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EXTENSION OF GRACE'S THEOREM TO BI-COMPLEX POLYNOMIALS

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Abstract. In this paper, we prove some results concerning the zeros of Bi-complex polynomials. These results as special cases include Grace's theorem and related results.

1. Introduction and Historical Background

Let $\mathbb{C} = \{z : z = x + iy; x, y \in \mathbb{R} \,\text{and}\, i = \sqrt{-1}\}$ be the set of complex numbers. For $z_1, z_2 \in \mathbb{C}$, the set $\mathbb{BC}$ of bi-complex numbers is defined as $\mathbb{BC} = \{Z : Z = z_1 + jz_2; z_1, z_2 \in \mathbb{C}\}$, where $ij = ji = k$ and $i^2 = j^2 = -k^2 = -1$. Here k is known as a hyperbolic imaginary unit. Thus more precisely bi-complex numbers are complex numbers with complex coefficients.

Addition and multiplication on $\mathbb{BC}$ is defined in the similar fashion as is defined on $\mathbb{C}$ and it is easy to observe that the set $\mathbb{BC}$ forms a commutative ring. However due to the presence of zero-divisors, $\mathbb{BC}$ is not a field. The set of zero-divisors in $\mathbb{BC}$ is given as:

$$\mathcal{O} = \{z_1 + jz_2 \in \mathbb{BC} : z_1^2 + z_2^2 = 0\} = \{a(1 \pm ij) : a \in \mathbb{C}\}.$$

For $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we have $Z = z_1 + jz_2 = x_1 + ix_2 + jy_1 + jiy_2$. Thus $\mathbb{BC}$ can be viewed as a real vector space isomorphic to $\mathbb{R}^4$ via the map $x_1 + ix_2 + jy_1 + jiy_2 \to (x_1, x_2, y_1, y_2)$.

As (for reference see [3]) the structure of $\mathbb{BC}$ consists of two imaginary units and one hyperbolic unit in it, therefore there are three possible conjugations on this structure:

1.: $\bar{Z} := \bar{z_1} + j\bar{z_2}$ (the bar-conjugation);
2.: $Z^\dagger := z_1 - jz_2$ (the †-conjugation);
3.: $Z^* := (\overline{Z})^\dagger = \overline{Z^\dagger} = \overline{z_1} - j\overline{z_2}$ (the ∗-conjugation).

One of the most important presentation of bi-complex numbers is the idempotent representation. The bi-complex numbers $e = \frac{1+ij}{2}$, $e^\dagger = \frac{1-ij}{2}$ are linearly independent in the linear space $\mathbb{BC}$ over $\mathbb{C}$. From the simple calculations, it can be easily seen that $e + e^\dagger = 1$, $e - e^\dagger = ij$, $e.e^\dagger = 0$, $e^2 = e$ and $(e^\dagger)^2 = e^\dagger$. Also it can be easily verified that any bi-complex number $Z = z_1 + jz_2$ can be uniquely written as $Z = (z_1 - iz_2)e + (z_1 + iz_2)e^\dagger$ and this unique representation of the bi-complex numbers is known as their idempotent representation.

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If $Z = z_1 + jz_2 = \zeta_1 e + \zeta_2 e^\dagger$, then the norm function $\|.\| : \mathbb{BC} \to \mathbb{R}^+$, where R^+ denotes the set of all non-negative real numbers, is defined as:

$$\|Z\| = \{|z_1|^2 + |z_2|^2\}^{\frac{1}{2}} = \Big\{\frac{|\zeta_1|^2 + |\zeta_2|^2}{2}\Big\}^{\frac{1}{2}}.$$

From the idempotent representation of any bi-complex number $Z = z_1 + jz_2$ as $Z = (z_1 - iz_2)e + (z_1 + iz_2)e^\dagger$, we get the idea of defining two spaces $\mathbb{A} = \{z_1 - iz_2 : z_1, z_2 \in \mathbb{C}\}$ and $\overline{\mathbb{A}} = \{z_1 + iz_2 : z_1, z_2 \in \mathbb{C}\}$, known as auxiliary complex spaces. Though $\mathbb{A}$ and $\overline{\mathbb{A}}$ contain same elements as in $\mathbb{C}$ but these convenient notations are used for special representation of elements in the sense that each $Z = z_1 + jz_2 = (z_1 - iz_2)e + (z_1 + iz_2)e^\dagger \in \mathbb{BC}$ associates the points $(z_1 - iz_2) \in \mathbb{A}$ and $(z_1 + iz_2) \in \overline{\mathbb{A}}$. Also to each point $(z_1 - iz_2, z_1 + iz_2) \in \mathbb{A} \times \overline{\mathbb{A}}$, there is a unique point in $\mathbb{BC}$.

