KYUNGPOOK Math. J. 62(2022), 271-287
https://doi.org/10.5666/KMJ.2022.62.2.271
pISSN 1225-6951 eISSN 0454-8124

A New Geometric Constant in Banach Spaces Related to the Isosceles Orthogonality

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Abstract. In this paper, starting with the geometric constants that can characterize Hilbert spaces, combined with the isosceles orthogonality of Banach spaces, the orthogonal geometric constant $\Omega_X(\alpha)$ is defined, and some theorems on the geometric properties of Banach spaces are derived. Firstly, this paper reviews the research progress of orthogonal geometric constants in recent years. Then, this paper explores the basic properties of the new geometric constants and their relationship with conventional geometric constants, and deduces the identity of $\Omega_X(\alpha)$ and $\gamma_X(\alpha)$. Finally, according to the identities, the relationship between these the new orthogonal geometric constant and the geometric properties of Banach Spaces (such as uniformly non-squareness, smoothness, convexity, normal structure, etc.) is studied, and some necessary and sufficient conditions are obtained.

1. Introduction

As we all know, the geometric theory of Banach spaces has been fully developed and synthesizes the properties of concrete spaces such as the classical sequence spaces $c_0, l_p(1 \leq p < \infty)$ and the function space $C[a,b]$. After fifty years of exploration and research, scholars found that some abstract properties of Banach spaces can be quantitatively described by some special constants. At present, there are many papers on geometric constants, but how to use geometric constants to classify Banach spaces is an important problem. For example, Clarkson introduced the module of convexity to be used to characterize uniformly convex spaces [17], and the von-Neumann constant to be used to characterize uniformly non-square spaces and inner product spaces [6]. After, in order to study the normal structure of spaces,

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Received February 25, 2022; revised April 8, 2022; accepted April 18, 2022.
2020 Mathematics Subject Classification: 46B20.
Key words and phrases: Banach spaces, isosceles orthogonality, geometric constants, geometric properties.
This work was supported by the National Natural Science Foundation of P. R. China (Nos. 11971493 and 12071491).

James introduced the James constant [13]. After the appearance of these constants, many scholars paid attention to them, and obtained many wonderful properties. Although the study of geometric constants has gone through more than half a century, many new geometric constants constantly appear in our field of vision. Since the 1960s, not only the geometric theory of Banach spaces has been fully developed, but also its research methods have been applied to matrix theory, differential equations and so on.

As the geometric properties of general Hilbert spaces, orthogonal relation has strong geometric intuition. For the general Banach spaces, due to the lack of the definition of inner product, scholars introduced a variety of orthogonality equivalent to the traditional orthogonal relationship in Hilbert spaces. For example, in the real normed space $(X, \|\cdot\|)$, James [14] defined isosceles orthogonality: $x \perp_I y$ if and only if $\|x+y\| = \|x-y\|$. In 1935, Birkhoff [2] defined Birkhoff orthogonality: $x \perp_B y$ if and only if $\|x\| \leq \|x+ty\|$. For another example, Roberts [21] defined Robert orthogonality which contains both isosceles and Birkhoff orthogonality: $x \perp_R y$ if and only if $\|x+\lambda y\| = \|x-\lambda y\|$. In addition to the above three orthogonalities, Balestro [4] introduced Pythagorean orthogonality: $x \perp_P y$ if and only if $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. In Hilbert space, these orthogonalities can be simplified to the orthogonal relation in the traditional sense. However, these orthogonalities are different in general Banach spaces. In order to study the differences between these orthogonalities, a large number of orthogonal geometric constants have been defined and studied [2, 15, 18], including

$$BR(X) = \sup_{\alpha>0}\left\{\frac{\|x+\alpha y\| - \|x-\alpha y\|}{\alpha} : x, y \in S_X, x \perp_B y\right\}$$

and

$$BI(X) = \sup\left\{\frac{\|x+y\| - \|x-y\|}{\|x\|} : x, y \in S_X, x, y \neq 0, x \perp_B y\right\}.$$

The introduction of these orthogonal geometric constants not only enriches the theory of Banach spaces, but also provides important tools for the study of quasi Banach spaces.

Although there are a large number of studies on the differences between these orthogonalities, there are few studies involving Pythagorean orthogonality, especially the differences between isosceles and Pythagorean orthogonality. Therefore, this paper defines a new orthogonal geometric constant with the help of the properties of isosceles orthogonality, as follows:

$$\Omega_X(\alpha) = \sup\left\{\frac{\|\alpha x+y\|^2 + \|x+\alpha y\|^2}{\|x+y\|^2} : x \perp_I y, (x,y) \neq (0,0)\right\}, \text{ where } 0 \leq \alpha < 1.$$

Then, the inequalities between the new constant and the James constant, von-Nuemann constant and the module of convexity are discussed. Finally, the judgment theorems of the geometric properties of Banach spaces are obtained, including uniform non-squareness, uniform convexity, uniform smoothness, strict convexity and uniform normal structure.

2. Notations and Preliminaries

In this section, let's recall some concepts of geometric properties of Banach spaces and significant functions.

Definition 2.1. ([5]) Let Xbe Banach space, then the module of convexity is defined as

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in S_X, \|x-y\| = \varepsilon\right\}, \text{ where } \varepsilon \in [0,2].$$

Definition 2.2. ([13]) Let X be Banach space, then the James constant is defined as

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in S_X\}.$$

Definition 2.3. ([6]) Let X be Banach space, then the von-Neumann constant is defined as

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2\|x\|^2 + 2\|y\|^2} : x, y \in X, (x,y) \neq (0,0)\right\}.$$

And the modified von-Neumann constant is defined as

$$C'_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{4} : x, y \in S_X\right\}.$$

Some famous conclusions about $C_{NJ}(X)$ are listed below:
(i) $C_{NJ}(X) \leq J(X)$ [23];
(ii) $1 \leq C_{NJ}(X) \leq 2$ [16];
(iii) X is a Hilbert space if and only if $C_{NJ}(X) = 1$ [16];
(iv) X is uniformly non-square if and only if $C_{NJ}(X) < 2$ [22].

