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ZPI Property In Amalgamated Duplication Ring

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Abstract. Let A be a commutative ring. We say that A is a ZPI ring if every proper ideal of A is a finite product of prime ideals [5]. In this paper, we study when the amalgamated duplication of A along an ideal I, $A \bowtie I$ to be a ZPI ring. We show that if I is an idempotent ideal of A, then A is a ZPI ring if and only if $A \bowtie I$ is a ZPI ring.

1. Introduction

All rings considered in this paper are commutative and unitary. Let A and B be commutative rings with identity, $f : A \to B$ a ring homomorphism and J an ideal of B. Then the subring $A \bowtie^f J$ of $A \times B$ is defined as follows:

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A \text{ and } j \in J\}.$$

We call the ring $A \bowtie^f J$ the *amalgamation of A with B along J with respect to f*. This construction was introduced and studied by D'Anna, Finacchiaro and Fontana [1, 2]. The study of the amalgamation ring widespread and can improve early studies on classical constructions like $A + XB[X]$, $A + XB[[X]]$ and $D + M$ which are in fact, special cases of amalgamated algebra rings. However, we will be mostly interested in the amalgamated duplication ring which is a particular case of the amalgamated algebra ring. Let A be a commutative ring and I an ideal of A. The following ring construction called *the amalgamated duplication of A along I* was introduced by D'Anna in [3]. It is the subring $A \bowtie I$ of $A \times A$ consisting of all pairs $(x, y) \in A \times A$ with $x - y \in I$. Motivations and additional applications of the amalgamated duplication are discussed in detail in [3, 4]. Recall that a commutative ring A is called a *ZPI ring* if every proper ideal of A is a finite product of prime ideals [5, 8, 9].

In this paper we study the ZPI property and we give a necessary and sufficient condition for the amalgamated duplication of A along an ideal I, $A \bowtie I$ to be a ZPI

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ring, where I is an idempotent ideal (i.e., $I^2 = I$). We show that if A is a ZPI ring, then A/I is a ZPI ring, and we prove that the reverse is not true in general. Let I be an idempotent ideal of A. We show that A is a ZPI ring if and only if $A \bowtie I$ is a ZPI ring. We end this paper by a sufficient condition for the amalgamated algebra along an ideal to be a ZPI ring. Let A and B be commutative rings with identity, $f : A \to B$ a ring homomorphism and J an idempotent ideal of B. We show that if J is included in the radical of Jacobson of B, then $A \bowtie^f J$ is a ZPI ring if and only if A is a ZPI ring and $f(A) + J$ is Noetherian.

2. Main Results

In this paper we study the ZPI properties on amalgamated duplication of A along an ideal I, $A \bowtie I$. First let us recall the following notions. Let A be a commutative ring and I be an ideal of A. Let $A \bowtie I$ be the subring of $A \times A$ consisting of the elements $(a, a + i)$ for $a \in A$ and $i \in I$. Then the ring $A \bowtie I$ is called the *amalgamated duplication of A along an ideal I*. Recall that a commutative ring A is said to be *ZPI* if every proper ideal of A is a finite product of prime ideals of A. It was shown in [7, Theorem 9.10] that A is a ZPI ring if and only if A is Noetherian and for all maximal ideal M of A, there is no ideal properly contained between M^2 and M.

Example 2.1. Let $A = \mathbb{Z}[\![X]\!]$. We will show that A is not a ZPI ring. Indeed, let $M = (X, 2)\mathbb{Z}[\![X]\!]$ and $I = (X^2, 2X, 2)\mathbb{Z}[\![X]\!]$. Since $2 \in I \backslash M^2$, then $M^2 \subset I$. Moreover, $I \subset M$, because $X \in M \backslash I$. Thus $M^2 \subset I \subset M$, and hence A is not a ZPI ring.