The cartesian set $\mathbb{BC}$ determined by $X_1 \subset \mathbb{A}$ and $X_2 \subset \overline{\mathbb{A}}$ is defined as

$$X_1 \times_e X_2 := \{z_1 + jz_2 \in \mathbb{BC} : z_1 + jz_2 = w_1 e + w_2 e^\dagger, (w_1, w_2) \in X_1 \times X_2\}.$$

An open discus $D(a; r_1, r_2)$ with centre $a = a_1 e + a_2 e^\dagger$ and radii $r_1 > 0$, $r_2 > 0$ is defined as

$$\begin{aligned} D(a; r_1, r_2) &= B(a_1, r_1) \times_e B(a_2, r_2) \\ &= \{w_1 e + w_2 e^\dagger \in \mathbb{BC} : |w_1 - a_1| < r_1, |w_2 - a_2| < r_2\} \end{aligned}$$

and a closed discus $\overline{D}(a; r_1, r_2)$ with centre $a = a_1 e + a_2 e^\dagger$ and radii $r_1 > 0$, $r_2 > 0$ is defined as

$$\begin{aligned} \overline{D}(a; r_1, r_2) &= \overline{B}(a_1, r_1) \times_e \overline{B}(a_2, r_2) \\ &= \{w_1 e + w_2 e^\dagger \in \mathbb{BC} : |w_1 - a_1| \le r_1, |w_2 - a_2| \le r_2\}. \end{aligned}$$

Where $B(z, r)$ and $\overline{B}(z, r)$ respectively represent open and closed ball with centre z and radius r.

It is worth here to mention that $\overline{D}(a; r_1, r_2)$, the product of two discs respectively of radii r_1 and r_2, geometrically represents a duocylinder or double cylinder in 4-dimensional Euclidean space. This duocylinder or double cylinder in 4-dimensional Euclidean space is analogous to a cylinder in 3- dimensional Euclidean space, which is the cartesian product of a disc with a line segment (for reference see [6]). If both $r_1 > 0$ and $r_2 > 0$ are equal to r, then the discus is called a $\mathbb{BC} - Disc$ and is denoted by $D(a; r, r) = D(a; r)$.

A bi-complex polynomial of degree n is a function of the form

$$P(Z) = \sum_{i=0}^{n} A_i Z^i, \ A_n \neq 0,$$

where A_i, $i = 0, 1, 2, \ldots, n$ are bi-complex numbers and Z is a bi-complex variable. Now if we write $Z = z_1 + jz_2 = \zeta_1 e + \zeta_2 e^\dagger$ and $A_i = \alpha_i e + \beta_i e^\dagger$ for all $i = 0, 1, 2, \ldots, n$, then $Z^i = {\zeta_1}^i e + {\zeta_2}^i e^\dagger$ and we can re-write our polynomial in the idempotent representation as

$$P(Z) = \sum_{i=0}^{n} (\alpha_i \zeta_1^i) e + \sum_{i=0}^{n} (\beta_i \zeta_2^i) e^\dagger = f_1(\zeta_1) e + f_2(\zeta_2) e^\dagger.$$

Now if we denote the sets of distinct zeros of f_1 and f_2 by S_1 and S_2, and if S denotes the set of distinct zeros of the polynomial P, then

$$S = S_1 e + S_2 e^\dagger.$$

Therefore the following three cases fully describe the structure of the null-set of the polynomial $P(Z)$ of degree n (for details see [3])

1. If both polynomials f_1 and f_2 are of degree at least one, and if $S_1 = \{\mathfrak{z}_{1,1}, \mathfrak{z}_{1,2}, \dots, \mathfrak{z}_{1,k}\}$ and $S_2 = \{\mathfrak{z}_{2,1}, \mathfrak{z}_{2,2}, \dots, \mathfrak{z}_{2,l}\}$, then the set of distinct zeros of the polynomial $P(z)$ is given by
$$S = \{Z_{s,t} = \mathfrak{z}_{1,s}e + \mathfrak{z}_{2,t}e^{\dagger} : s = 1, \dots, k, t = 1.\dots, l\}.$$
2. If f_1 is identically zero, then $S_1 = \mathbb{C}$ and $S_2 = \{\mathfrak{z}_{2,1}, \mathfrak{z}_{2,2}, \dots, \mathfrak{z}_{2,l}\}$, with $l \leq n$. Therefore
$$S = \{Z_t = \lambda e + \mathfrak{z}_{2,t}e^{\dagger} : \lambda \in \mathbb{C}, t = 1.\dots, l\}.$$
Similarly, If f_2 is identically zero, then $S_2 = \mathbb{C}$ and $S_1 = \{\mathfrak{z}_{1,1}, \mathfrak{z}_{1,2}, \dots, \mathfrak{z}_{1,k}\}$, with $k \leq n$. Hence
$$S = \{Z_s = \mathfrak{z}_{1,s}e + \lambda e^{\dagger} : \lambda \in \mathbb{C}, s = 1.\dots, k\}.$$
3. If all the coefficients A_i with the exception $A_0 = \alpha_0 e + \beta_0 e^{\dagger}$ are complex multiples of e (respectively of $e^{\dagger}$), but $\beta_0 \neq 0$ (respectively $\alpha_0 \neq 0$), then polynomial P has no zeros.

