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# Uniformly Close-to-Convex Functions with Respect to Conjugate Points

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Abstract. In this paper, we introduce a new subclass of $k$-uniformly close-to-convex functions with respect to conjugate points. We study characterization, coefficient estimates, distortion bounds, extreme points and radii problems for this class. We also discuss integral means inequality with the extremal functions. Our findings may be related with the previously known results.

## 1. Introduction

Let $\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ be the open unit disk and $\mathcal{H}(\mathbb{U})$ denote the class of all analytic functions $f$ defined in the open unit disc $\mathbb{U}$. For a positive integer $n$ and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] := \left\{f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, z \in \mathbb{U}\right\}. \tag{1.1}$$

We define the class $\mathcal{A}$ by

$$\mathcal{A} := \left\{f \in \mathcal{H}[0, 1] : \ f(z) = z + a_2 z^2 + \cdots \qquad (z \in \mathbb{U})\right\}. \tag{1.2}$$

Let $\mathcal{P}$ denote the class of Carathéodory functions $p$ such that $p(0) = 1, \mathrm{Re}\,(p(z)) > 0$ and

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, z \in \mathbb{U}.$$

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The Möbius function $L_0(z) = \frac{1+z}{1-z}$, $z \in \mathbb{U}$ or its rotation acts as an extremal function for the class $\mathcal{P}$ and maps the open unit disc onto the right half-plane. Recall also the class $\mathcal{P}(\gamma) \subset \mathcal{P}$, $0 \leq \gamma < 1$, consisting of functions $p \in \mathcal{P}$ such that $\Re(p(z)) > \gamma$ in $\mathbb{U}$. For $f, g \in \mathcal{H}(\mathbb{U})$, we say that the function $f$ *is subordinate to the function* $g$ and write $f(z) \prec g(z)$, if there exists a *Schwarz function* $w$, that is, $w \in \mathcal{H}(\mathbb{U})$, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g\big(w(z)\big)$. For a univalent function $g$, $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. For reference, see [10]. Applying subordination, Janowski [6] introduced the class $\mathcal{P}[A, B]$ for $-1 \leq B < A \leq 1$. A function $p$ analytic in $\mathbb{U}$ such that $p(0) = 1$ belongs to the class $\mathcal{P}[A, B]$, if

$$p(z) \prec \frac{1 + Az}{1 + Bz} \qquad (z \in \mathbb{U}) \quad \text{or} \quad p(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \qquad (z \in \mathbb{U}),$$

where $w$ is a *Schwarz function*. Geometrically, the image $p(\mathbb{U})$ of $p \in \mathcal{P}[A, B]$ lies inside the open unit disc centered on the real axis with diameter ends at $p(-1)$ and $p(1)$. Clearly, $\mathcal{P}[A, B] \subset \mathcal{P}\left(\frac{1-A}{1-B}\right)$. The class $\mathcal{P}[A, B]$ is related with the class $\mathcal{P}$ of functions with positive real parts by $p \in \mathcal{P}$ if and only if

$$\frac{(A + 1)\, p(z) - (A - 1)}{(B + 1)\, p(z) - (B - 1)} \in \mathcal{P}[A, B].$$

The simplest representation of the conic domain $\Delta_k$ is

$$\Delta_k = \left\{ w = u + iv : u > k\sqrt{(u - 1)^2 + v^2}, k \geq 0 \right\}.$$

This domain represents the right half plane for $k = 0$, a hyperbola for $0 < k < 1$, a parabola for $k = 1$ and an ellipse for $k > 1$. A normalized analytic function $f$ is close-to-convex, if and only if there exists a function $g \in \mathcal{C}$ such that

$$\Re \left\{ \frac{zf'(z)}{g(z)} \right\} \in \mathcal{P}, \;\; z \in \mathbb{U}.$$

We denote the class of close-to-convex functions by $\mathcal{K}$. This class was introduced by Kaplan [16]. Sakaguchi [13] defined the class of starlike functions with respect to symmetric points as follows:

**Definition 1.1.** Let $f \in \mathcal{S}$. Then $f \in \mathcal{S}_{\mathrm{CP}}$, if it satisfies the condition:

$$\operatorname{Re}\left( \frac{zf'(z)}{f(z) + \overline{f(\overline{z})}} \right) > 0, \qquad (z \in \mathbb{U}).$$

For $f \in \mathcal{A}$, $f \in \mathcal{C}_{\mathrm{CP}}$ [18] if and only if $zf' \in \mathcal{S}_{\mathrm{CP}}$, where $\mathcal{C}_{\mathrm{CP}}$ represents the class of convex functions with respect to conjugate points. Various authors studied the class $\mathcal{C}_{\mathrm{CP}}$ of functions convex with respect toconjugate points and its subclasses, for detail, see [11, 12, 16, 21]. Obviously, it is a subclass of close-to-convex functions and hence it represents univalent functions. Moreover, this class includes the class of convex functions and odd starlike functions with respect to the origin, see [13].

