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# ON THE DOMINATION NUMBER OF A GRAPH AND ITS SQUARE GRAPH

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Abstract. For a given graph $G = (V, E)$, a dominating set is a subset $V'$ of the vertex set $V$ so that each vertex in $V \setminus V'$ is adjacent to a vertex in $V'$. The minimum cardinality of a dominating set of $G$ is called the *domination number* of $G$ and is denoted by $\gamma(G)$. For an integer $k \geq 1$, the *k-th power* $G^k$ of a graph $G$ with $V(G^k) = V(G)$ for which $uv \in E(G^k)$ if and only if $1 \leq d_G(u, v) \leq k$. Note that $G^2$ is the square graph of a graph $G$. In this paper, we obtain some tight bounds for the sum of the domination numbers of a graph and its square graph in terms of the order, order and size, and maximum degree of the graph $G$. Also, we characterize such extremal graphs.

## 1. Introduction

By a graph, we mean a finite, simple and connected graph. For a graph $G$, its vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. The number of vertices $|V(G)|$ of a graph $G$ is called the *order* of $G$ and is denoted by $n = n(G)$ and the number of edges $|E(G)|$ of a graph $G$ is called the *size* of $G$ and is denoted by $m = m(G)$. The *neighborhood* $N(v) = N_G(v)$ of a vertex $v$ consists of the vertices adjacent to $v$ and $|N(v)|$ is called the *degree* of $v$ and is denoted by $d_G(v)$ or $d(v)$. The *k-neighborhood* $N_G^k[v]$ of a vertex $v \in V(G)$ is the set of all vertices at distance at most $k$ from $v$. The *minimum degree* and *maximum degree* of the graph $G$ is denoted by $\delta(G)$ and $\Delta(G)$ respectively. A vertex of degree one is called a *pendant vertex* and a vertex which is adjacent to a pendant vertex is called a *support vertex*. If $U$ is a proper subset of $V$, then $G \setminus U$ denotes the subgraph of $G$ with vertex set $V \setminus U$ and whose edges are all those of $G$ which are not incident with any vertex in $U$. For a subset $S$ of $V$, the subgraph induced by $S$ is denoted by $G\langle S\rangle$. Let $C_n$ denote the *cycle* on $n$ vertices, $K_n$ denotes the *complete graph* of order $n$, and $K_{p,q}$ denote the *complete bipartite graph*. Note that $K_{1,q}$ is called a *star*. A graph $G$ is said to be *connected* if there exists a path between any two vertices of $G$. For two vertices $u$ and $v$ in a connected graph $G$, the *distance* $d(u, v)$ between $u$ and $v$ is the length of a shortest $u$-$v$ path in $G$. The *diameter* of $G$ is defined as $\max\{d(u, v) : u, v \in V(G)\}$ and is denoted by $diam(G)$. For a vertex $v$ of $G$, its *eccentricity* $e(v)$ is defined by $e(v) =\max\{d(u, v) : u \in V(G)\}$. A *forest* is an acyclic graph. A *galaxy* is a forest in which each component is a star. A *tree* is a connected acyclic graph. A *path* is a tree on $n$ vertices with maximum degree is two and is denoted by $P_n$. A *bistar* is a tree of diameter three. A graph is called *unicyclic* if $G$ contain exactly one cycle. A subset $S$ of $V$ is called an *independent set* of $G$ if no two

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vertices of $S$ are adjacent in $G$.

The *complement* $\overline{G}$ of a graph $G$ is the graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. The *Corona* of two graphs $G_1$ and $G_2$, denoted by $G_1 \circ G_2$, is the graph obtained by taking one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$ in which the $i^{th}$ vertex of $G_1$ is joined to every vertex in the $i^{th}$ copy of $G_2$. The *Cartesian Product* of simple graphs $G$ and $H$ is the simple graph $G \times H$ with vertex set $V(G) \times V(H)$, in which $(u, v)$ is adjacent to $(u', v')$ if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$. The *line graph* $L(G)$ of a graph $G$ is a graph whose vertex set is $E(G)$ and two vertices of $L(G)$ are adjacent if and only if the corresponding edges share a common end in $G$.