Definition 2.4. ([25]) Let X be Banach space, then the function $\gamma_X(t) : [0,1] \to [0,4]$ is defined as

$$\gamma_X(t) = \sup\left\{\frac{\|x+ty\|^2 + \|x-ty\|^2}{2} : x, y \in S_X\right\}.$$

Definition 2.5. ([8]) Let X be Banach space, then the module of smoothness $\rho_X(t)$ is defined as

$$\rho_X(t) = \sup\left\{\frac{\|x+ty\| + \|x-ty\|}{2} - 1 : x, y \in S_X,\right\}, \text{ where } t \in [0,+\infty).$$

In addition, in order to better characterize the properties of Banach spaces, Zbganu [7] generalized the constant $C_{NJ}(X)$ in 2001 and introduced the following constant:

Definition 2.6. ([7]) Let X be Banach space, then the Zbganu constant is defined as

$$C_Z(X) = \sup\left\{\frac{\|x+y\|\|x-y\|}{\|x\|^2+\|y\|^2} : x, y \in X, (x, y) \neq (0, 0)\right\}.$$

Alonso and Martin [24] proved the existence of Banach space X such that $C_Z(X) < C_{NJ}(X)$. Now, we review some definitions of the properties of Banach spaces.

Definition 2.7. ([13]) The Banach space X is called uniformly non-square if there exists $\delta \in (0, 1)$ such that for any $x, y \in S_X$, either $\frac{\|x+y\|}{2} \leq 1-\delta$ or $\frac{\|x-y\|}{2} \leq 1-\delta$.

Definition 2.8. ([5]) The Banach spac X is called strictly convex if $\|x\| = \|y\| = 1$ and $x \neq y$ imply $\|x+y\| < 2$.

Definition 2.9. ([5]) The Banach spac X is said to be uniformly convex whenever given $0 < \varepsilon \leq 2$, there exists $\delta > 0$ such that if $x, y \in S_X$ and $\|x-y\| \geq \varepsilon$, then $\left\|\frac{x+y}{2}\right\| \leq 1-\delta$.

Definition 2.10. ([1]) Let X be Banach space, then $diamA = \sup\{\|x-y\| : x, y \in A\}$ is called the diameter of A and $r(A) = \inf\{\sup\{\|x-y\|\} : y \in A\}$ is called the Chebyshev radius of A. X is said to have normal structure provided $r(A) < diamA$ for every bounded closed convex subset A of X with $diamA > 0$. X is said to have uniform normal structure if $\inf\left\{\frac{diamA}{r(A)}\right\} > 1$ with $diamA > 0$.

Next, list some conclusions about the geometric properties of Banach spaces as follows:

Lemma 2.11. *Let X be Banach space, then*

(i) If $\delta_X(1) > 0$, then X has normal structure [10].

(ii) X is uniformly non-square if and only if $J(X) < 2$ [11].

(iii) If $J(X) < \frac{1+\sqrt{5}}{2}$, then X has uniform normal structure [9].

(iv) X is uniformly smooth if and only if $\lim_{t\to 0^+} \frac{\gamma_X(t)-1}{t} = 0$ [25].

(v) If $2\gamma_X(t) < 1+(1+t)^2$ for some $t \in (0, 1]$, then X has uniform normal structure [25].

(vi) X is strictly convex if and only if $\delta_X(2) = 1$ [25].

(vii) X is uniformly convex if and only if $\sup\{\varepsilon \in [0, 2] : \delta_X(\varepsilon) = 0\} = 0$ [10].

As we all know, for general normed spaces, the parallelogram rule can describe inner product spaces. In [3], this rule is extended to the following form:

Lemma 2.12. ([3]) *Let X be a real normed linear space, then $(X, \|\cdot\|)$ is an inner product space if and only if for any $x, y \in S_X$, there exist $\alpha, \beta \neq 0$ such that*

$$\|\alpha x+\beta y\|^2 + \|\alpha x-\beta y\|^2 \sim 2(\alpha^2+\beta^2),$$

where $\sim$ stands for $=, \leq$ or $\geq$.

In the middle of last century, in order to extend the orthogonal relation of inner product spaces to any Banach spaces, many scholars introduced new orthogonality. For example, as the orthogonality of general Banach spaces, James defined isosceles orthogonality in 1945:

Definition 2.13. ([14]) Let X be Banach space, $x, y \in X$, if $\|x+y\| = \|x-y\|$, then x is called to be isosceles orthogonal to y, denoted as $x \perp_I y$.

As a special case of isosceles orthogonality, Roberts also introduced Roberts orthogonality:

Definition 2.14. ([21]) Let X be Banach space, $x, y \in X$, if $\|x+\lambda y\| = \|x-\lambda y\|$ for any $\lambda \in R$, then x is called to be Roberts orthogonal to y, denoted as $x \perp_R y$.

In addition to isosceles orthogonality, Birkhoff also defined Birkhoff orthogonality in 1935:

Definition 2.15. ([2]) Let X be Banach space, $x, y \in X$, if $\|x\| \leq \|x+ty\|$ for any $t \in R$, then x is called to be Birkhoff orthogonal to y, denoted as $x \perp_B y$.

Moreover, Balestro [4] introduced Pythagorean orthogonality, which is equivalent to orthogonality in the traditional sense in the inner product space.

Definition 2.16. ([4]) Let X be Banach space, $x, y \in X$, if $\|x+y\|^2 = \|x\|^2 + \|y\|^2$, then x is called to be Pythagorean orthogonal to y, denoted as $x \perp_P y$.

In recent years, based on the geometric constants describing properties of Banach spaces, scholars have defined many new geometric constants with the help of Birkhoff orthogonality and Roberts orthogonality, and explored the properties of Banach spaces [19, 12].

Definition 2.17. ([19]) Let X be Banach space, then the Birkhoff orthogonal geometric constant $BR(X)$ is defined as

$$BR(X) = \sup_{\alpha>0} \left\{ \frac{\|x+\alpha y\| - \|x-\alpha y\|}{\alpha} : x, y \in S_X, x \perp_B y \right\}.$$

From the definition of $BR(X)$, we can see that it can describe the difference between Roberts and Birkhoff orthogonality. Meanwhile Birkhoff orthogonality is homogeneous, it can be thought that $BR(X)$ also measure the difference between Birkhoff and isosceles orthogonalities. In [19], the author proves that $BR(X) = 0$ and X is Hilbert spaces, and deduces the properties of the corresponding points when $BR(X)$ reaches the supremum.