A normalized analytic function $f \in \mathcal{Q}$ the class of quasi convex, if and only if $zf' \in \mathcal{K}$. Silverman [19] introduced the class $\mathcal{T}$ of analytic univalent functions $f$ having non-negative coefficients in the series representation:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0,\ z \in \mathbb{U}). \tag{1.3}$$

The class $\mathcal{TS}^*(\gamma)$ a subclasses of $\mathcal{T}$ is defined by $\mathcal{TS}^*(\gamma) = \mathcal{S}^*(\gamma) \cap \mathcal{T}$. Selvaraj and Selvakumaran [14] investigated the classes $\mathcal{S}_{\mathrm{CP}}(\delta, k, \gamma)$ and $\mathcal{C}_{\mathrm{CP}}(\delta, k, \gamma)$ defined below:

$$\operatorname{Re} \frac{2zF_\delta'(z)}{(1-\delta)\, F(z) + \delta z F'(z)} > k \left| \frac{2zF_\delta'(z)}{(1-\delta)\, F(z) + \delta z F'(z)} - 1 \right| + \gamma \qquad (z \in \mathbb{U}),$$

where

$$F(z) = f(z) + \overline{f(\overline{z})} \tag{1.4}$$

and

$$F_\delta(z) = (1-\delta)\, f(z) + \delta z f'(z), \tag{1.5}$$

$0 \leq \delta \leq 1, 0 \leq \gamma < 1, k \geq 0$. Here we let $\mathcal{TS}_{\mathrm{CP}}(\delta, k, \gamma) = \mathcal{T} \cap \mathcal{S}_{\mathrm{CP}}(\delta, k, \gamma)$. A function $f \in \mathcal{C}_{\mathrm{CP}}(\delta, k, \gamma)$ [18] if and only if $zf' \in \mathcal{S}_{\mathrm{CP}}(\delta, k, \gamma)$. For $\delta = 0$, we have the classes $\mathcal{S}_{\mathrm{CP}}(k, \gamma)$ and $\mathcal{C}_{\mathrm{CP}}(k, \gamma)$ and also the classes

$$\mathcal{TS}_{\mathrm{CP}}(k, \gamma) = \mathcal{T} \cap \mathcal{S}_{\mathrm{CP}}(k, \gamma) \ \text{ and } \ \mathcal{TC}_{\mathrm{CP}}(k, \gamma) = \mathcal{T} \cap \mathcal{C}_{\mathrm{CP}}(k, \gamma). \tag{1.6}$$

These and related classes have been introduced by various authors, for example, see [2, 3, 4, 5, 7, 9, 15, 20].

**Definition 1.2.** A function $f \in \mathcal{K}_{\mathrm{CP}}(\delta, k, \gamma, \beta)$ if we have

$$\operatorname{Re} \frac{2zF_\delta'(z)}{(1-\delta)\, G(z) + \delta z G'(z)} > k \left| \frac{2zF_\delta'(z)}{(1-\delta)\, G(z) + \delta z G'(z)} - 1 \right| + \beta \qquad (z \in \mathbb{U}),$$

where $F_\delta$ is given in (1.5), $G(z) = g(z) + \overline{g(\overline{z})} : g \in \mathcal{TC}_{\mathrm{CP}}(k, \gamma)$ and the class $\mathcal{TC}_{\mathrm{CP}}(k, \gamma)$ is defined in (1.6), $0 \leq \delta \leq 1, 0 \leq \gamma < 1, k \geq 0$ and $0 \leq \beta < 1$.

Here we let

$$\mathcal{TK}_{\mathrm{CP}}(\delta, k, \gamma, \beta) = \mathcal{K}_{\mathrm{CP}}(\delta, k, \gamma, \beta) \cap \mathcal{T},$$

where $\mathcal{T}$ is defined by (1.3). In our investigation of the class $\mathcal{TK}_{\mathrm{CP}}(\delta, k, \gamma, \beta)$, we obtain necessary and sufficient conditions, coefficient estimates, distortion bounds, extreme points, radii of close-to-convexity, starlikeness and convexity. We also obtain integral means inequality and other related properties.

## 2. Some Useful Lemmas

We now state some useful lemmas which we may need to establish our results in the sequel.