In this paper, we extend some results concerning complex polynomials to Bi-complex polynomials. Before discussing these results, we first recall the following basic definitions. Let $\mathbb{P}_n$ be the class of complex polynomials of degree n. Let $f, g \in \mathbb{P}_n$ be such that for $A_j, B_j \in \mathbb{C}$, $j = 0, 1, 2, \dots, n$, $f(z) = \sum_{j=0}^{n} \binom{n}{j} A_j z^j$ and $g(z) = \sum_{j=0}^{n} \binom{n}{j} B_j z^j$, $A_n B_n \neq 0$, then these two polynomials are said to be *Apolar*, if their coefficients satisfy the equation

$$A_0 B_n - \binom{n}{1} A_1 B_{n-1} + \binom{n}{2} A_2 B_{n-2} + \dots + (-1)^n A_n B_0 = 0. \tag{1.1}$$

Clearly, for a given polynomial there are number of polynomials apolar to it. Also the Hadamard product of these complex polynomials f and g is defined as

$$h(z) := (f * g)(z) = \sum_{j=0}^{n} \binom{n}{j} A_j B_j z^j.$$

1.1. Apolarity of Bi-complex polynomials. Following the approach of complex polynomials, we can say that two bi-complex polynomials

$$F(Z) = \sum_{k=0}^{n} \binom{n}{k} A_k Z^k = \Big(\sum_{k=0}^{n} \binom{n}{k} \alpha_k \zeta_1^k\Big)e + \Big(\sum_{k=0}^{n} \binom{n}{k} \beta_k \zeta_2^k\Big)e^{\dagger} = f_1(\zeta_1)e + f_2(\zeta_2)e^{\dagger}$$

and

$$G(Z) = \sum_{k=0}^{n} \binom{n}{k} B_k Z^k = \Big(\sum_{k=0}^{n} \binom{n}{k} \gamma_k \zeta_1^k\Big)e + \Big(\sum_{k=0}^{n} \binom{n}{k} \delta_k \zeta_2^k\Big)e^{\dagger} = g_1(\zeta_1)e + g_2(\zeta_2)e^{\dagger},$$

where $A_i = \alpha_i e + \beta_i e^\dagger$, $B_i = \gamma_i e + \delta_i e^\dagger$ for $i = 1, 2, \ldots, n$ and $Z = \zeta_1 e + \zeta_1 e^\dagger$, are apolar, if

$$\begin{aligned}
&A_0B_n - \binom{n}{1}A_1B_{n-1} + \binom{n}{2}A_2B_{n-2} - \ldots + (-1)^n A_nB_0\\
&\quad = (\alpha_0 e + \beta_0 e^\dagger)(\gamma_n e + \delta_n e^\dagger) - \binom{n}{1}(\alpha_1 e + \beta_1 e^\dagger)(\gamma_{n-1} e + \delta_{n-1} e^\dagger) +\\
&\qquad \binom{n}{2}(\alpha_2 e + \beta_2 e^\dagger)(\gamma_{n-2} e + \delta_{n-2} e^\dagger) - \ldots + (-1)^n(\alpha_n e + \beta_n e^\dagger)(\gamma_0 e + \delta_0 e^\dagger)\\
&\quad = \big(\alpha_0\gamma_n - \binom{n}{1}\alpha_1\gamma_{n-1} + \binom{n}{2}\alpha_2\gamma_{n-2} - \ldots + (-1)^n\alpha_n\gamma_0\big)e\\
&\qquad + \big(\beta_0\delta_n - \binom{n}{1}\beta_1\delta_{n-1} + \binom{n}{2}\beta_2\delta_{n-2} - \ldots + (-1)^n\beta_n\delta_0\big)e^\dagger\\
&\quad = 0.
\end{aligned}$$