If $\alpha(G)$ is a graph parameter, then the lower and upper bounds on the sum $\alpha(G) + \alpha(\overline{G})$ in terms of $n$ are of prime importance in graph theory. The first of its kind with reference to chromatic number $\chi(G)$ of $G$ was studied by Nordhaus and Gaddum on complementary graphs (a graph and its complement) and published in American Mathematical Monthly in 1956. They proved lower and upper bounds on the sum and on the product of $\chi(G)$ and $\chi(\overline{G})$ in terms of the order $n$ of $G$. The original relations presented by Nordhaus and Gaddum [15] in 1956 are as follows.

THEOREM 1.1. [15] *If $G$ is a graph of order $n$, then*

$$2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1 \text{ and}$$
$$n \leq \chi(G).\chi(\overline{G}) \leq \frac{(n+1)^2}{4}.$$

*Furthermore, these bounds are best possible for infinitely many values of $n$.*

Since then, any bound on the sum and / or the product of an invariant in a graph $G$ and the same invariant in the complement $\overline{G}$ of $G$ is called a Nordhaus-Gaddum type inequality or relation. The theory of domination is one of the fast growing research areas in graph theory. The concept of dominating set was introduced by C. Berge [2] and by Ore [16] in 1962. A subset $S$ of $V$ is called a *dominating set* of $G$ if every vertex not in $S$ is adjacent to some vertex in $S$. The *domination number* of $G$ is the minimum cardinality taken over all dominating sets of $G$ and is denoted by $\gamma(G)$. A dominating set $S$ of minimum cardinality is called a $\gamma$-set of $G$. The Nordhaus-Gaddum type for domination number proved by Jaeger and Payan [8] in 1972 are as follows.

THEOREM 1.2. [8] *For any graph $G$ with at least two vertices,*

$$3 \leq \gamma(G) + \gamma(\overline{G}) \leq n + 1 \text{ and}$$
$$2 \leq \gamma(G).\gamma(\overline{G}) \leq n.$$

This has been extended to other graph theoretic parameters. A survey of these results is published in [1]. Like $\overline{G}$, there are several derived graphs in the literature. In [11–14], the authors obtained similar results for line graphs, total graphs, shadow graphs and block graphs. Since power graph is one of the derived graphs, we extend the Nordhaus-Gaddum type result to square graph for the parameter domination number.

The paper proceeds as follows. In Section 2, first we collect some results which will be used in our investigations. In Section 3, we obtain lower and upper bounds for the sum $\gamma(G) + \gamma(G^2)$ in terms of order $n$ where $G^2$ is the square graph of a graph $G$. In Section 4, we obtain similar results in terms of order $n$ and size $m$. In Section 5, we obtain the similar for planar graphs. Finally, in Section 6, we present the same type of results in terms of order $n$ and maximum degree $\Delta(G)$.

## 2. Preliminary results

The following results will be used in our investigations.

THEOREM 2.1. [16] *If a graph $G$ of order $n$ and has no isolated vertices, then $\gamma(G) \leq n/2$.*

THEOREM 2.2. [5,17] *For a graph $G$ with even order $n$ and no isolated vertices, $\gamma(G) = n/2$ if and only if the components of $G$ are the cycle $C_4$ or the corona $H \circ K_1$ for any connected graph $H$.*

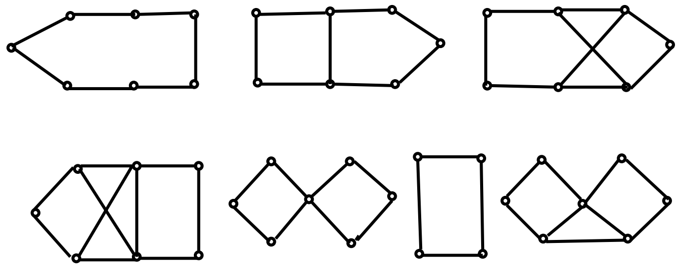

FIGURE 1. Graphs in the family $\mathcal{A}$.