**Lemma 2.1.** *If $\gamma \in \mathcal{R}$ and $\omega \in \mathcal{C}$ then*

$$\mathrm{Re}\,\omega \geq \gamma \;\Leftrightarrow\; |\omega + (1-\gamma)| - |\omega - (1-\gamma)| \geq 0.$$

**Lemma 2.2.** *If $\gamma \in \mathcal{R}$ and $\omega \in \mathcal{C}$ then*

$$\mathrm{Re}\,\omega \geq k\,|\omega - 1| + \gamma \;\Leftrightarrow\; \mathrm{Re}\left|\omega\left(1+ke^{i\theta}\right) - ke^{i\theta}\right| \geq \gamma, -\pi \leq \theta \leq \pi.$$

**Lemma 2.3.** ([14])*Let $f \in \mathcal{TC}_{\mathrm{CP}}(k,\gamma)$. Then*

$$b_n \leq \frac{1-\gamma}{(1+k)n^2 - n(k+\gamma)},\; n \geq 2.$$

**Lemma 2.4.** ([14]) *Let $f, g \in \mathcal{S}$ with $f \prec g$. Then for $\eta > 0$ and $z = re^{i\theta}$, we have*

$$\int_0^{2\pi} \left|g\left(re^{i\theta}\right)\right|^{\eta} d\theta \leq \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^{\eta} d\theta.$$

## 3. Main Results

We employ the similar technique of Aqlan et al. [1] to find the coefficient estimates for the functions in the class $\mathcal{TK}_{\mathrm{CP}}(\delta,k,\gamma,\beta)$.

**Theorem 3.1.** *A function $f \in \mathcal{TK}_{\mathrm{CP}}(\delta,k,\gamma,\beta)$, if and only if it satisfies the coefficient conditions*

$$\sum_{n=2}^{\infty}\left[1-\delta+n\delta\right]\left[2\left(1+k\right)na_n - \left(k+\beta\right)b_n\right] \leq 2\left(1-\beta\right), \tag{3.1}$$

*for $0 \leq \delta \leq 1, 0 \leq \gamma < 1, k \geq 0, 0 \leq \beta < 1$ and $b_n$ is given by*

$$b_n \leq \frac{1-\gamma}{(1+k)n^2 - n(k+\gamma)}, n = 2, 3, \ldots.$$

*Proof.* Suppose that $f \in \mathcal{A}$ satisfies the condition given in (3.1). We prove that $f \in \mathcal{TK}_{\mathrm{CP}}(\delta,k,\gamma,\beta)$. In view of Lemma 2.1, it is enough to show that

$$\mathrm{Re}\left\{\frac{2\left(1+ke^{i\theta}\right)zF'_{\delta}(z)}{\left(1-\delta\right)G(z) + \delta\, zG'(z)} - ke^{i\theta}\right\} > \beta, -\pi < \theta < \pi, \tag{3.2}$$

where $F_\delta$ is given by (1.5), $G(z) = g(z) - g(-z)$, and $g \in \mathcal{C}_{\mathrm{CP}}(\delta, k, \gamma)$. We now consider that

$$\operatorname{Re}\left\{\frac{H(z)}{J(z)}\right\} \geq \beta, \tag{3.3}$$

where

$$H(z) = 2z - 2\left(1 + ke^{i\theta}\right)\sum_{n=2}^{\infty} n\theta_n a_n z^n + 2ke^{i\theta}\sum_{n=2}^{\infty}\theta_n b_n z^n, \tag{3.4}$$

$$J(z) = (1-\delta)\, G(z) + \delta\, zG'(z) = 2z - \sum_{n=2}^{\infty} 2\theta_n b_n z^n, \theta_n = 1 - \delta + n\delta. \tag{3.5}$$

To obtain (3.2) or (3.3), in view of Lemma 2.1, we need to prove that

$$|D(z)| = |N(z)| - |M(z)| = |H(z) + (1-\beta)\, J(z)| - |H(z) - (1-\beta)\, J(z)| \geq 0.(z \in \mathbb{U}), \tag{3.6}$$

Also for $H$ and $J$ as above, we see that

$$|N(z)| \geq (4 - 2\beta)\,|z| - \sum_{n=2}^{\infty}\left[2na_n + 2(1-\beta)\, b_n\right]\theta_n\,|z^n| - k\sum_{n=2}^{\infty}\left[2na_n - 2b_n\right]\theta_n\,|z|^n. \tag{3.7}$$

Again we consider that

$$|M(z)| \leq 2\beta\,|z| + \sum_{n=2}^{\infty}\left[2na_n - 2(1-\beta)\, b_n\right]\theta_n\,|z^n| + k\sum_{n=2}^{\infty}\left[2na_n - 2b_n\right]\theta_n\,|z|^n,$$

or we can write

$$-|M(z)| \geq -2\beta\,|z| - \sum_{n=2}^{\infty}\left[2na_n - 2(1-\beta)\, b_n\right]\theta_n\,|z^n| - k\sum_{n=2}^{\infty}\left[2na_n - 2b_n\right]\theta_n\,|z|^n. \tag{3.8}$$