Lemma 2.2. *Let A be a ZPI ring and I an ideal of A. Then A/I is a ZPI ring.*

Proof. Let J be an ideal of A/I. Then $J = B/I$ is such that B is an ideal of A containing I. Since A is a ZPI ring, $B = P_1 \cdots P_k$ where P_i is a prime ideal of A for each $1 \leq i \leq k$. Thus $J = P_1 \cdots P_k/I = P_1/I \cdots P_k/I$ and therefore A/I is a ZPI ring. □

The following example proves that the reverse of the previous lemma is not true in general.

Example 2.3. $A = \mathbb{Z}[\![X]\!]$. Then by Example 2.1, A is not a ZPI ring. Let $I = X\mathbb{Z}[\![X]\!]$. Since $A/I \simeq \mathbb{Z}$, then A/I is a ZPI ring.

Let A and B be commutative rings with identity, $f : A \to B$ a ring homomorphism and J an ideal of B. Then the subring $A \bowtie^f J$ of $A \times B$ is defined as follows:

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A \text{ and } j \in J\}.$$

We call the ring $A \bowtie^f J$ *amalgamation of A with B along J with respect to f*. Let p_A and p_B be the restrictions to $A \bowtie^f J$ of $A \times B$ onto A and B, respectively. Let

$\pi : B \to B/J$ be the canonical projection and $\widehat{f} = \pi \circ f$. Then $A \bowtie^f J$ is the pullback $\widehat{f} \times_{B/J} \pi$ of $\widehat{f}$ and π:

$$\begin{array}{ccc} A \bowtie^f J = \widehat{f} \times_{B/J} \pi & \xrightarrow{p_A} & A \\ {\scriptstyle p_B}\downarrow & & {\scriptstyle \widehat{f}}\downarrow \\ B & \xrightarrow{\pi} & B/J. \end{array}$$

Proposition 2.4. *Let A and B be commutative rings with identity, $f : A \to B$ a ring homomorphism and J an ideal of B. If $A \bowtie^f J$ is a ZPI ring, then A and $f(A) + J$ are ZPI rings.*

Proof. By [2, Proposition 5.1] $\dfrac{A \bowtie^f J}{(0, J)} \simeq A$ and $\dfrac{A \bowtie^f J}{(f^{-1}(J), 0)} \simeq f(A) + J$. Then by Lemma 2.2, A and $f(A) + J$ are ZPI rings. □

Let P be a prime ideal of A and Q be a prime ideal of B. We note $P'_f = \{(p, f(p) + j), \mid p \in P \text{ and } j \in J\}$ and $\overline{Q}_f = \{(a, f(a) + j) \mid a \in A, j \in J \text{ and } f(a) + j \in Q\}$. According to [1, Proposition 2.6], the set of maximal ideals of $A \bowtie^f J$ is $\text{Max}(A \bowtie^f J) = \{P'_f \mid P \in \text{Max(A)}\} \cup \{\overline{Q}_f \mid Q \in \text{Max(B)} \backslash V(J)\}$, where $V(J) = \{Q \in \text{Spec}(B) \mid J \subseteq Q\}$. Note that when $A = B$, $f = id_A$ and $J = I$, then we obtain $A \bowtie^f J = A \bowtie I$ the amalgamated duplication of A along an ideal I.

Remark 2.5. Let I be an ideal of a commutative ring A. Then the maximal ideals of $A \bowtie I$ are:

1. $N \bowtie I$, where N is a maximal ideal of A.
2. $\{(q + i, q) \mid q \in Q, i \in I \text{ and } I \nsubseteq Q\}$, where Q is a maximal ideal of A.

Proof. Let M be a maximal ideal of $A \bowtie I$. Then $M = N \bowtie I$ where N is a maximal ideal of A or $M = \{(a, a + i)$, with $a \in A$, $i \in I$ and $a + i \in Q\}$ for some maximal ideal Q of A such that $I \nsubseteq Q$. Since $a + i \in Q$, there exists $q \in Q$ such that $a = q - i$. Thus $M = \{(q - i, q) \mid i \in I \text{ and } q \in Q\}$. This implies that $M = \{(q + i, q) \mid q \in Q, i \in I \text{ and } I \nsubseteq Q\}$. □

Recall that an ideal I of a commutative ring A is said to be *idempotent* if $I^2 = I$.