That is, if

$$\alpha_0\gamma_n - \binom{n}{1}\alpha_1\gamma_{n-1} + \binom{n}{2}\alpha_2\gamma_{n-2} - \ldots + (-1)^n\alpha_n\gamma_0 = 0 \tag{1}$$

and

$$\beta_0\delta_n - \binom{n}{1}\beta_1\delta_{n-1} + \binom{n}{2}\beta_2\delta_{n-2} - \ldots + (-1)^n\beta_n\delta_0 = 0. \tag{2}$$

From (1) and (2), it follows that two bi-complex polynomials

$$F(Z) = \sum_{k=0}^{n}\binom{n}{k}A_kZ^k = \Big(\sum_{k=0}^{n}\binom{n}{k}\alpha_k\zeta_1^k\Big)e + \Big(\sum_{k=0}^{n}\binom{n}{k}\beta_k\zeta_2^k\Big)e^\dagger = f_1(\zeta_1)e + f_2(\zeta_2)e^\dagger$$

and

$$G(Z) = \sum_{k=0}^{n}\binom{n}{k}B_kZ^k = \Big(\sum_{k=0}^{n}\binom{n}{k}\gamma_k\zeta_1^k\Big)e + \Big(\sum_{k=0}^{n}\binom{n}{k}\delta_k\zeta_2^k\Big)e^\dagger = g_1(\zeta_1)e + g_2(\zeta_2)e^\dagger,$$

are Apolar, if the coefficients of their corresponding idempotent parts satisfy the following equations simultaneously

$$\alpha_0\gamma_n - \binom{n}{1}\alpha_1\gamma_{n-1} + \binom{n}{2}\alpha_2\gamma_{n-2} - \ldots + (-1)^n\alpha_n\gamma_0 = 0$$

and

$$\beta_0\delta_n - \binom{n}{1}\beta_1\delta_{n-1} + \binom{n}{2}\beta_2\delta_{n-2} - \ldots + (-1)^n\beta_n\delta_0 = 0.$$

In other words, two bi-complex polynomials $F(Z) = f_1(\zeta_1)e + f_2(\zeta_2)e^\dagger$ and $G(Z) = g_1(\zeta_1)e + g_2(\zeta_2)e^\dagger$ are apolar if their corresponding idempotent parts are apolar simultaneously.

1.2. Hadamard product of Bi-complex polynomials. Following the approach of complex functions, we define the Hadamard product of two bi-complex polynomials

$$F(Z) = \sum_{k=0}^{n}\binom{n}{k}A_kZ^k = \Big(\sum_{k=0}^{n}\binom{n}{k}\alpha_k\zeta_1^k\Big)e + \Big(\sum_{k=0}^{n}\binom{n}{k}\beta_k\zeta_2^k\Big)e^\dagger = f_1(\zeta_1)e + f_2(\zeta_2)e^\dagger$$

and

$$G(Z) = \sum_{k=0}^{n}\binom{n}{k}B_kZ^k = \Big(\sum_{k=0}^{n}\binom{n}{k}\gamma_k\zeta_1^k\Big)e + \Big(\sum_{k=0}^{n}\binom{n}{k}\delta_k\zeta_2^k\Big)e^\dagger = g_1(\zeta_1)e + g_2(\zeta_2)e^\dagger$$

by

$$H(Z) = F(Z) * G(Z) = \sum_{j=0}^{n} \binom{n}{j} A_j B_j Z^j.$$

Which further gives after substituting $A_i = \alpha_i e + \beta_i e^\dagger$, $B_i = \gamma_i e + \delta_i e^\dagger$ for $i = 1, 2, \ldots, n$ and $Z = \zeta_1 e + \zeta_1 e^\dagger$

$$\begin{aligned} H(Z) &= \sum_{j=0}^{n} \binom{n}{j} (\alpha_j e + \beta_j e^\dagger)(\gamma_j e + \delta_j e^\dagger)(\zeta_1 e + \zeta_2 e^\dagger)^j \\ &= \sum_{j=0}^{n} \binom{n}{j} (\alpha_j e + \beta_j e^\dagger)(\gamma_j e + \delta_j e^\dagger)(\zeta_1^j e + \zeta_2^j e^\dagger) \\ &= \sum_{j=0}^{n} \binom{n}{j} \{(\alpha_j \gamma_j \zeta_1^j) e + (\beta_j \delta_j \zeta_2^j) e^\dagger\} \\ &= \Big(\sum_{j=0}^{n} \binom{n}{j} \alpha_j \gamma_j \zeta_1^j\Big) e + \Big(\sum_{j=0}^{n} \binom{n}{j} \beta_j \delta_j \zeta_2^j\Big) e^\dagger \\ &= (f_1 * g_1)(\zeta_1) e + (f_2 * g_2)(\zeta_2) e^\dagger \\ &= h_1(\zeta_1) e + h_2(\zeta_2) e^\dagger. \end{aligned}$$

Thus the covolution or Hadamard product of two bi-complex polynomials $F(Z) = f_1(\zeta_1)e + f_2(\zeta_2)e^\dagger$ and $G(Z) = g_1(\zeta_1)e + g_2(\zeta_2)e^\dagger$is defined by

$$\begin{aligned} H(Z) &= F(Z) * G(Z) \\ &= h_1(\zeta_1) e + h_2(\zeta_2) e^\dagger, \end{aligned} \tag{3}$$

where $h_1(\zeta_1) = (f_1 * g_1)(\zeta_1)$ and $h_2(\zeta_2) = (f_2 * g_2)(\zeta_2)$.