Keeping in view (3.6),(3.7) and (3.8), we have

$$|D(z)| \geq 4(1-\beta)\, r - 4\sum_{n=2}^{\infty}\left[na_n + nka_n - (\beta + k)\, b_n\right]\theta_n r^n. \tag{3.9}$$

For the limiting value of $r$, that is, $r \longrightarrow 1$ and in view of (3.1) and (3.9), we obtain

$$|D(z)| \geq 4(1-\beta) - 4\sum_{n=2}^{\infty}\left[n(1+k)\, a_n - (\beta + k)\, b_n\right]\theta_n.$$

Again this inequlity alongwith (3.1) implies (3.6). Hence $f \in \mathcal{TK}_{\mathrm{CP}}(\delta, k, \gamma, \beta)$. Conversely, suppose that $f \in \mathcal{TK}_{\mathrm{CP}}(\delta, k, \gamma, \beta)$. By Lemma 2.2, we have (3.2), i.e.,

$$\operatorname{Re}\left\{\left(1+ke^{i\theta}\right)\psi(z)-ke^{i\theta}\right\} \geq \beta \tag{3.10}$$

or

$$\psi(z)=\frac{2zF'_{\delta}(z)}{(1-\delta)\,G(z)+\delta\,zG'(z)} = \frac{z-\sum\limits_{n=2}^{\infty} nz^n a_n-\sum\limits_{n=2}^{\infty} n(n-1)\delta\, a_n z^n}{z-\sum\limits_{n=2}^{\infty}\theta_n b_n z^n} = \frac{1-\sum\limits_{n=2}^{\infty} na_n\theta_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}\theta_n b_n z^{n-1}}$$

or

$$\psi(z)-\beta=\frac{2(1-\beta)-2\sum\limits_{n=2}^{\infty}\left[na_n-\beta b_n\right]\theta_n z^{n-1}}{2-2\sum\limits_{n=2}^{\infty}\theta_n b_n z^{n-1}} \tag{3.11}$$

and

$$\psi(z)-1=\frac{-2\sum\limits_{n=2}^{\infty}\left[na_n-\beta b_n\right]\theta_n z^{n-1}}{2-2\sum\limits_{n=2}^{\infty}\theta_n b_n z^{n-1}}. \tag{3.12}$$

Using (3.11) and (3.12) in (3.10), we get

$$\operatorname{Re}\left[\frac{2(1-\beta)-2\sum\limits_{n=2}^{\infty}\theta_n\left[na_n-\beta b_n\right]z^{n-1}}{2-2\sum\limits_{n=2}^{\infty}\theta_n b_n z^{n-1}}-\frac{2ke^{i\theta}\sum\limits_{n=2}^{\infty}\left[na_n-b_n\right]\theta_n z^{n-1}}{2-2\sum\limits_{n=2}^{\infty}\theta_n b_n z^{n-1}}\right]\geq 0.$$

We choose $z=r>0$ to have

$$\operatorname{Re}\left[\frac{2(1-\beta)-2\sum\limits_{n=2}^{\infty}\left[na_n-\beta b_n\right]\theta_n r^{n-1}}{2-2\sum\limits_{n=2}^{\infty}\theta_n b_n r^{n-1}}-\frac{2ke^{i\theta}\sum\limits_{n=2}^{\infty}\left[na_n-b_n\right]\theta_n r^{n-1}}{2-2\sum\limits_{n=2}^{\infty}\theta_n b_n r^{n-1}}\right]\geq 0.$$

For $\operatorname{Re}\left(-e^{i\theta}\right)>-\left|e^{i\theta}\right|=-1$ and $\theta_n=1-\delta+n\delta$, the above inequality becomes

$$\operatorname{Re}\left[\frac{2(1-\beta)-\sum\limits_{n=2}^{\infty}\left[2(1+k)\,na_n-(k+\beta)\,b_n\right]\theta_n r^{n-1}}{1-\sum\limits_{n=2}^{\infty}\theta_n b_n r^{n-1}}\right]\geq 0.$$

Again taking $r \longrightarrow 1^{-}$, we get

$$\sum_{n=2}^{\infty} (1-\delta+n\delta) \left[2\,(1+k)\,na_n - (k+\beta)\,b_n\right] \leq 2\,(1-\beta)\,,$$

where

$$b_n \leq \frac{1-\gamma}{(1+k)n^2 - n(k+\gamma)}, n = 2, 3, ....$$

This completes the proof. □

As an immediate consequence of the above Theorem 3.1, we have the following theorem for the coefficients estimates of $f \in \mathcal{TK}_{\mathrm{CP}}(\delta, k, \gamma, \beta)$.

