8188	0.8348	15	4	0.7567	0.7584	0.7615	0.7661	0.8323
15	5	0.9019	0.8894	0.8787	0.8697	0.8587	15	5	0.7749	0.7769	0.7761	0.7770	0.8264
15	6	0.9652	0.9507	0.9378	0.9265	0.8953	15	6	0.8141	0.8074	0.8025	0.7995	0.8287
15	7	1.023	1.009	0.9966	0.9849	0.9427	15	7	0.8569	0.8464	0.8377	0.8310	0.8391
15	8	1.074	1.063	1.052	1.042	1.000	15	8	0.9051	0.8915	0.8798	0.8700	0.8577
15	9	1.118	1.111	1.104	1.098	1.066	15	9	0.9560	0.9406	0.9269	0.9150	0.8852
15	10	1.156	1.154	1.153	1.151	1.141	15	10	1.007	0.9919	0.9778	0.9650	0.9229
15	11	1.193	1.196	1.199	1.202	1.225	15	11	1.058	1.044	1.031	1.019	0.9729
15	12	1.233	1.239	1.246	1.254	1.317	15	12	1.105	1.096	1.087	1.078	1.039
15	13	1.287	1.291	1.298	1.307	1.419	15	13	1.152	1.149	1.146	1.143	1.128
15	14	1.368	1.360	1.361	1.368	1.530	15	14	1.206	1.211	1.216	1.221	1.259
15	15	1.497	1.461	1.442	1.439	1.654	15	15	1.340	1.336	1.339	1.346	1.495

where

$$\delta_{r,n,m,k} = -\log(1-\lambda\mathcal{D}_1(r,n,m,k)-\omega\mathcal{D}_2(r,n,m,k))+2b(r)J_0(r,n,m,k)+6c(r)J_1(r,n,m,k),$$

$$J_\ell(r,n,m,k) = \int_0^1 \tfrac{z^\ell(a(r)z+b(r)z^2+c(r)z^3)}{a(r)+2b(r)z+3c(r)z^2}dz, \ell = 0,1,$$

$$H_n^\star = H_n[\tfrac{1}{m+1}] - H_n[\tfrac{2}{m+1}],$$

$$H_n^{\star\star} = H_n[\tfrac{1}{m+1}] - 2H_n[\tfrac{2}{m+1}] + H_n[\tfrac{3}{m+1}],$$

$H_n[m]$ denotes the generalized harmonic numbers, which is calculated by $H_n[m] = \sum_{i=1}^n \frac{1}{i^m}$, and $H(Y_{[r,n,m,k]})$ is the Shannon entropy that was derived in Theorem 3.2 of [9].

Proof. For simplicity, write $\mathcal{D}_1(r,n,m,k) = \mathcal{D}_r^{(1)}$ and $\mathcal{D}_2(r,n,m,k) = \mathcal{D}_r^{(2)}$, then by using (1.1) and (3.1), the K-L distance from $Y_{[r,n,m,k]}$ to $Y_{r,n,m,k}$ is given by

$$K(Y_{[r,n,m,k]}, Y_{r,n,m,k}) = \int_{-\infty}^{\infty} f_{[r,n,m,k]}(y) \log\left(\tfrac{f_{[r,n,m,k]}(y)}{f_{r,n,m,k}(y)}\right) dy$$

$$= -H(Y_{[r,n,m,k]}) - I(Y_{[r,n,m,k]}, Y_{r,n,m,k})$$

$$= -H(Y_{[r,n,m,k]}) - \int_{-\infty}^{\infty} f_{[r,n,m,k]}(y) \log f_{r,n,m,k}(y) dy$$

$$(4.2) \quad = -H(Y_{[r,n,m,k]}) - \log\left(\tfrac{C_{r-1}}{(r-1)!}\right) - (\gamma_r - 1)A - (r-1)B - \mathrm{E}(\log f_Y(Y_{[r,n,m,k]})),$$

where

$$A=\int_{-\infty}^{\infty} f_Y(y)\left[1+\lambda\mathcal{D}_r^{(1)}(1-2F_Y(y))+\omega\mathcal{D}_r^{(2)}(2F_Y(y)-3F_Y^2(y))\right]$$
$$\times\log\overline{F}_Y(y)dy,$$
$$B=\int_{-\infty}^{\infty} f_Y(y)\left[1+\lambda\mathcal{D}_r^{(1)}(1-2F_Y(y))+\omega\mathcal{D}_r^{(2)}(2F_Y(y)-3F_Y^2(y))\right]$$
$$\times\log\left(\tfrac{1-\overline{F}_Y^{m+1}(y)}{m+1}\right)dy.$$

Upon taking $z=F_Y(y)$ in A, we get

$$A=-1+\frac{\lambda\mathcal{D}_r^{(1)}}{2}+\frac{\omega\mathcal{D}_r^{(2)}}{3}. \tag{4.3}$$