Theorem 2.6. *Let I be an idempotent ideal of A. Then the following assertions are equivalent:*

1. *A is a ZPI ring.*
2. *$A \bowtie I$ is a ZPI ring.*

Proof. (2) ⇒ (1). Follows from Proposition 2.4.

(1) $\Rightarrow$ (2). Let M be a maximal ideal of $A \bowtie I$. Suppose that there exists an ideal J of $A \bowtie I$ such that $M^2 \subseteq J \subseteq M$. By Remark 2.5, $M = N \bowtie I$ for some maximal ideal N of A or $M = \{(q+i, q) \mid q \in Q,\, i \in I\}$ for some maximal ideal Q of A such that $I \nsubseteq Q$.

First case: $M = N \bowtie I$, where N is a maximal ideal of A.
Claim $(0, I) \subseteq J$.
Proof of claim. Let $(0, a) \in (0, I)$. Since I is an idempotent ideal of A, there exist $\alpha_1, ...\alpha_n, \beta_1, ..., \beta_n \in I$ such that $a = \alpha_1\beta_1 + \cdots + \alpha_n\beta_n$. Thus

$$(0, a) = (0, \alpha_1)(0, \beta_1) + \cdots + (0, \alpha_n)(0, \beta_n) \in (N \bowtie I)^2 \subseteq J.$$

Now, since $M^2 \subseteq J \subseteq M$, then $N^2 \subseteq P_A(J) \subseteq N$. This implies that $P_A(J) = N^2$ or $P_A(J) = N$, because A is a ZPI ring.

1. $P_A(J) = N$. We will prove that $J = M = N \bowtie I$. It suffices to show that $N \bowtie I \subseteq J$. Let $(a, a+i) \in N \bowtie I$. Then $a \in P_A(J)$; so there exists $j \in J$ such that $(a, a+j) \in J$. We have $(a, a+i) = (a, a+i+j-j) = (a, a+j)+(0, i-j)$. By claim above, $(0, I) \subseteq J$; so $(a, a+i) \in J$. Thus $J = N \bowtie I$.

2. $P_A(J) = N^2$. We will prove that $J = M^2 = (N \bowtie I)^2$. It suffices to show that $J \subseteq (N \bowtie I)^2$. Let $(a, a+i) \in J$. Then $a \in N^2$; so $a = \alpha_1\beta_1 + \cdots + \alpha_n\beta_n$ for some $\alpha_1, ...\alpha_n, \beta_1, ..., \beta_n \in N$. Thus

   $$(a, a+i) = (\alpha_1, \alpha_1)(\beta_1, \beta_1) + \cdots + (\alpha_n, \alpha_n)(\beta_n, \beta_n) + (0, i).$$

   Since $(0, I) \subseteq (N \bowtie I)^2$, then $(a, a+i) \in (N \bowtie I)^2$. This implies that $J \subseteq (N \bowtie I)^2$. Hence $J = (N \bowtie I)^2$.

Second case: $M = \{(q+i, q) \mid q \in Q,\, i \in I\}$ for some maximal ideal Q of A such that $I \nsubseteq Q$.
Claim $(I, 0) \subseteq M^2 \subseteq J$.
Proof of claim. Let $(a, 0) \in (I, 0)$. Since I is an idempotent ideal, $(a, 0) = (\alpha_1, 0)(\beta_1, 0) + \cdots + (\alpha_n, 0)(\beta_n, 0)$ for some $\alpha_k, \beta_k \in I$. As for all $1 \leq k \leq n$, $(\alpha_k, 0) \in M$, then $(a, 0) \in M^2$. This implies that $(I, 0) \subseteq M \subseteq J$.