**Theorem 3.2.** *For a function* $f : f \in \mathcal{TK}_{\mathrm{CP}}(\delta, k, \gamma, \beta)$, *we have*

$$a_n \leq \frac{(1-\beta)}{(1+k)\,n\,(1-\delta+n\delta)} + \frac{(1-\gamma)\,(k+\beta)}{[2(1+k)n-(k+\beta)]\,(1+k)\,n^2\,(1-\delta+n\delta)}$$

*for* $0 \leq \delta \leq 1, 0 \leq \gamma < 1, k \geq 0, 0 \leq \beta < 1$.

*Proof.* For $f \in \mathcal{TK}_{\mathrm{CP}}(\delta, k, \gamma, \beta)$, we use Theorem 3.1 to have

$$\sum_{n=2}^{\infty} 2\,(1+k)\,n\,(1-\delta+n\delta)\,a_n \leq 2\,(1-\beta) + (k+\beta) \sum_{n=2}^{\infty} (1-\delta+n\delta)\,b_n. \tag{3.13}$$

Using Lemma 2.3 in (3.13), we have

$$a_n \leq \frac{2\,(1-\beta)\left[(1+k)n^2 - n(k+\gamma)\right] + (1-\gamma)\,(k+\beta)\,(1-\delta+n\delta)}{2\,(1+k)\,n\,[(1+k)n^2 - n(k+\gamma)]\,(1-\delta+n\delta)}.$$

This implies that

$$a_n \leq \frac{2\,(1-\beta)\left[(1+k)n^2 - n(k+\gamma)\right] + (1-\gamma)\,(k+\beta)\,(1-\delta+n\delta)}{2\,(1+k)\,n\,[(1+k)n^2 - n(k+\gamma)]\,(1-\delta+n\delta)}.$$

The equality holds for the function

$$f\,(z) = z - \frac{2\,(1-\beta)\left[(1+k)n^2 - n(k+\gamma)\right] + (1-\gamma)\,(k+\beta)\,(1-\delta+n\delta)}{2\,(1+k)\,n\,[(1+k)n^2 - n(k+\gamma)]\,(1-\delta+n\delta)} z^n.$$

□

## 4. Growth and Distortion Problems

Now we solve the growth problems for functions in the class $\mathcal{TK}_{\mathrm{CP}}(\delta, k, \gamma, \beta)$.

**Theorem 4.1.** *If a function $f \in \mathcal{TK}_{\mathrm{CP}}(\delta, k, \gamma, \beta)$, then for $z = r\exp(i\theta) \in \mathbb{U}$, we have*

$$r - \left[\frac{(1-\beta)}{2(1+k)(1+\delta)} + \frac{(1-\gamma)(k+\beta)}{8(1+k)(k-\gamma+2)}\right] r^2$$
$$\leq |f(z)| \leq r + \left[\frac{1-\beta}{2(1+k)(1+\delta)} + \frac{(1-\gamma)(k+\beta)}{8(1+k)(k-\gamma+2)}\right] r^2, \tag{4.1}$$

*and*

$$1 - \left[\frac{1-\beta}{(1+k)(1+\delta)} + \frac{(1-\gamma)(k+\beta)}{4(1+k)(k-\gamma+2)}\right] r$$
$$\leq |f(z)| \leq r + \left[\frac{1-\beta}{(1+k)(1+\delta)} + \frac{(1-\gamma)(k+\beta)}{4(1+k)(k-\gamma+2)}\right] r. \tag{4.2}$$

*Proof.* For $f \in \mathcal{TK}_{\mathrm{CP}}(\delta, k, \gamma, \beta)$, the inequality (3.1) yields

$$\sum_{n=2}^{\infty} [1-\delta+n\delta]\,[2(1+k)\,na_n - (k+\beta)\,b_n] \leq 2(1-\beta)$$

$$4(1+k)(1+\delta)\sum_{n=2}^{\infty} a_n \leq 2(1-\beta) + (1+\delta)(k+\beta)\sum_{n=2}^{\infty} b_n$$

This implies that

$$\sum_{n=2}^{\infty} a_n < \frac{1-\beta}{2(1+k)(1+\delta)} + \frac{(1-\gamma)(k+\beta)}{8(1+k)(k-\gamma+2)}.$$

Thus from $|f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n$ with $0 < |z| = r < 1$, we have

$$|f(z)| \leq r + \sum_{n=2}^{\infty} a_n r^2 \leq r + \left[\frac{1-\beta}{2(1+k)(1+\delta)} + \frac{(1-\gamma)(k+\beta)}{8(1+k)(k-\gamma+2)}\right] r^2 \tag{4.3}$$