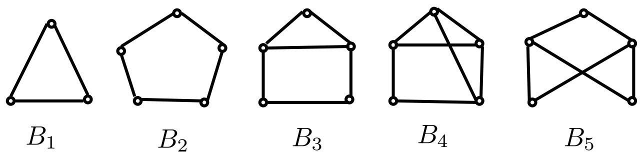


FIGURE 2. Graphs in the family $\mathcal{B}$.

In [4,18] E. J. Cockayne, T. W. Haynes, S. T. Hedetniemi, B. Randerath and L. Volkmann defined six classes of graphs using the following families of graphs which were useful for characterize the connected graphs for which $\gamma(G) = \lfloor n/2 \rfloor$ . Let
$$\mathcal{G}_1 = \{C_4\} \cup \{G : G = H \circ K_1 \textit{ where } H \textit{ is connected}\}$$
and
$$\mathcal{G}_2 = \mathcal{A} \cup \mathcal{B} - \{C_4\}.$$
For any graph $H$, let $\mathcal{S}(H)$ denote the set of connected graphs, each of which can be formed from $H \circ K_1$ by adding a new vertex $x$ and edges joining $x$ to one or more vertices of $H$. Then define
$$\mathcal{G}_3 = \bigcup_H \mathcal{S}(H),$$
where the union is taken over all graphs $H$. Let $y$ be a vertex of a copy of $C_4$ and, for $G \in \mathcal{G}_3$, let $\theta(G)$ be the graph obtained by joining $G$ to $C_4$ with the single edge $xy$, where $x$ is the new vertex added in forming $G$. Then define
$$\mathcal{G}_4 = \{\theta(G) : G \in \mathcal{G}_3\}.$$
Next, let $u, v, w$ be a vertex sequence of a path $P_3$ or a cycle $C_3$. For any graph $H$, let $\mathcal{P}(H)$ be the set of connected graphs which formed from $H \circ K_1$ by joining at least one of $u$ and $w$ to one or more vertices of $H$. Then define
$$\mathcal{G}_5 = \bigcup_H \mathcal{P}(H).$$
Let $H$ be a graph and $X \in \mathcal{B}$. Let $\mathcal{R}(H, X)$ be the set of connected graphs which may be formed from $H \circ K_1$ by joining each vertex of $U \subset V(X)$ to one or more vertices of $H$ such that no set with fewer than $\gamma(X)$ vertices of $X$ dominates $V(X) - U$. Then define
$$\mathcal{G}_6 = \bigcup_{H,X} \mathcal{R}(H, X).$$

THEOREM 2.3. [4,18] *A connected graph $G$ of order $n$ satisfies $\gamma(G) = \lfloor \frac{n}{2} \rfloor$ if and only if $G \in \mathcal{G} = \bigcup_{i=1}^{6} \mathcal{G}_i$.*

Theorem 2.4. [2] *For any graph $G$ of order $n$ and size $m$,*
$$n - m \leq \gamma(G) \leq n + 1 - \sqrt{1 + 2m}.$$
*Furthermore, $\gamma(G) = n - m$ if and only if $G$ is a galaxy.*

Theorem 2.5. [7] *For any connected graph $G$, $\lceil \frac{diam(G)+1}{3} \rceil \leq \gamma(G)$.*

Theorem 2.6. [2,19] *For any graph $G$ of order $n$ and maximum degree $\Delta(G)$, $\lceil \frac{n}{1+\Delta(G)} \rceil \leq \gamma(G) \leq n - \Delta(G)$.*

Theorem 2.7. [10] *If $G$ is a 3-regular planar graph with diameter 2, then $G$ is isomorphic to the cartesian product $K_2 \times K_3$.*

Theorem 2.8. [10] *If $G$ is a 4-regular planar graph with diameter two, then $G$ is isomorphic to any one of the graphs given in Figure 3.*

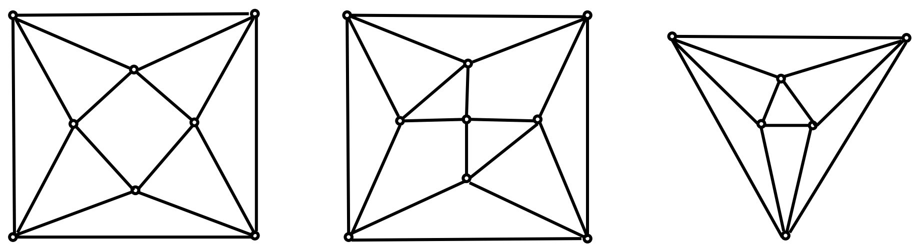


Figure 3. 4-regular Planar Graphs of diameter 2.