We set the projection:

$$\begin{array}{cccc} H & : & A \bowtie I & \rightarrow & A \\ & & (a, a+i) & \mapsto & a+i. \end{array}$$

Now, we have $Q^2 = H(M^2) \subseteq H(J) \subseteq H(M) = Q$. Since A is a ZPI ring, then $H(J) = Q$ or $H(J) = Q^2$.

1. $H(J) = Q$. We will show that $M = J$. Let $(q+i, q)$ be an element of M. Since q$\in Q = H(J)$, there exist $a \in A$, $i' \in I$ such that $q = a+i'$ with $(a, a+i') \in J$. By the claim above $(q+i, q) = (a+i+i', a+i') = (a, a+i') + (i+i', 0) \in J$. Hence $M = J$.

2. If $H(J) = Q^2$. We show that $J = M^2$. Let $(a, a+i) \in J$. Then $a + i \in Q^2$ which implies that $a+i = \alpha_1\beta_1 + \cdots + \alpha_n\beta_n$ for some $\alpha_1, ...\alpha_n, \beta_1, ..., \beta_n \in Q$. We have

$$\begin{aligned}(a, a+i) &= (\alpha_1\beta_1 + \cdots + \alpha_n\beta_n - i, \alpha_1\beta_1 + \cdots + \alpha_n\beta_n) \\ &= (\alpha_1\beta_1, \alpha_1\beta_1) + \cdots + (\alpha_n\beta_n, \alpha_n\beta_n) + (-i, 0) \\ &= (\alpha_1, \alpha_1)(\beta_1, \beta_1) + \cdots + (\alpha_n, \alpha_n)(\beta_n, \beta_n) + (-i, 0).\end{aligned}$$

For all $1 \leq k \leq n, (\alpha_k, \alpha_k)(\beta_k, \beta_k) \in M^2$, then the claim above $(I, 0) \subseteq M^2$. This implies that $(a, a+i) \in M^2$, and $M^2 = J$.

Hence $A \bowtie I$ is a ZPI ring. □

Question 2.7. Is the property I idempotent in Theorem 2.6 is necessary?

Recall that an integral domain is said to be a *Dedekind domain* if every proper ideal of A is a finite product of prime ideals. Note that $A \bowtie I$ is an integral domain if and only if A is an integral domain and $I = (0)$.

Corollary 2.8. *Let A be an integral domain. Then the following assertions are equivalent:*

1. *A is a Dedekind domain.*
2. *$A \bowtie 0 = \{(a, a) \mid a \in A\}$ is a Dedekind domain.*

Proof. (1) $\Rightarrow$ (2) Let $I = (0)$. Then I is an idempotent ideal of A. Since A is a ZPI ring, then by Theorem 2.6, $A \bowtie I$ is a ZPI ring. Moreover, A is an integral domain and $I = (0)$, then $A \bowtie I$ is an integral domain. Hence $A \bowtie I$ is a Dedekind domain.

(2) $\Rightarrow$ (1) Since $A \bowtie 0$ is a ZPI ring, then by Theorem 2.6, A is a ZPI ring. As $A \bowtie 0$ is an integral domain, then A is an integral domain. Hence A is a Dedekind domain. □

Proposition 2.9. *Let A and B be commutative rings with identity, $f : A \to B$ a ring homomorphism and J an ideal of B. If J is included in the radical of Jacobson of B, then* $\text{Max}(A \bowtie^f J) = \{P'_f \mid P \in \text{Max}(A)\}$.

Proof. Since $J \subseteq \bigcap_{Q \in Max(B)} Q$, then for all $Q \in \text{Max(B)}$, $J \subseteq Q$; so $\{\overline{Q}_f,\ Q \in \text{Max(B)} \backslash V(J)\} = \emptyset$. □

Let A and B be commutative rings with identity, $f : A \to B$ a ring homomorphism and J an ideal of B. According to [6, Proposition 3.2], $A \bowtie^f J$ is a Noetherian ring if and only if A and $f(A) + J$ are Noetherian.