Similarly for the lower bounds on the left side of (4.1), we can write

$$|f(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n \geq r - \left[\frac{1-\beta}{2(1+k)(1+\delta)} + \frac{(1-\gamma)(k+\beta)}{8(1+k)(k-\gamma+2)}\right] r^2. \tag{4.4}$$

The inequalities (4.3) and (4.4) lead to (4.1). On the same lines, we can find the lower and upper bounds of $f'$ which are given by (4.2). The bounds on $f$ and $f'$ represented by (4.1) and (4.2) respectively are sharp for the extremal function $f$ given below by (4.5) and its derivative

$$f(z) = z - \left[\frac{1-\beta}{2(1+k)(1+\delta)} + \frac{(1-\gamma)(k+\beta)}{8(1+k)(k-\gamma+2)}\right] z^2, \quad z = \pm r. \tag{4.5}$$

$\square$

## 5. Extreme Points

Here we find the extremal points of functions belonging to the newly introduced class.

**Theorem 5.1.** *Let* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{2(1-\beta)\left[(1+k)n^2 - n(k+\gamma)\right] + (1-\gamma)(k+\beta)(1-\delta+n\delta)}{2(1+k)n\left[(1+k)n^2 - n(k+\gamma)\right](1-\delta+n\delta)} z^n, n \geq 2,$$

*where* $\theta_n = 1-\delta+n\delta$ *and* $= 1-(-1)^n$. *Then* $f \in \mathcal{TK}_{\mathrm{CP}}(\delta,k,\gamma,\beta)$, *if and only if it takes the form*

$$f(z) = \sum_{n=2}^{\infty} \mu_n f_n(z), \text{ where } \mu_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \mu_n = 1.$$

*Proof.* Consider

$$f(z) = \sum_{n=2}^{\infty} \mu_n f_n(z) = \mu_1 f_1(z) + \sum_{n=2}^{\infty} \mu_n f_n(z), \quad \mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n.$$

This proves that

$$f(z) = z - \sum_{n=2}^{\infty} \mu_n \frac{2(1-\beta)\left[(1+k)n^2 - n(k+\gamma)\right] + (1-\gamma)(k+\beta)(1-\delta+n\delta)}{2(1+k)n\left[(1+k)n^2 - n(k+\gamma)\right](1-\delta+n\delta)} z^n.$$

Since

$$\sum_{n=2}^{\infty} \mu_n \frac{2(1-\beta)\left[(1+k)n^2 - n(k+\gamma)\right] + (1-\gamma)(k+\beta)(1-\delta+n\delta)}{2(1+k)n\left[(1+k)n^2 - n(k+\gamma)\right](1-\delta+n\delta)} \times$$

$$\frac{2(1+k)n\left[(1+k)n^2 - n(k+\gamma)\right](1-\delta+n\delta)}{2(1-\beta)\left[(1+k)n^2 - n(k+\gamma)\right] + (1-\gamma)(k+\beta)(1-\delta+n\delta)} = \sum_{n=2}^{\infty} \mu_n = 1 - \mu_1 \leq 1.$$

Therefore, by Defintion 1.2, we have $f \in \mathcal{TK}_{\mathrm{CP}}(\delta,k,\gamma,\beta)$. Conversely, suppose that $f \in \mathcal{TK}_{\mathrm{CP}}(\delta,k,\gamma,\beta)$. Then by Theorem 3.1, we have

$$a_n \leq \frac{2(1-\beta)\left[(1+k)n^2 - n(k+\gamma)\right] + (1-\gamma)(k+\beta)(1-\delta+n\delta)}{2(1+k)n\left[(1+k)n^2 - n(k+\gamma)\right](1-\delta+n\delta)}, \ n \geq 2.$$

where $\theta_n = 1-\delta+n\delta$ and $= 1-(-1)^n$. By letting $1 - \sum_{n=2}^{\infty} \delta_n = \delta_1$ and

$$\delta_n = \left\{ \frac{2(1-\beta)\left[(1+k)n^2 - n(k+\gamma)\right] + (1-\gamma)(k+\beta)(1-\delta+n\delta)}{2(1+k)n\left[(1+k)n^2 - n(k+\gamma)\right](1-\delta+n\delta)} \right\} a_n, \ n \geq 2,$$

we have
$$\sum_{n=1}^{\infty} \delta_n f_n(z) = \delta_1 f_1(z) - \sum_{n=2}^{\infty} \delta_n f_n(z) = f(z).$$