Recently, in order to explore the difference between Birkhoff orthogonality and isosceles orthogonality, Ji [15] and Mizuguchi [18] have defined two geometric constants, as shown below:

Definition 2.18. ([15]) Let X be Banach space, then the isosceles orthogonal geometric constant $D(X)$ is defined as

$$D(X) = \inf\left\{\inf_{\lambda\in R} \|x+\lambda y\| : x, y \in S_X, x \perp_I y\right\}.$$

Definition 2.19. ([18]) Let X be Banach space, then the isosceles orthogonal geometric constant $IB(X)$ is defined as

$$IB(X) = \inf\left\{\frac{\inf_{\lambda\in R} \|x+\lambda y\|}{\|x\|} : x, y \in X, x, y \neq 0, x \perp_I y\right\}.$$

Based on the parallelogram law and isosceles orthogonality, Liu [20] introduced a new geometric constant $\Omega(X)$, gave properties of this geometric constant, and used it to characterized the inner product space.

Definition 2.20. ([20]) Let X be Banach space, then the isosceles orthogonal geometric constant $\Omega(X)$ is defined as

$$\Omega(X) = \sup\left\{\frac{\|2x+y\|^2 + \|x+2y\|^2}{\|x+y\|^2} : x, y \in X, (x, y) \neq (0, 0), x \perp_I y\right\}.$$

In this paper, for narrative convenience, we let X be real Banach space with $dimX \geq 2$. The unit ball and the unit sphere of X are denoted by B_X and S_X, respectively.

3. The Isosceles Orthogonal Geometric Constant of Quadratic Form

As is known to all, for the general Banach space X, Pythagorean orthogonality and isosceles orthogonality are not equivalent. But when X is an inner product space, for any two non-zero vectors $x, y \in X$ and $x \perp_I y$, it is easy to know $x \perp y$, that is, $x \perp_P y$. Hence $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ and

$$\begin{aligned}\|\alpha x+y\|^2 + \|x+\alpha y\|^2 &= \|x\|^2 + \|\alpha y\|^2 + \|y\| + 2\alpha\langle x, y\rangle + \|\alpha x\|^2 + \|y\|^2 + 2\alpha\langle x, y\rangle \\ &= \|x\|^2 + \alpha^2\|y\|^2 + \alpha^2\|x\|^2 + \|y\|^2 \\ &= (1+\alpha^2)(\|x\|^2 + \|y\|^2),\end{aligned}$$

which implie that

$$\frac{\|\alpha x+y\|^2 + \|x+\alpha y\|^2}{\|x+y\|^2} = \frac{\|\alpha x+y\|^2 + \|x+\alpha y\|^2}{\|x\|^2 + \|y\|^2} = 1+\alpha^2$$

for any $\alpha \in R$. Therefore, in order to explore the difference between Pythagorean orthogonality and isosceles orthogonality, this paper defines the isosceles orthogonal geometric constant of quadratic form, as follows:

Definition 3.21. Let X be Banach space, then the isosceles orthogonal geometric constant of quadratic form is defined as

$$\Omega_X(\alpha) = \sup\left\{ \frac{\|\alpha x + y\|^2 + \|x + \alpha y\|^2}{\|x + y\|^2} : x \perp_I y, (x, y) \neq (0, 0) \right\}, \text{ where } 0 \leq \alpha < 1.$$

Theorem 3.22. *Let X ba Banach space, then* $1 + \alpha^2 \leq \Omega_X(\alpha) \leq 2$.

Proof. Letting $x_0 = 0, y_0 \neq 0$, then $x_0 \perp_I y_0$ and

$$\Omega_X(\alpha) \geq \frac{\|\alpha x_0 + y_0\|^2 + \|x_0 + \alpha y_0\|^2}{\|x_0 + y_0\|^2} = 1 + \alpha^2.$$

Letting $x, y \in X$ and $x \perp_I y$, then $\alpha x + y = \frac{1+\alpha}{2} \cdot (x + y) - \frac{1-\alpha}{2} \cdot (x - y)$ and $x + \alpha y = \frac{1+\alpha}{2} \cdot (x + y) + \frac{1-\alpha}{2} \cdot (x - y)$, thus

$$\|\alpha x + y\| \leq \frac{1+\alpha}{2}\|x + y\| + \frac{1-\alpha}{2}\|x - y\| = \|x + y\|$$

and

$$\|x + \alpha y\| \leq \frac{1+\alpha}{2}\|x + y\| + \frac{1-\alpha}{2}\|x - y\| = \|x + y\|,$$

that is, $\Omega_X(\alpha) \leq 2$. □

Example 3.23. Let l_1 be the linear space of all sequences in R such that $\sum_{i=1}^{\infty} |x_i| < \infty$ with the norm defined by

$$\|x\|_1 = \sum_{i=1}^{\infty} |x_i|.$$

Choose $x = (1, 1, 0, \cdots)$, $y = (1, -1, 0, \cdots)$, then $x \perp_I y$ and $\|x+y\|_1 = \|\alpha x+y\|_1 = \|x + \alpha y\|_1 = 2$, that is, $\Omega_{l_1}(\alpha) \geq 2$. Hence $\Omega_{l_1}(\alpha) = 2$.

Let l_∞ be the linear space of all bounded sequences in R with the norm defined by

$$\|x\|_\infty = \sup_{1 \leq n \leq \infty} |x_n|.$$

Choose $x = (1, 0, \cdots)$, $y = (0, -1, 0, \cdots)$, then $x \perp_I y$ and $\|x+y\|_\infty = \|\alpha x+y\|_\infty = \|x + \alpha y\|_\infty = 1$, that is, $\Omega_{l_\infty}(\alpha) \geq 2$. Hence $\Omega_{l_\infty}(\alpha) = 2$.