Theorem 2.9. [10] *There exist no 5-regular planar graphs with diameter 2.*

Definition 2.10. A graph obtained by joining at least one new isolated vertices $S$ to each pendant vertex of a graph G is denoted by $G(S)$. In this notation, $P_2(S)$ is a bistar.

Lemma 2.11. *For any tree $G$, $\gamma(G) = 2$ if and only if $G \cong P_2(S)$ or $P_3(S)$ or $P_4(S)$.*

*Proof.* Assume that $\gamma(G) = 2$. Clearly $diam(G) = 3$ or 4or 5. If $diam(G) = 3$, then $G$ is bistar. Suppose $diam(G) = 4$, let $v_1v_2v_3v_4v_5$ be a diametrical path in $G$. Clearly $d(v_1) = d(v_5) = 1$ and $v_2$, $v_4$ are support vertices. Since any dominating set of $G$ must contain $v_2$ and $v_4$, $d(v_3) = 2$. Hence $G \cong P_3(S)$. If $diam(G) = 5$, then by a similar argument, $G \cong P_4(S)$. The converse is obvious. □

## 3. Bounds in terms of order

In this section, we obtain lower and upper bounds for the sum $\gamma(G) + \gamma(G^2)$ in terms of order $n$ where $G^2$ is the square of a graph $G$ for which has no isolated vertices. Since any dominating set of $G$ is also a dominating set of the square graph $G^2$,
$$\gamma(G^2) \leq \gamma(G) \tag{3.1}$$
and hence by Theorem 2.1,
$$1 \leq \gamma(G^2) \leq \tfrac{n}{2}. \tag{3.2}$$
Some properties for square graphs in domination theory using Eqs.(3.1) and (3.2) are listed in the following:
(3) For all connected graphs $G$ of order at most 5, $\gamma(G^2) = 1$.
(4) $\gamma(G^2) = 1$ if and only if $e(v) \leq 2$ for some $v \in V(G)$.
(5) $\gamma(G^2) = \frac{n}{2}$ if and only if $G \cong K_2$.

(6) $\gamma(G^2) = \frac{n-1}{2}$ if and only if $G \cong P_3$ or $C_3$.
(7) $2 \leq \gamma(G) + \gamma(G^2) \leq n$ and the lower bound is attained if and only if $\Delta(G) = n-1$ and the upper bound is attained if and only if $G \cong K_2$.

Proposition 3.1. *If $H'$ is an induced subgraph of $G$, then $\gamma(G^2) \leq \gamma(H'^2) + \gamma([G \setminus H']^2)$.*

*Proof.* Let $H'$ be an induced subgraph of $G$. Then $G \setminus H'$ is also a subgraph of $G$ which is disjoint from $H'$. If $S_1,\ S_2$ are $\gamma$-sets of $H'^2$ and $(G \setminus H')^2$ respectively, then $S_1 \cup S_2$ is a dominating set of $G^2$.

$$\begin{aligned} \text{Therefore } \gamma(G^2) &\leq |S_1 \cup S_2| \\ &= |S_1| + |S_2| - |S_1 \cap S_2| \\ &\leq |S_1| + |S_2| \\ &= \gamma(H'^2) + \gamma([G \setminus H']^2). \end{aligned}$$ □

Theorem 3.2. *For any connected graph $G$ of even order $n \geq 6$, $\gamma(G^2) \leq \frac{n-2}{2}$ and the equality holds if and only if $G$ is either $P_6$ or $C_6$.*