□

## 6. Radius Problems

**Theorem 6.1.** *If $f \in \mathcal{TK}_{\mathrm{CP}}(\delta, k, \gamma, \beta)$, then $f \in \mathcal{K}(\mu)$ for $|z| < r_0(\delta, k, \gamma, \beta)$, we have*
$$r_0 = \inf\left\{\frac{(1-\mu)\,2\,(1+k)\left[(1+k)n^2 - n(k+\gamma)\right](1-\delta+n\delta)}{2\,(1-\beta)\left[(1+k)n^2 - n(k+\gamma)\right] + (1-\gamma)(k+\beta)(1-\delta+n\delta)}\right\}^{\frac{1}{n-1}}.$$

*Proof.* By a simple computation, we can write
$$|f'(z) - 1| = \left|-\sum_{n=2}^{\infty} n a_n z^{n-1}\right| \le \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

For $f \in \mathcal{K}(\mu)$ ,the class of close-to-convex of order $\mu$, we have the condition
$$\sum_{n=2}^{\infty} \left(\frac{n}{1-\mu}\right) a_n |z|^{n-1} \le 1. \tag{6.1}$$

For the coefficient condition required by the Theorem 3.1, the inequality (6.1) is true if
$$|z| \le \left\{\frac{2\,(1-\mu)(1+k)\left[(1+k)n^2 - n(k+\gamma)\right](1-\delta+n\delta)}{2\,(1-\beta)\left[(1+k)n^2 - n(k+\gamma)\right] + (1-\gamma)(k+\beta)(1-\delta+n\delta)}\right\}^{\frac{1}{n-1}},$$

or we can write
$$r_0 = \inf\left\{\frac{2\,(1-\mu)(1+k)\left[(1+k)n^2 - n(k+\gamma)\right](1-\delta+n\delta)}{2\,(1-\beta)\left[(1+k)n^2 - n(k+\gamma)\right] + (1-\gamma)(k+\beta)(1-\delta+n\delta)}\right\}^{\frac{1}{n-1}}.$$

This completes the proof. □

**Theorem 6.2.** *If $f \in \mathcal{TK}_{\mathrm{CP}}(\delta, k, \gamma, \beta)$,then $f \in \mathcal{S}^*(\mu)$, $0 \le \mu < 1$ for $|z| < r_1$, such that*
$$r_1 = \inf\left\{\frac{2\,(1-\mu)\,n\,(1+k)\left[(1+k)n^2 - n(k+\gamma)\right]\theta_n}{\left[2\,(1-\beta)\left[(1+k)n^2 - n(k+\gamma)\right] + (1-\gamma)(k+\beta)\,\theta_n\right](n-\mu)}\right\}^{\frac{1}{n-1}}, \quad n \ge 2$$

*where $\theta_n = 1 - \delta + n\delta$.*

*Proof.* A simple computation leads to

$$\left|\frac{zf'(z)}{f(z)}-1\right|=\left|\frac{z-\sum\limits_{n=2}^{\infty}na_nz^n}{z-\sum\limits_{n=2}^{\infty}a_nz^n}-1\right|\leq\frac{\sum\limits_{n=2}^{\infty}(na_n-a_n)\,|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n\,|z|^{n-1}}.$$

For a starlike function $f$ of order $\mu$, we have the condition:

$$\sum_{n=2}^{\infty}\left(\frac{n-\mu}{1-\mu}\right)a_n\,|z|^{n-1}\leq 1. \tag{6.2}$$

From the coefficient condition required by the Theorem 3.1, the inequality (6.2) is true if

$$|z|\leq\left\{\frac{2\,(1-\mu)\,n\,(1+k)\left[(1+k)n^2-n(k+\gamma)\right]\theta_n}{\left[2\,(1-\beta)\left[(1+k)n^2-n(k+\gamma)\right]+(1-\gamma)\,(k+\beta)\,\theta_n\right](n-\mu)}\right\}^{\frac{1}{n-1}},\quad n\geq 2.$$

where $\theta_n=1-\delta+n\delta$. This expression yields that

$$r_1=\inf\left\{\frac{2\,(1-\mu)\,n\,(1+k)\left[(1+k)n^2-n(k+\gamma)\right]\theta_n}{\left[2\,(1-\beta)\left[(1+k)n^2-n(k+\gamma)\right]+(1-\gamma)\,(k+\beta)\,\theta_n\right](n-\mu)}\right\}^{\frac{1}{n-1}},\quad n\geq 2.$$

where $\theta_n=1-\delta+n\delta$. This is the desired proof. □

**Theorem 6.3.** *If $f\in\mathcal{TK}_{\mathrm{CP}}(\delta,k,\gamma,\beta)$, then $f\in\mathcal{C}(\mu)$ for $|z|<r_2$, where*

$$r_2=\inf\left\{\frac{2\,(1-\mu)\,n^2\,(1+k)\left[(1+k)n^2-n(k+\gamma)\right](1-\delta+n\delta)}{\left[2\,(1-\beta)\left[(1+k)n^2-n(k+\gamma)\right]+(1-\gamma)\,(k+\beta)\,(1-\delta+n\delta)\right](n-\mu)}\right\}^{\frac{1}{1-n}},\ n\geq 2.$$