Example 3.24. Let $C[a, b]$ be the linear space of all real valued continuous functions on $[a, b]$ with the norm defined by

$$\|x\| = \sup_{t \in [a,b]} |x(t)|.$$

We choose $x_0 = \frac{1}{a-b}(t-b), y_0 = \frac{-1}{a-b}(t-b)+1 \in S_{C[a,b]}$, then $x_0 \perp_I y_0$, thus

$$\begin{aligned}\Omega_{C[a,b]}(\alpha) &\geq \frac{\|\alpha x_0 + y_0\|^2 + \|x_0 + \alpha y_0\|^2}{\|x_0 + y_0\|^2} \\ &= \sup_{t\in[a,b]} \left|\frac{1-\alpha}{a-b}(t-b)+\alpha\right|^2 + \sup_{t\in[a,b]} \left|\frac{1+\alpha}{a-b}(t-b)-\alpha\right|^2 = 2,\end{aligned}$$

which implies that $\Omega_{C[a,b]}(\alpha) = 2$.

Theorem 3.25. *Let X be Banach space, then*
(i) $\Omega_X(\alpha)$ is convex and continuous with respect to $\alpha \in [0,1)$.
(ii) $\Omega_X(\alpha)$ is a non-decreasing function with respect to $\alpha \in [0,1)$.
(iii) $\frac{\Omega_X(\alpha)-2}{1-\alpha}$ is a non-increasing function with respect to $\alpha \in [0,1)$.

Proof. (i) Since $\|\cdot\|_2$ is convex and $\|\alpha x+y\|^2+\|x+\alpha y\|^2 = \|(\|\alpha x+y\|, \|x+\alpha y\|)\|_2^2$, then $\Omega_X(\alpha)$ is obviously convex and continuous.

(ii) In order to prove this theorem, we need to extend the definition interval of the constant $\Omega_X(\alpha)$ to $(-1,1)$, and it is easy to know that $\Omega_X(-\alpha) = \Omega_X(-\alpha)$, $\alpha \in [0,1)$. Setting $0 \leq \alpha_1 < \alpha_2 < 1$, then

$$\begin{aligned}\Omega_X(\alpha_1) &= \Omega_X\left(\frac{\alpha_2+\alpha_1}{2\alpha_2}\cdot\alpha_2 + \frac{\alpha_2-\alpha_1}{2\alpha_2}\cdot(-\alpha_2)\right) \\ &\leq \frac{\alpha_2+\alpha_1}{2\alpha_2}\Omega_X(\alpha_2) + \frac{\alpha_2-\alpha_1}{2\alpha_2}\Omega_X(-\alpha_2) = \Omega_X(\alpha_2),\end{aligned}$$

thus $\Omega_X(\alpha)$ is a non-decreasing function.

(iii) Setting $0 \leq \alpha_1 < \alpha_2 < 1$. In order to ensure the continuity of $\Omega_X(\alpha)$, its supplementary definition is $\Omega_X(1) = 2$. Then

$$\begin{aligned}\frac{\Omega_X(\alpha_2)-2}{1-\alpha_2} &= \frac{\Omega_X\left(\frac{\alpha_2-\alpha_1}{1-\alpha_1}\cdot 1 + \left(1-\frac{\alpha_2-\alpha_1}{1-\alpha_1}\right)\cdot\alpha_1\right)-2}{1-\alpha_2} \\ &\leq \frac{2\cdot\frac{\alpha_2-\alpha_1}{1-\alpha_1} + \left(1-\frac{\alpha_2-\alpha_1}{1-\alpha_1}\right)\Omega_X(\alpha_1)-2}{1-\alpha_2} = \frac{\Omega_X(\alpha_1)-2}{1-\alpha_1},\end{aligned}$$

thus $\frac{\Omega_X(\alpha)-2}{1-\alpha}$ is a non-increasing function. □

Theorem 3.26. *Let X be Banach space, then the following conditions are equivalent:*
(i) X is Hilbert space.
(ii) $\Omega_X(\alpha) = 1+\alpha^2$ for any $\alpha \in [0,1)$.
(iii) $\Omega_X(\alpha_0) = 1+\alpha_0^2$ for some $\alpha_0 \in [0,1)$.

Proof. Suppose (i) holds, then (ii) clearly holds by the definition of $\Omega_X(\alpha)$.

Suppose (ii) holds, then (iii) is clearly established.

Suppose (iii) holds, then $x+y \perp_I x-y$ for any $x, y \in S_X$. Hence

$$\frac{\|\alpha_0(x+y)+(x-y)\|^2+\|(x+y)+\alpha_0(x-y)\|^2}{\|(x+y)+(x-y)\|^2} \leq 1+\alpha_0^2,$$

that is, $\|(\alpha_0+1)x+(1-\alpha_0)y\|^2+\|(\alpha_0+1)x-(1-\alpha_0)y\|^2 \leq 4(1+\alpha_0^2)$. Letting $a=\alpha_0+1, b=1-\alpha_0$, then $a, b \neq 0$ and $\|ax+by\|^2+\|ax-by\|^2 \leq 2(a^2+b^2)$, which implies that (i) holds. □

4. Conclusions Related to Other Geometric Constants

In this section, we will study some inequalities for $\Omega_X(\alpha)$ and some geometric constnats, including the James constant $J(X)$, the von-Neumann constant $C_{NJ}(X)$, the module of convexity $\delta_X(\varepsilon)$ and so on. Moreover, these inequalities will help us to discuss the relations between $\Omega_X(\alpha)$ and some properties of Banach spaces in the next section.