On the other hand, by taking the transformation $t=\frac{1-\overline{F}_Y^{m+1}(y)}{m+1}$ in B, we get the representation

$$B=B_1+\lambda\mathcal{D}_r^{(1)}(B_1-2B_2)+\omega\mathcal{D}_r^{(2)}(2B_2-3B_3), \tag{4.4}$$

where

$$B_1=\int_0^{\frac{1}{m+1}}(1-(m+1)t)^{\frac{1}{m+1}-1}\log t dt=-H_n[\tfrac{1}{m+1}]-\log(1+m),$$

$$\begin{aligned}B_2&=\int_0^{\frac{1}{m+1}}(1-(m+1)t)^{\frac{1}{m+1}-1}(1-(1-(m+1)t)^{\frac{1}{m+1}})\log t dt\\&=\frac{1}{2}\left(-2H_n[\tfrac{1}{m+1}]+H_n[\tfrac{2}{m+1}]-\log(1+m)\right)\end{aligned}$$

and

$$\begin{aligned}B_3&=\int_0^{\frac{1}{m+1}}(1-(m+1)t)^{\frac{1}{m+1}-1}(1-(1-(m+1)t)^{\frac{1}{m+1}})^2\log t dt\\&=-H_n[\tfrac{1}{m+1}]+H_n[\tfrac{2}{m+1}]-\frac{1}{3}H_n[\tfrac{3}{m+1}]-\frac{1}{3}\log(1+m).\end{aligned}$$

Now, by incorporating the values of B_1, B_2 and B_3 in (4.4), we get

$$\begin{aligned}B=&-H_n[\tfrac{1}{m+1}]-\log(1+m)+\lambda\mathcal{D}_r^{(1)}\left(H_n[\tfrac{1}{m+1}]-H_n[\tfrac{2}{m+1}]\right)\\&+\omega\mathcal{D}_r^{(2)}\left(H_n[\tfrac{1}{m+1}]-2H_n[\tfrac{2}{m+1}]+H_n[\tfrac{3}{m+1}]\right).\end{aligned} \tag{4.5}$$

Finally, upon incorporating the values of (4.3), (4.5) and $H(Y_{[r,n,m,k]})$ in (4.2), we get (4.1). The theorem is proved. □

Table 3 provides the K-L distance from $Y_{[r,n,m,k]}$ to $Y_{r,n,m,k}$ in the models, OSs and SOSs under IFGM. The entries were computed using (4.1). From Table 3, the following properties can be extracted:

1. For fixed n, the maximum value of $K(Y_{[r,n,0,1]}, Y_{r,n,0,1})$ attains at lower and upper extremes. Moreover, its smallest value occurs at the central terms.
2. For fixed n, the maximum value of $K(Y_{[r,n,1,1]}, Y_{r,n,1,1})$ occurs at lower extreme terms. In contrast, its smallest value occurs at the central terms.
3. For the two models OSs and SOSs the K-L distance decreases with increasing ω.
4. Generally, for $r \leq n-2$, we have $K(Y_{[r,n,1,1]}, Y_{r,n,1,1}) > K(Y_{[r,n,0,1]}, Y_{r,n,0,1})$.

Table 3: $K(Y_{[r,n,0,1]}, Y_{r,n,0,1})$ and $K(Y_{[r,n,1,1]}, Y_{r,n,1,1})$ under IFGM

$K(Y_{[r,n,0,1]}, Y_{r,n,0,1})$		$\lambda = 0.75$					$K(Y_{[r,n,1,1]}, Y_{r,n,1,1})$		$\lambda = 0.75$				
		ω							ω				
n	**r**	-1.5	-1.0	-0.5	0.5	1.0	**n**	**r**	-1.5	-1.0	-0.5	0.5	1.0
3	1	0.7366	0.6729	0.6107	0.4913	0.4344	3	1	1.7939	1.6722	1.5519	1.3166	1.2019
3	2	0.2347	0.2256	0.2167	0.2001	0.1923	3	2	0.6667	0.6061	0.5470	0.4339	0.3800
3	3	0.6858	0.6377	0.5926	0.5104	0.4732	3	3	0.4439	0.4266	0.4102	0.3804	0.3669
8	1	3.4601	3.2848	3.1107	2.7662	2.5961	8	1	7.4755	7.2409	7.0069	6.5408	6.3089
8	2	2.6171	2.3965	2.1783	1.7494	1.5393	8	2	6.0416	5.6818	5.3235	4.6123	4.2599
8	3	2.0387	1.8605	1.6847	1.3411	1.1737	8	3	4.9346	4.5474	4.1625	3.4012	3.0254
8	4	1.6052	1.5101	1.4166	1.2346	1.1463	8	4	3.9233	3.5901	3.2597	2.6084	2.2883
8	5	1.3843	1.3640	1.3442	1.3056	1.2869	8	5	2.9737	2.7524	2.5335	2.1042	1.8941
8	6	1.5248	1.5202	1.5158	1.5074	1.5033	8	6	2.1568	2.0658	1.9815	1.8121	1.7298
8	7	2.2560	2.1551	2.0573	1.8704	1.7810	8	7	1.6945	1.6931	1.6919	1.6895	1.6884
8	8	3.9939	3.6294	3.2778	2.6088	2.2907	8	8	2.3683	2.2257	2.0880	1.8262	1.7017

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