2. Results and Discussion

To prove our results, we need the following lemmas due to Price [3].

Lemma 2.1. *Let $X = X_1 e + X_2 e^\dagger := \{\zeta_1 e + \zeta_2 e^\dagger : \zeta_1 \in X_1, \zeta_2 \in X_2\}$ be a domain in $\mathbb{BC}$. A bi-complex function $F = f_1 e + f_2 e^\dagger : X \to \mathbb{BC}$ is holomorphic if and only if both the component functions f_1 and f_2 are holomorphic in X_1 and X_2 respectively.*

Lemma 2.2. *Let F be a bi-complex holomorphic function defined in a domain $X = X_1 e + X_2 e^\dagger := \{\zeta_1 e + \zeta_2 e^\dagger : \zeta_1 \in X_1, \zeta_2 \in X_2\}$ such that $F(Z) = f_1(\zeta_1)e + f_2(\zeta_2)e^\dagger$, for all $Z = \zeta_1 e + \zeta_2 e^\dagger \in X$. Then, $F(Z)$ has a zero in X if and only if $f_1(\zeta_1)$ and $f_2(\zeta_2)$ both have a zero at ζ_1 in X_1 and at ζ_2 in X_2 respectively.*

The main aim of writing this paper is to extend Grace's theorem [1] and related results proved for complex polynomials to bi-complex polynomials. We first prove the following result, which extends Grace's theorem to bi-complex polynomials.

Theorem 2.3. *If $F(Z)$ and $G(Z)$ are apolar bi-complex polynomials and if any one of them has all its zeros in a closed discus $\overline{D}(c; r_1, r_2)$, then the other will have atleast one zero in $\overline{D}$.*

Proof. Let the two bi-complex polynomials in their idempotent representation be

$$F(Z) = f_1(\zeta_1)e + f_2(\zeta_2)e^{\dagger}$$

and

$$G(Z) = g_1(\zeta_1)e + g_2(\zeta_2)e^{\dagger}.$$

Assume that the bi-complex polynomial $F(Z)$ has all its zeros in discus

$$\overline{D}(c; r_1, r_2),$$

where $c = c_1e + c_2e^{\dagger}$. This implies by Lemma 2.2 that $f_1(\zeta_1)$ and $f_2(\zeta_2)$ have all their zeros in

$$X_1 = \{\zeta_1 \in \mathbb{A} : |\zeta_1 - c_1| \leq r_1\} \subset \mathbb{C}$$

and

$$X_2 = \{\zeta_2 \in \overline{\mathbb{A}} : |\zeta_2 - c_2| \leq r_2\} \subset \mathbb{C}$$

respectively. Now it is given that $F(Z)$ and $G(Z)$ are apolar bi-complex polynomials. Therefore the polynomial $f_1(\zeta_1)$ is apolar to polynomial $g_1(\zeta_1)$ and the polynomial $f_2(\zeta_2)$ is apolar to the polynomial $g_2(\zeta_2)$ simultaneously. Hence by Grace's theorem for complex polynomials, we conclude that atleast one zero of $g_1(\zeta_1)$ and atleast one zero of $g_2(\zeta_2)$ lie in X_1 and X_2 respectively. Hence by lemma 2.2, bi-complex polynomial

$$G(Z) = g_1(\zeta_1)e + g_2(\zeta_2)e^{\dagger}$$

has at least one zero in

$$X_1e + X_2e^{\dagger} = \overline{D}(c; r_1, r_2).$$

This completes the proof of the Theorem. □

Next we prove the following result, which extends a result due to Szegö [4] to bi-complex polynomials.

THEOREM 2.4. *From the two bi-complex polynomials $F(z) := \sum_{j=0}^{n} \binom{n}{j} A_j Z^j$ and $G(z) := \sum_{j=0}^{n} \binom{n}{j} B_j Z^j$, let us form the composite bi-complex polynomial*

$$H(Z) := \sum_{j=0}^{n} \binom{n}{j} A_j B_j Z^j.$$

If all the zeros of $F(z)$ lie in a closed discus $\overline{D}(c; r_1, r_2)$, then every zero $w = w_1e + w_2e^{\dagger}$ of $H(Z)$ has the form $w = -\mu\vartheta$, where $\mu = \mu_1e + \mu_2e^{\dagger}$ is a suitably chosen point in $\overline{D}$ and $\vartheta = \vartheta_1e + \vartheta_2e^{\dagger}$ is a zero of $G(Z)$.