Theorem 4.27. *Let X be Banach space, then $\Omega_X(\alpha)=\frac{(1+\alpha)^2}{2}\gamma_X\left(\frac{1-\alpha}{1+\alpha}\right)$.*

Proof. Letting $x, y \in X$ and $x \perp_I y$, we set $u=\frac{x+y}{2}, v=\frac{x-y}{2}$, then

$$x+\alpha y=(1+\alpha)u+(1-\alpha)v, \alpha x+y=(1+\alpha)u-(1-\alpha)v,$$

thus $\|u\|=\|v\|$ and

$$\begin{aligned}\frac{\|x+\alpha y\|^2+\|\alpha x+y\|^2}{\|x+y\|^2} &= \frac{\|(1+\alpha)u+(1-\alpha)v\|^2+\|(1+\alpha)u-(1-\alpha)v\|^2}{4\|u\|^2} \\ &= \frac{(1+\alpha)^2}{4}\cdot\frac{\left\|u+\frac{1-\alpha}{1+\alpha}v\right\|^2+\left\|u-\frac{1-\alpha}{1+\alpha}v\right\|^2}{\|u\|^2}.\end{aligned}$$

Let $x'=\frac{u}{\|u\|}, y'=\frac{v}{\|v\|}$, then $x', y' \in S_X$ and

$$\begin{aligned}\frac{\left\|u+\frac{1-\alpha}{1+\alpha}v\right\|^2+\left\|u-\frac{1-\alpha}{1+\alpha}v\right\|^2}{\|u\|^2} &= \left\|x'+\frac{1-\alpha}{1+\alpha}y'\right\|^2+\left\|x'-\frac{1-\alpha}{1+\alpha}y'\right\|^2 \\ &\leq 2\gamma_X\left(\frac{1-\alpha}{1+\alpha}\right),\end{aligned}$$

that is, $\Omega_X(\alpha) \leq \frac{(1+\alpha)^2}{2}\gamma_X\left(\frac{1-\alpha}{1+\alpha}\right)$.

Letting $x, y \in S_X$, we set $u = \frac{x+y}{2}$, $v = \frac{x-y}{2}$, then $u+v, u-v \in S_X$. Since

$$\begin{aligned}\frac{\left\|x + \frac{1-\alpha}{1+\alpha}y\right\|^2 + \left\|x - \frac{1-\alpha}{1+\alpha}y\right\|^2}{2} &= \frac{\left\|u+v + \frac{1-\alpha}{1+\alpha}(u-v)\right\|^2 + \left\|u+v - \frac{1-\alpha}{1+\alpha}(u-v)\right\|^2}{2\|u+v\|^2} \\ &= \frac{2}{(1+\alpha)^2} \cdot \frac{\|u+\alpha v\|^2 + \|\alpha u + v\|^2}{\|u+v\|^2} \\ &\leq \frac{2}{(1+\alpha)^2}\Omega_X(\alpha),\end{aligned}$$

then $\Omega_X(\alpha) \geq \frac{(1+\alpha)^2}{2}\gamma_X\left(\frac{1-\alpha}{1+\alpha}\right)$. □

Example 4.28. Let l_p ($1 < p < \infty$) be the linear space of all sequences in R such that $\sum_{i=1}^{\infty} |x_i|^p < \infty$ with the norm defined by

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}.$$

Choose $x_0 = \left(\frac{1}{2^{\frac{1}{p}}}, \frac{1}{2^{\frac{1}{p}}}, 0, \cdots, 0\right)$, $y_0 = \left(\frac{1}{2^{\frac{1}{p}}}, -\frac{1}{2^{\frac{1}{p}}}, 0, \cdots, 0\right)$, then

$$\gamma_{l_p}(t) \geq \frac{\|x_0 + ty_0\|^2 + \|x_0 - ty_0\|^2}{2} = \left(\frac{(1+t)^p + (1-t)^p}{2}\right)^{\frac{2}{p}},$$

which implies that

$$\Omega_{l_p}(\alpha) \geq \frac{(1+\alpha)^2}{2} \cdot \left(\frac{\left(1 + \frac{1-\alpha}{1+\alpha}\right)^p + \left(1 - \frac{1-\alpha}{1+\alpha}\right)^p}{2}\right)^{\frac{2}{p}} = 2^{1-\frac{2}{p}}(1+\alpha^p)^{\frac{2}{p}}.$$

In particular, if $2 \leq p < \infty$, then since $\gamma_{l_p}(t) = \left(\frac{(1+t)^p + (1-t)^p}{2}\right)^{\frac{2}{p}}$ [5], we can deduce that

$$\Omega_{l_p}(\alpha) = \frac{(1+\alpha)^2}{2}\gamma_X\left(\frac{1-\alpha}{1+\alpha}\right) = 2^{1-\frac{2}{p}}(1+\alpha^p)^{\frac{2}{p}}.$$

Corollary 4.29. *Let X be Banach space, then $(1-\alpha)^2 C'_{NJ}(X) \leq \Omega_X(\alpha) \leq (1+\alpha^2)C_{NJ}(X)$.*

Proof. Since

$$C_{NJ}(X) \geq \frac{\gamma_X\left(\frac{1-\alpha}{1+\alpha}\right)}{1 + \left(\frac{1-\alpha}{1+\alpha}\right)^2} = \frac{(1+\alpha)^2\gamma_X\left(\frac{1-\alpha}{1+\alpha}\right)}{2+2\alpha^2} = \frac{\Omega_X(\alpha)}{1+\alpha^2},$$

then $\Omega_X(\alpha) \leq (1+\alpha^2)C_{NJ}(X)$.

Note that $x+y \perp_I x-y$ for any $x, y \in S_X$, then

$$\begin{aligned}\Omega_X(\alpha) &\geq \frac{\|x+y+\alpha(x-y)\|^2 + \|\alpha(x+y)+x-y\|^2}{\|x+y+x-y\|^2} \\ &= \frac{\|(1+\alpha)x+(1-\alpha)y\|^2 + \|(1+\alpha)x-(1-\alpha)y\|^2}{4} \\ &= \frac{(1+\alpha)^2}{4}(\|x+ky\|^2 + \|x-ky\|^2),\end{aligned}$$

where $k = \frac{1-\alpha}{1+\alpha}$. Since

$$\|x+ky\| = \left\|\frac{1+k}{2}(x+y) + \frac{1-k}{2}(x-y)\right\| \geq \left|\frac{1+k}{2}\|x+y\| - \frac{1-k}{2}\|x-y\|\right|$$

and

$$\|x-ky\| = \left\|\frac{1-k}{2}(x+y) + \frac{1+k}{2}(x-y)\right\| \geq \left|\frac{1-k}{2}\|x+y\| - \frac{1+k}{2}\|x-y\|\right|,$$

then we have

$$\begin{aligned}\|x+ky\|^2 + \|x-ky\|^2 &\geq \frac{1+k^2}{2}(\|x+y\|^2 + \|x-y\|^2) - (1-k^2)\|x+y\|\|x-y\| \\ &\geq k^2(\|x+y\|^2 + \|x-y\|^2).\end{aligned}$$