*Proof.* Consider

$$\left|\frac{zf''(z)}{f'(z)}\right|=\left|\frac{-\sum\limits_{n=2}^{\infty}n\,(n-1)\,a_nz^{n-1}}{1-\sum\limits_{n=2}^{\infty}na_nz^{n-1}}\right|\leq\frac{\sum\limits_{n=2}^{\infty}n\,(n-1)\,b_n\,|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}na_n\,|z|^{n-1}}.$$

For $f\in\mathcal{C}(\mu)$, the class of convex functions of order $\mu$, we have the condition:

$$\sum_{n=2}^{\infty}\left(\frac{n-\mu}{1-\mu}\right)na_n\,|z|^{n-1}\leq 1. \tag{6.3}$$

From the condition required by the Theorem 3.1, the inequality (6.3) is true if

$$|z|\leq\left\{\frac{2\,(1-\mu)\,n^2\,(1+k)\left[(1+k)n^2-n(k+\gamma)\right]\theta_n}{\left[2\,(1-\beta)\left[(1+k)n^2-n(k+\gamma)\right]+(1-\gamma)\,(k+\beta)\,\theta_n\right](n-\mu)}\right\}^{\frac{1}{n-1}},\ n\geq 2,$$

where $\theta_n=1-\delta+n\delta$. This expression yields the required result. □

## 7. Integral Means Inequality

In [19], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal for the family $\mathcal{T}$. He applied this function to resolve his integral means inequality, conjectured in [17] and settled in [18], that is

$$\int_0^{2\pi} |f(z)|^\eta \, d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta \, d\theta, , f_2 \in \mathcal{T} \text{ and } 0 < r < 1$$

In [18], he also proved his conjecture for the subclasses $\mathcal{T}(\gamma)$ and $\mathcal{C}(\gamma)$ of $\mathcal{T}$. We solve Silverman's conjecture for the class of functions $\mathcal{TK}_{\mathrm{CP}}(\delta, k, \gamma, \beta)$. We only need the concept of subordination between analytic functions already defined above and a subordination theorem of Littlewood [8].

**Theorem 7.1.** *If $f \in \mathcal{TK}_{\mathrm{CP}}(\delta, k, \gamma, \beta), 0 \leq \delta \leq 1, 0 \leq \beta < 1, k \geq 0$ and*

$$f_2(z) = z - \left[\frac{1-\beta}{2(1+k)(1+\delta)} + \frac{(1-\gamma)(k+\beta)}{8(1+k)(k-\gamma+2)}\right] z^2, \tag{7.1}$$

*then for $z = re^{i\theta}$, and $0 < r < 1$ we have*

$$\int_0^{2\pi} |f(z)|^\eta \, d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta \, d\theta.$$

*Proof.* For $f \in \mathcal{TK}_{\mathrm{CP}}(\delta, k, \gamma, \beta)$, the relation (7.1) is equivalent to

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} a_n z^{n-1}\right|^\eta d\theta \leq \int_0^{2\pi} \left|1 - \frac{(1-\beta)z}{2(1+k)(1+\delta)} - \frac{(1-\gamma)(k+\beta)z}{8(1+k)(k-\gamma+2)}\right|^\eta d\theta.$$

By Lemma 2.4, it is enough to prove that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \left[\frac{1-\beta}{2(1+k)(1+\delta)} + \frac{(1-\gamma)(k+\beta)}{8(1+k)(k-\gamma+2)}\right] z$$

Suppose that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{(1-\beta)\omega(z)}{2(1+k)(1+\delta)} - \frac{(1-\gamma)(k+\beta)\omega(z)}{8(1+k)(k-\gamma+2)}, \omega(0) = 0, |\omega(z)| \leq 1.$$

This can be written as

$$|\omega(z)| \leq |z| \sum_{n=2}^{\infty} \frac{2(1+k)n^2\left[(1+k)n^2 - n(k+\gamma)\right](1-\delta+n\delta)a_n}{2(1-\beta)\left[(1+k)n^2 - n(k+\gamma)\right] + (1-\gamma)(k+\beta)(1-\delta+n\delta)} \leq |z|.$$

This completes the proof by Theorem 3.1. □

**8. Concluding Remarks and Observations**

In this research, we used the idea associated with the conic domains and introduced a new subclass of $k$-uniformly close-to-convex functions with respect to conjugate points. We study characterization, coefficient estimates, distortion bounds, extreme points and radii problems for this class. We also discussed the integral means inequality with the extremal functions. Our findings may be related with the previously known results.

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