Proof. Let the two bi-complex polynomials in their idempotent representation be

$$F(Z) = f_1(\zeta_1)e + f_2(\zeta_2)e^{\dagger}$$

and

$$G(Z) = g_1(\zeta_1)e + g_2(\zeta_2)e^{\dagger}.$$

Now, we have the composite bi-complex polynomial as

$$\begin{aligned} H(Z) &= F(Z) * G(Z) \\ &= \sum_{j=0}^{n} \binom{n}{j} A_j B_j Z^j \\ &= h_1(\zeta_1)e + h_2(\zeta_2)e^{\dagger}, \end{aligned}$$

where $h_1(\zeta_1) = (f_1 * g_1)(\zeta_1)$ and $h_2(\zeta_2) = (f_2 * g_2)(\zeta_2)$. Since $\vartheta = \vartheta_1e + \vartheta_2e^{\dagger}$ is a zero of

$$G(Z) = g_1(\zeta_1)e + g_2(\zeta_2)e^{\dagger},$$

therefore ϑ_1 and ϑ_2 are the zeros of $g_1(\zeta_1)$ and $g_2(\zeta_2)$ respectively. Also $\mu = \mu_1 e + \mu_2 e^\dagger$ is a suitably chosen point in $\overline{D}$, therefore

$$\mu_1 \in X_1 = \{\zeta_1 \in \mathbb{A} : |\zeta_1 - c_1| \leq r_1\} \subset \mathbb{C}$$

and

$$\mu_2 \in X_2 = \{\zeta_2 \in \overline{\mathbb{A}} : |\zeta_2 - c_2| \leq r_2\} \subset \mathbb{C}.$$

Hence with the help of Szegö's theorem [4] for complex polynomials, it follows that all the zeros of

$$h_1(\zeta_1) = (f_1 * g_1)(\zeta_1)$$

and

$$h_2(\zeta_2) = (f_2 * g_2)(\zeta_2)$$

are respectively of the forms $w_1 = -\mu_1\vartheta_1$ and $w_2 = -\mu_2\vartheta_2$. This implies from Lemma 2.2 that all the zeros of the bi-complex polynomial

$$H(Z) = h_1(\zeta_1)e + h_2(\zeta_2)e^\dagger$$

are of the form

$$\begin{aligned} w &= w_1 e + w_2 e^\dagger \\ &= (-\mu_1\vartheta_1)e + (-\mu_2\vartheta_2)e^\dagger \\ &= -\{\mu_1\vartheta_1 e + \mu_2\vartheta_2 e^\dagger\} \\ &= -\mu\vartheta. \end{aligned}$$

□

We also prove the following result, which extends a result due to Cohn and Egervary ([2], p. 66) to bi-complex polynomials.

Theorem 2.5. *If all the zeros of a bi-complex polynomial* $F(Z) := \sum_{j=0}^{n} \binom{n}{j} A_j Z^j$ *lie in open discus* $D(c; r_1, r_2)$ *and if all the zeros of the bi-complex polynomial* $G(Z) := \sum_{j=0}^{n} \binom{n}{j} B_j Z^j$ *lie in closed discus* $\overline{D}(c; s_1, s_2)$*, then all the zeros of the composite bi-complex polynomial*

$$H(Z) := \sum_{j=0}^{n} \binom{n}{j} A_j B_j Z^j$$

lie in open discus $D(c; r_1 s_1, r_2 s_2)$*.*

Proof. Here the two bi-complex polynomials in their idempotent representation are

$$F(Z) = f_1(\zeta_1)e + f_2(\zeta_2)e^\dagger$$

and

$$G(Z) = g_1(\zeta_1)e + g_2(\zeta_2)e^\dagger.$$

Also, we have the composite bi-complex polynomial as

$$\begin{aligned} H(Z) &= F(Z) * G(Z) \\ &= \sum_{j=0}^{n} \binom{n}{j} A_j B_j Z^j \\ &= h_1(\zeta_1)e + h_2(\zeta_2)e^\dagger, \end{aligned}$$

where $h_1(\zeta_1) = (f_1 * g_1)(\zeta_1)$ and $h_2(\zeta_2) = (f_2 * g_2)(\zeta_2)$. Now, if $\vartheta = \vartheta_1 e + \vartheta_2 e^\dagger$ is a zero of