Hence $\Omega_X(\alpha) \geq \frac{(1-\alpha)^2}{4}(\|x+y\|^2+\|x-y\|^2)$, that is, $\Omega_X(\alpha) \geq (1-\alpha)^2 C'_{NJ}(X)$. □

Corollary 4.30. *Let X be Banach space, then*

$$\frac{(1+\alpha)^2}{2}J^2(X) - 2\alpha(1+\alpha)J(X) + 2\alpha^2 \leq \Omega_X(\alpha) \leq \frac{1+\alpha^2}{4}J^2(X) + 2\alpha J(X) + 1 + \alpha^2.$$

Proof. Letting $x, y \in S_X$, then since $\|x+y\| = \left\|x + \frac{1-\alpha}{1+\alpha}y + \frac{2\alpha}{1+\alpha}y\right\| \leq \left\|x + \frac{1-\alpha}{1+\alpha}y\right\| + \frac{2\alpha}{1+\alpha}$ and $\|x-y\| = \left\|x - \frac{1-\alpha}{1+\alpha}y - \frac{2\alpha}{1+\alpha}y\right\| \leq \left\|x - \frac{1-\alpha}{1+\alpha}y\right\| + \frac{2\alpha}{1+\alpha}$, we have

$$\begin{aligned}&\min\{\|x+y\|, \|x-y\|\}^2 \\ &\leq \min\left\{\left\|x + \frac{1-\alpha}{1+\alpha}y\right\| + \frac{2\alpha}{1+\alpha}, \left\|x - \frac{1-\alpha}{1+\alpha}y\right\| + \frac{2\alpha}{1+\alpha}\right\}^2 \\ &\leq \frac{\left(\left\|x + \frac{1-\alpha}{1+\alpha}y\right\| + \frac{2\alpha}{1+\alpha}\right)^2 + \left(\left\|x - \frac{1-\alpha}{1+\alpha}y\right\| + \frac{2\alpha}{1+\alpha}\right)^2}{2} \\ &= \frac{\|x + \frac{1-\alpha}{1+\alpha}y\|^2 + \|x - \frac{1-\alpha}{1+\alpha}y\|^2}{2} + \frac{2\alpha}{1+\alpha}(\|x + \frac{1-\alpha}{1+\alpha}y\| + \|x - \frac{1-\alpha}{1+\alpha}y\|) + \frac{4\alpha^2}{(1+\alpha)^2} \\ &\leq \gamma_X\left(\frac{1-\alpha}{1+\alpha}\right) + \frac{4\alpha}{1+\alpha}\sqrt{\gamma_X\left(\frac{1-\alpha}{1+\alpha}\right)} + \frac{4\alpha^2}{(1+\alpha)^2},\end{aligned}$$

which shows that $J(X) \le \sqrt{\gamma_X\left(\frac{1-\alpha}{1+\alpha}\right)} + \frac{2\alpha}{1+\alpha} = \sqrt{\frac{2}{(1+\alpha)^2}\Omega_X(\alpha)} + \frac{2\alpha}{1+\alpha}$. Hence

$$\frac{(1+\alpha)^2}{2}J^2(X) - 2\alpha(1+\alpha)J(X) + 2\alpha^2 \le \Omega_X(\alpha).$$

In addition, since $\|x+ky\| = \left\|\frac{1+k}{2}(x+y) + \frac{1-k}{2}(x-y)\right\| \le \frac{1+k}{2}\|x+y\| + \frac{1-k}{2}\|x-y\|$ and $\|x-ky\| = \left\|\frac{1-k}{2}(x+y) + \frac{1+k}{2}(x-y)\right\| \le \frac{1-k}{2}\|x+y\| + \frac{1+k}{2}\|x-y\|$, then

$$\begin{aligned}\|x+ky\|^2 + \|x-ky\|^2 &\le \frac{1+k^2}{2}(\|x+y\|^2 + \|x-y\|^2) + (1-k^2)\|x+y\|\|x-y\| \\ &\le \frac{1+k^2}{2}(J^2(X)+4) + 2(1-k^2)J(X),\end{aligned}$$

that is, $\gamma_X\left(\frac{1-\alpha}{1+\alpha}\right) \le \frac{1+\alpha^2}{2(1+\alpha)^2}J^2(X) + \frac{4\alpha}{(1+\alpha)^2}J(X) + \frac{2+2\alpha^2}{(1+\alpha)^2}$. Hence $\Omega_X(\alpha) \le \frac{1+\alpha^2}{4}J^2(X) + 2\alpha J(X) + 1 + \alpha^2$. □

Corollary 4.31. *Let X be Banach space, then*

$$\frac{(1-\alpha)^2}{2}(1+\frac{\varepsilon}{2}-\delta_X(\varepsilon))^2 \le \Omega_X(\alpha) \le (1+\alpha^2)(1-\delta_X(\varepsilon)+\frac{\alpha\varepsilon}{1+\alpha^2})^2 + \frac{(1-\alpha^2)^2\varepsilon^2}{4+4\alpha^2},$$

where $\varepsilon \in (0,2]$.