Proposition 2.10. *Let A and B be commutative rings with identity, $f : A \to B$ a ring homomorphism and J an idempotent ideal of B. Assume that J is included in the radical of Jacobson of B. Then the following assertions are equivalent:*

1. *$A \bowtie^f J$ is a ZPI ring.*
2. *A is a ZPI ring and $f(A)+J$ is a Noetherian ring.*

Proof. (1) $\Rightarrow$ (2) Follows from Proposition 2.4.

(2) $\Leftarrow$ (1) Since A and $f(A)+J$ are Noetherian, then $A \bowtie^f J$ is Noetherian. It suffices to prove that for all maximal ideal M of $A \bowtie^f J$ there is no ideal properly contained between M and M^2. Assume that there exists an ideal I of $A \bowtie^f J$ such that $M^2 \subseteq I \subseteq M$, for some maximal ideal M of $A \bowtie^f J$. Since J is included in the radical of Jacobson of B, then by Proposition 2.9, $\mathrm{Max}(A \bowtie^f J) = \{P'_f \mid P \in \mathrm{Max(A)}\}$; so $M = P \bowtie^f J$, for some maximal ideal P of A. We will show that $(0, J) \subseteq I$. Let $(0, a) \in (0, J)$. Since J is an idempotent ideal of B, there exist $\alpha_1, ...\alpha_n, \beta_1, ..., \beta_n \in J$ such that $a = \alpha_1\beta_1 + \cdots + \alpha_n\beta_n$. Thus

$$(0, a) = (0, \alpha_1)(0, \beta_1) + \cdots + (0, \alpha_n)(0, \beta_n) \in (P \bowtie^f J)^2 \subseteq I.$$

Now, since $M^2 \subseteq I \subseteq M$, then $P^2 \subseteq P_A(I) \subseteq P$. This implies that $P_A(I) = P^2$ or $P_A(I) = P$, because A is a ZPI ring.

First case: $P_A(I) = P$. We will prove that $I = M = P \bowtie^f J$. It suffices to show that $P \bowtie^f J \subseteq I$. Let $(a, f(a)+j) \in P \bowtie^f J$. Then $a \in P_A(I)$; so there exists an $i \in J$ such that $(a, f(a)+i) \in I$. We have $(a, f(a)+j) = (a, f(a)+j+i-i) = (a, f(a)+i) + (0, j-i)$. Since $(0, J) \subseteq I$, $(a, f(a)+j) \in I$. Thus $I = P \bowtie^f J$.

Second case: $P_A(I) = P^2$. We will prove that $I = M^2 = (P \bowtie^f J)^2$. It suffices to show that $I \subseteq (P \bowtie^f J)^2$. Let $(a, f(a)+j) \in I$. Then $a \in P^2$; so $a = \alpha_1\beta_1 + \cdots + \alpha_n\beta_n$ for some $\alpha_1, ...\alpha_n, \beta_1, ..., \beta_n \in P$. Thus

$$(a, f(a)+j) = (\alpha_1, f(\alpha_1))(\beta_1, f(\beta_1)) + \cdots + (\alpha_n, f(\alpha_n))(\beta_n, f(\beta_n)) + (0, j).$$

Since $(0, J) \subseteq (P \bowtie^f J)^2$, then $(a, f(a)+j) \in (P \bowtie^f J)^2$. This implies that $I \subseteq (P \bowtie^f J)^2$. Hence $I = (P \bowtie^f J)^2$. □

Let A be a commutative ring. We denote by $\Gamma(A) := \{(a, f(a)) \mid a \in A\}$ the Graph of A.

Example 2.11. Let A and B be commutative rings with identity, $f : A \to B$ a ring homomorphism and $J = (0)$. It is easy to see that J is an idempotent ideal of B included in the radical of Jacobson of B. By Proposition 2.10, $\Gamma(A)$ is a ZPI ring if and only if A is a ZPI ring and $f(A)$ is Noetherian.

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