$$G(Z) = g_1(\zeta_1)e + g_2(\zeta_2)e^\dagger$$

and $\mu = \mu_1 e + \mu_2 e^\dagger$ is a suitably chosen point in $D(c; r_1, r_2)$, then from the proof of theorem 2.4, we have that every zero of

$$h_1(\zeta_1) = (f_1 * g_1)(\zeta_1)$$

and

$$h_2(\zeta_2) = (f_2 * g_2)(\zeta_2)$$

are respectively of the forms $w_1 = -\mu_1\vartheta_1$ and $w_2 = -\mu_2\vartheta_2$. This implies

$$\begin{aligned}|w_1| &= |-\mu_1\vartheta_1| \\ &= |\mu_1||\vartheta_1| \\ &< r_1 s_1.\end{aligned}$$

Similarly $|w_2| < r_2 s_2$. Thus we conclude that all the zeros of $h_1(\zeta_1)$ and all the zeros of $h_2(\zeta_2)$ lie in

$$X_1 = \{\zeta_1 \in \mathbb{A} : |\zeta_1 - c_1| < r_1 s_1\} \subset \mathbb{C}$$

and

$$X_2 = \{\zeta_2 \in \overline{\mathbb{A}} : |\zeta_2 - c_2| < r_2 s_2\} \subset \mathbb{C}$$

respectively. Hence by lemma 2.2, bi-complex polynomial

$$H(Z) = h_1(\zeta_1)e + h_2(\zeta_2)e^\dagger$$

has all its zeros in

$$X_1 e + X_2 e^\dagger = D(c; r_1 s_1, r_2 s_2).$$

This completes the proof. □

Finally we prove the following result, which extends a result due to Walsh [5] to bi-complex polynomials.

Theorem 2.6. *From two bi-complex polynomials*

$$F(Z) := \sum_{j=0}^{n} A_j Z^j$$

and

$$G(z) := \sum_{j=0}^{n} A_j Z^j$$

of degree n, let us form the composite bi-complex polynomial as

$$H(z) := \sum_{j=0}^{n} (n-j)! B_{n-j} F^j(Z) = \sum_{j=0}^{n} (n-j)! A_{n-j} G^j(Z)$$

of degree n. if all the zeros of $F(Z)$ lie in a discus $\overline{D}(c; r_1, r_2)$, then all the zeros of $H(Z)$ has the form $w = \vartheta + \mu$, where ϑ is a zero of $G(Z)$ and μ is suitably chosen point in $\overline{D}$.

Proof. From the hypothesis, we have

$$F(Z) := \sum_{i=0}^{n} A_i Z^i \quad \text{and} \quad G(Z) := \sum_{i=0}^{n} B_i Z^i.$$

Therefore,

$$F^k(z) = \sum_{i=k}^{n} \frac{i!}{(i-k)!} A_i Z^{(i-k)}, \;\; k = 1, 2, ..., n$$

and

$$G^k(z) = \sum_{i=k}^{n} \frac{i!}{(i-k)!} B_i Z^{(i-k)}, \;\; k = 1, 2, ..., n.$$

Now we have

$$\begin{aligned}\sum_{k=0}^{n}(n-k)!B_{n-k}F^k(Z) = n!B_nF(Z) + (n-1)!B_{n-1}F'(Z) + ...+ \\ B_1F^{(n-1)}(Z) + B_0F^n(Z).\end{aligned}$$

This gives

$$\begin{aligned}\sum_{k=0}^{n}(n-k)!B_{n-k}F^k(Z) &= n!B_n[A_0 + A_1 + ... + A_{n-1}Z^{n-1} + A_nZ^n]+ \\ &\quad (n-1)!B_{n-1}[A_1 + 2A_2Z + ... + (n-1)A_{n-1}Z^{n-2}+ \\ &\quad nA_nZ^{n-1}] + ... + B_1[(n-1)!A_{n-1} + n!A_nZ] + B_0n!A_n \\ &= [n!A_0B_n + (n-1)!A_1B_{n-1} + ... + (n-1)!A_{n-1}B_1+ \\ &\quad n!A_nB_0] + Z[n!A_1B_n + 2(n-1)!A_2B_{n-1} + ... + n!A_nB_1]+ \\ &\quad ... + Z^{n-1}[n!A_{n-1}B_n + n(n-1)!A_nB_{n-1}] + Z^n[n!A_nB_n]. \end{aligned} \tag{4}$$