Proof. Since

$$\begin{aligned}2\gamma_X(t) &\ge \|x+ty\|^2 + \|x-ty\|^2 \\ &\ge \frac{1+t^2}{2}(\|x+y\|^2 + \|x-y\|^2) - (1-t^2)\|x+y\|\|x-y\| \\ &\ge \frac{t^2}{2}(\|x+y\| + \|x-y\|)^2,\end{aligned}$$

then $\gamma_X(t) \ge t^2(\rho_X(1)+1)^2$, that is,

$$\Omega_X(\alpha) = \frac{(1+\alpha)^2}{2}\gamma_X\left(\frac{1-\alpha}{1+\alpha}\right) \ge \frac{(1-\alpha)^2}{2}(\rho_X(1)+1)^2.$$

Note that $\rho_X(1) = \sup\left\{\frac{\varepsilon}{2} - \delta_X(\varepsilon) : 0 \le \varepsilon \le 2\right\}$ [17], then

$$\Omega_X(\alpha) \ge \frac{(1-\alpha)^2}{2}\left(1+\frac{\varepsilon}{2}-\delta_X(\varepsilon)\right)^2.$$

Since $\|x+y\| \le 2 - 2\delta_X(\|x-y\|)$ for any $x, y \in S_X$, then

$$\begin{aligned}&\|x+ty\|^2 + \|x-ty\|^2 \\ &\le \frac{1+t^2}{2}(\|x+y\|^2 + \|x-y\|^2) + (1-t^2)\|x+y\|\|x-y\| \\ &\le \frac{1+t^2}{2}(4(1-\delta_X(\|x-y\|))^2 + \|x-y\|^2) + 2(1-t^2)(1-\delta_X(\|x-y\|))\|x-y\|,\end{aligned}$$

which implies that $\gamma_X(t) \leq (1+t^2)(1-\delta_X(\varepsilon))^2 + (1-t^2)(1-\delta_X(\varepsilon))\varepsilon + \frac{1+t^2}{4}\varepsilon^2$. Thus $\Omega_X(\alpha) = \frac{(1+\alpha)^2}{2}\gamma_X\left(\frac{1-\alpha}{1+\alpha}\right) \leq (1+\alpha^2)(1-\delta_X(\varepsilon))^2 + 2\alpha\varepsilon(1-\delta_X(\varepsilon)) + \frac{1+\alpha^2}{4}\varepsilon^2$. □

5. Conclusions Related to the Properties of Banach Spaces

In this part, with the help of the inequality of the new constant and the definition of the geometric properties of Banach spaces, the characterization theorems of the new constant for the properties of uniformly non-square, uniformly convex, strictly convex and uniform normal structure of Banach spaces are derived.

Theorem 5.32. *Let X be Banach space, then*

(i) If X is not uniformly non-square, then $\Omega_X(\alpha) = 2$ for any $\alpha \in [0,1)$.

(ii) If $\Omega_X(\alpha_0) < \frac{(1+\alpha_0)^2}{4} + 1$ for some $\alpha_0 \in [0,1)$, then X has uniform normal structure.

(iii) If $\Omega_X(\alpha_0) < \frac{9(1-\alpha_0)^2}{8}$ for some $\alpha_0 \in [0,1)$, then X has normal structure.

Proof. (i) Since X is not uniformly non-square, then $\gamma_X\left(\frac{1-\alpha}{1+\alpha}\right) = \left(1+\frac{1-\alpha}{1+\alpha}\right)^2 = \frac{4}{(1+\alpha)^2}$, that is, $\Omega_X(\alpha) = \frac{(1+\alpha)^2}{2}\gamma_X\left(\frac{1-\alpha}{1+\alpha}\right) = 2$.

(ii) Since $\Omega_X(\alpha_0) < \frac{(1+\alpha_0)^2}{4} + 1$, then

$$2\gamma_X\left(\frac{1-\alpha_0}{1+\alpha_0}\right) = \frac{4}{(1+\alpha_0)^2}\Omega_X(\alpha_0) < 1 + \frac{4}{(1+\alpha_0)^2} = 1 + \left(1+\frac{1-\alpha_0}{1+\alpha_0}\right)^2,$$

which shows that X uniform normal structure.

(iii) Since $\Omega_X(\alpha_0) < \frac{9(1-\alpha_0)^2}{8}$, then $\frac{(1-\alpha_0)^2}{2}\left(1+\frac{\varepsilon}{2}-\delta_X(\varepsilon)\right)^2 < \frac{9(1-\alpha_0)^2}{8}$, that is, $\delta_X(\varepsilon) > \frac{\varepsilon-1}{2}$, thus $\delta_X(1) > 0$. Hence X has normal structure. □

Example 5.33. Let $l_p - l_q$ ($1 \leq q \leq p < \infty$) be R^2 with the norm defined by

$$\|(x_1,x_2)\| = \begin{cases} \|(x_1,x_2)\|_p \ , \ x_1x_2 \geq 0 \\ \|(x_1,x_2)\|_q \ , \ x_1x_2 < 0 \end{cases}.$$

Let $l_\infty - l_1$ be R^2 with the norm defined by

$$\|(x_1,x_2)\| = \begin{cases} \|(x_1,x_2)\|_\infty \ , \ x_1x_2 \geq 0 \\ \|(x_1,x_2)\|_1 \ , \ x_1x_2 < 0 \end{cases}.$$

We choose $x_0 = \left(\frac{1}{2^{\frac{1}{p}}}, \frac{1}{2^{\frac{1}{p}}}\right), y_0 = \left(\frac{1}{2^{\frac{1}{q}}}, -\frac{1}{2^{\frac{1}{q}}}\right)$, then

$$\gamma_{l_p-l_q}(t) \geq \frac{\|x_0+ty_0\|^2 + \|x_0-ty_0\|^2}{2} = 2^{-\frac{2}{p}}\left[\left(1+t\cdot 2^{\frac{1}{p}-\frac{1}{q}}\right)^p + \left(1-t\cdot 2^{\frac{1}{p}-\frac{1}{q}}\right)^p\right]^{\frac{2}{p}},$$

which implies that

$$\begin{aligned}\Omega_{l_p-l_q}(\alpha) &= \frac{(1+\alpha)^2}{2}\gamma_{l_p-l_q}\left(\frac{1-\alpha}{1+\alpha}\right)\\ &\geq 2^{-1-\frac{2}{p}}\left\{\left[1+2^{\frac{1}{p}-\frac{1}{q}}+\left(1-2^{\frac{1}{p}-\frac{1}{q}}\right)\alpha\right]^p+\left[1-2^{\frac{1}{p}-\frac{1}{q}}+\left(1+2^{\frac{1}{p}-\frac{1}{q}}\right)\alpha\right]^p\right\}^{\frac{2}{p}}.\end{aligned}$$