Also we have

$$\begin{aligned}\sum_{k=0}^{n}(n-k)!A_{n-k}G^k(Z) &= n!A_nG(Z) + (n-1)!A_{n-1}G'(Z) + ...+ \\ &\quad A_1G^{(n-1)}(Z) + A_0G^n(z) \\ &= n!A_n[B_0 + B_1 + ... + B_{n-1}Z^{n-1} + B_nZ^n]+ \\ &\quad (n-1)!A_{n-1}[B_1 + 2B_2Z + ... + (n-1)B_{n-1}Z^{n-2}+ \\ &\quad nB_nZ^{n-1}]+, ... + A_1[(n-1)!B_{n-1} + n!B_nZ] + A_0n!B_n \\ &= [n!A_nB_0 + (n-1)!A_{n-1}B_1 + ... + (n-1)!A_1B_{n-1}+ \\ &\quad n!A_0B_n] + Z[n!A_nB_1 + 2(n-1)!A_{n-1}B_2 + ... + n!A_1B_n]+ \\ &\quad ... + Z^{n-1}[n!A_nB_{n-1} + n(n-1)!A_{n-1}B_n] + Z^n[n!A_nB_n]. \end{aligned} \tag{5}$$

From (4) and (5), we conclude that

$$H(Z) = \sum_{k=0}^{n}(n-k)!B_{n-k}F^{(k)}(Z) = \sum_{k=0}^{n}(n-k)!A_{n-k}G^{(k)}(Z).$$

Consider $A_j = \alpha_j e + \beta_j e^\dagger$, $B_j = \gamma_j e + \delta_j e^\dagger$ and $F(Z) = f_1(\zeta_1)e + f_2(\zeta_2)e^\dagger$, therefore

$$\begin{aligned}
H(Z) &= \sum_{k=0}^{n}(n-k)!B_{n-k}F^{(k)}(Z) \\
&= \sum_{k=0}^{n}((n-k)!e + (n-k)!e^\dagger)(\gamma_{n-k}e + \delta_{n-k}e^\dagger)(f_1^{(k)}(\zeta_1)e + f_2^{(k)}(\zeta_2)) \\
&= \sum_{k=0}^{n}((n-k)!\gamma_{n-k}f_1^{(k)}(\zeta_1))e + ((n-k)!\delta_{n-k}f_2^{(k)}(\zeta_2)) \\
&= \sum_{k=0}^{n}\big((n-k)!\gamma_{n-k}f_1^{(k)}(\zeta_1)\big)e + \sum_{k=0}^{n}\big((n-k)!\delta_{n-k}f_2^{(k)}(\zeta_2)\big) \\
&= h_1(\zeta_1)e + h_2(\zeta_2)e^\dagger, \qquad (6)
\end{aligned}$$

where

$$h_1(\zeta_1) = \sum_{k=0}^{n}\big((n-k)!\gamma_{n-k}f_1^{(k)}(\zeta_1)\big)$$

and

$$h_2(\zeta_2) = \sum_{k=0}^{n}\big((n-k)!\delta_{n-k}f_2^{(k)}(\zeta_2)\big).$$

Let $\vartheta = \vartheta_1 e + \vartheta_2 e^\dagger$ be a zero of $G(Z) = g_1(\zeta_1)e + g_2(\zeta_2)e^\dagger$, therefore ϑ_1 and ϑ_2 are the zeros of $g_1(\zeta_1)$ and $g_2(\zeta_2)$ respectively. Also $\mu = \mu_1 e + \mu_2 e^\dagger$ is a suitably chosen point in $\overline{D}$, therefore

$$\mu_1 \in X_1 = \{\zeta_1 \in \mathbb{A} : |\zeta_1 - c_1| \le r_1\} \subset \mathbb{C}$$

and

$$\mu_2 \in X_2 = \{\zeta_2 \in \overline{\mathbb{A}} : |\zeta_2 - c_2| \le r_2\} \subset \mathbb{C}$$

respectively. Hence with the help of Walsh's theorem [5] for complex polynomials, we have that all the zeros of $h_1(\zeta_1)$ and $h_2(\zeta_2)$ are respectively of the forms

$$w_1 = \mu_1 + \vartheta_1$$

and

$$w_2 = \mu_2 + \vartheta_2.$$

This implies from Lemma 2.2 that all the zeros of bi-complex polynomial

$$H(Z) = h_1(\zeta_1)e + h_2(\zeta_2)e^\dagger$$

are of the form

$$\begin{aligned}
w &= w_1 e + w_2 e^\dagger \\
&= (\mu_1 + \vartheta_1)e + (\mu_2 + \vartheta_2)e^\dagger \\
&= (\mu_1 e + \vartheta_1 e^\dagger) + (\mu_2 e + \vartheta_2 e^\dagger) \\
&= \mu + \vartheta.
\end{aligned}$$

Hence the theorem is proved completely. □

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