In particular, since $\gamma_{l_2-l_1}(t)=1+t+t^2$ and $\gamma_{l_\infty-l_1}(t)=\frac{1}{2}(1+(1+t)^2)$[5], then

$$\Omega_{l_2-l_1}(\alpha)=\frac{(1+\alpha)^2}{2}\gamma_{l_2-l_1}\left(\frac{1-\alpha}{1+\alpha}\right)=\frac{3+\alpha^2}{2}$$

and

$$\Omega_{l_\infty-l_1}(\alpha)=\frac{(1+\alpha)^2}{2}\gamma_{l_\infty-l_1}\left(\frac{1-\alpha}{1+\alpha}\right)=\frac{(1+\alpha)^2}{4}+1.$$

Thus $\Omega_{l_2-l_1}(\alpha),\Omega_{l_\infty-l_1}(\alpha)<2$ for any $\alpha\in[0,1)$, which implies that l_2-l_1, $l_\infty-l_1$ both are uniformly non-square.

Corollary 5.34. *Let X be a finite dimensional Banach space, if $\Omega_X(\alpha_0)=2$ for some $\alpha_0\in[0,1)$, then X is not uniformly non-square.*

Proof. Since $\Omega_X(\alpha_0)=2$, then there exist $x_n\in S_X, y_n\in B_X$ such that $x_n\perp_I y_n$ and

$$\lim_{n\to\infty}\frac{\|x_n+\alpha_0y_n\|^2+\|\alpha_0x_n+y_n\|^2}{\|x_n+y_n\|^2}=2.$$

Since X is finite dimensional, then there exist $x_0,y_0\in B_X$ such that $x_0\perp_I y_0$ and

$$\lim_{k\to\infty}\|x_{n_k}\|=\|x_0\|,\ \lim_{k\to\infty}\|y_{n_k}\|=\|y_0\|.$$

Note that $\|x_n+\alpha_0y_n\|\leq\|x_n+y_n\|$, $\|\alpha_0x_n+y_n\|\leq\|x_n+y_n\|$ and

$$\frac{\|x_n+y_n\|^2+\|x_n+y_n\|^2}{\|x_n+y_n\|^2}\leq 2,$$

then $\|x_0+\alpha_0y_0\|=\|x_0+y_0\|$ and $\|\alpha_0x_0+y_0\|=\|x_0+y_0\|$.

Since $\|x_0+\alpha_0y_0\|\leq(1-\alpha_0)\|x_0\|+\alpha_0\|x_0+y_0\|$, then $\|x_0+y_0\|\leq\|x_0\|$. In addition, we also can prove $\|x_0+y_0\|\leq\|y_0\|$, then

$$\max\{\|x_0+y_0\|,\|x_0-y_0\|\}=\|x_0+y_0\|\leq\min\{\|x_0\|,\|y_0\|\}\leq 1<1+\delta$$

for any $\delta\in(0,1)$, which implies that X is not uniformly non-square. □

Corollary 5.35. *Let X be Banach space and $\Omega_X(0)>1$, then X is not uniformly convex. In particular, if $\Omega_X(\alpha)>1+\alpha^2$ for any $\alpha\in[0,1)$, then X is not strictly convex.*

Proof. If $\Omega_X(0) > 1$, then there exists $\varepsilon_0 \in (0, 2]$ such that $\Omega_X(0) \geq 1 + \frac{\varepsilon_0^2}{4}$.

Since $\Omega_X(\alpha) \leq (1+\alpha^2)\left(1 - \delta_X(\varepsilon) + \frac{\alpha\varepsilon}{1+\alpha^2}\right)^2 + \frac{(1-\alpha^2)^2\varepsilon^2}{4+4\alpha^2}$, then

$$1 + \frac{\varepsilon_0^2}{4} \leq (1 - \delta_X(\varepsilon_0))^2 + \frac{\varepsilon_0^2}{4},$$

that is, $\delta_X(\varepsilon_0) = 0$. Thus $\sup\{\varepsilon \in [0, 2] : \delta_X(\varepsilon) = 0\} \geq \varepsilon_0 > 0$, that is, X is not uniformly convex.

In particular, if $\Omega_X(\alpha) > 1 + \alpha^2$, we assume that X is strictly convex, then $\delta_X(2) = 1$. Hence

$$1 + \alpha^2 < (1+\alpha^2)\left(1 - 1 + \frac{2\alpha}{1+\alpha^2}\right)^2 + \frac{(1-\alpha^2)^2}{1+\alpha^2} = 1 + \alpha^2,$$

this is contraditory, then X is not strictly convex. □

Theorem 5.36. *Let X be Banach space, then X is uniformly smooth if and only if*

$$\lim_{\alpha \to 1^-} \frac{1 + \alpha - \Omega_X(\alpha)}{1 - \alpha^2} = \frac{1}{2}.$$

Proof. Letting $t = \frac{1-\alpha}{1+\alpha}$, then $\alpha = \frac{1-t}{1+t}$ and $\alpha \to 1^- \iff t \to 0^+$. Thus we can get

$$\begin{aligned} \frac{1 + \alpha - \Omega_X(\alpha)}{1 - \alpha^2} &= \frac{1 + \alpha - \frac{(1+\alpha)^2}{2}\gamma_X\left(\frac{1-\alpha}{1+\alpha}\right)}{1 - \alpha^2} \\ &= \frac{1 + \frac{1-t}{1+t} - \frac{2}{(1+t)^2}\gamma_X(t)}{1 - \left(\frac{1-t}{1+t}\right)^2} = \frac{1 + t - \gamma_X(t)}{2t}. \end{aligned}$$

Therefore $\lim_{t\to 0^+} \frac{1+t-\gamma_X(t)}{2t} = \frac{1}{2}$ if and only if $\lim_{t\to 0^+} \frac{1-\gamma_X(t)}{t} = 0$. That is,

$$\lim_{t \to 0^+} \frac{1 + t - \gamma_X(t)}{2t} = \frac{1}{2}$$

if and only if X is uniformly smooth. □

Data Availability.
No data were used to support this study.

Conflicts of Interest. The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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