*Proof.* The required upper bound follows from Eq.(3.2), Properties 3, 5 and 6. Assume that $\gamma(G^2) = \frac{n-2}{2}$. We claim that $\Delta(G) = 2$. If $\Delta(G) \geq 3$, then there is a vertex $v$ of degree at least three in $G$. Clearly, $|N_G^2[v]| \geq 5$ and let $G' = G^2 \setminus N_G^2[v]$. If $G'_1, G'_2, \ldots, G'_s$ are the components of $G'$ with $|V(G'_i)| = l_i,\ 1 \leq i \leq s$, then it is clear that $\sum |V(G'_i)| \leq n-5$ and by Proposition 3.1, $\gamma(G^2) \leq 1 + \frac{n-5}{2} = \frac{n-3}{2}$, a contradiction. Hence $G$ is either $P_n$ or $C_n$. Since $\gamma(P_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$, by Eq.(3.1) and hypothesis $n = 6$. Hence $G$ is either $P_6$ or $C_6$. Converse is obvious by verification. □

Theorem 3.3. *If $G$ is any connected graph of even order $n$ at least 4, then $\gamma(G) + \gamma(G^2) = n-1$ if and only if $G \cong C_4$ or $P_4$.*

*Proof.* Assume that $\gamma(G) + \gamma(G^2) = n-1$. Since $n$ is even, by Theorem 2.1, Eqs.(3.1) and (3.2),

$$\gamma(G) = \tfrac{n}{2} \text{ and } \gamma(G^2) = \tfrac{n-2}{2}. \tag{3.3}$$

By Theorem 2.2, $G \cong C_4$ or $H \circ K_1$. If $G \cong P_4$ or $C_4$, then $\gamma(G^2) = 1 = \frac{n-2}{2}$. Hence $G$ satisfies Eq.(3.3). Otherwise, $G \cong H \circ K_1$ with $|V(H)| \geq 3$. Then $|V(G)| \geq 6$ and hence by Theorem 3.2, $G \cong P_6$ or $C_6$ which are not corona for any connected graph $H$. Hence $G \cong C_4$ or $P_4$. Converse is obvious by verification. □

Theorem 3.4. *For any connected graph $G$ of odd order $n$ and with at least three vertices, then $\gamma(G) + \gamma(G^2) = n-1$ if and only if $G$ is either $P_3$ or $C_3$.*

*Proof.* It follows from Theorems 2.1, Eqs.(3.1), (3.2) and Property 6. □

Lemma 3.5. *Let $G \cong H \circ K_1$. If $\Delta(H) \geq 3$, then $\gamma(G^2) \leq \frac{n-6}{2}$.*

*Proof.* Assume that $\Delta(H) \geq 3$. Then there exists a vertex $v$ of degree at least three in $H$. Clearly, $v$ is adjacent to at least eight vertices in $G^2$, that is $|N_G^2[v]| \geq 8$. Let $H' = G\langle N[v] \rangle$. Clearly, $\gamma(H'^2) = 1$. Hence by Proposition 3.1, $\gamma(G^2) \leq 1 + \frac{n-8}{2} = \frac{n-6}{2}$. □

Proposition 3.6. *(i) For any path $P_k (k \geq 3)$, $\gamma([P_k \circ K_1]^2) = \lceil \frac{k}{3} \rceil$.*
*(ii) For any cycle $C_k$, $\gamma([C_k \circ K_1]^2) = \lceil \frac{k}{3} \rceil$.*

*Proof.* i) Let $G = P_k \circ K_1$ with $P_k = (v_1 v_2 \ldots v_{k-1} v_k)$ and $v'_i$ be the pendant vertex adjacent to $v_i,\ 1 \leq i \leq k$ in $G$. If $k \equiv 0 (mod\ 3)$, let $S_1 = \{v_2, v_5, v_8, \ldots, v_{k-1}\}$. If $k \equiv 1 (mod\ 3)$, let $S_2 = \{v_2, v_5, v_8, \ldots, v_{k-2}\} \cup \{v'_k\}$. If $k \equiv 2 (mod\ 3)$, let $S_3 = \{v_2, v_5, v_8, \ldots, v_{k-3}\} \cup \{v_k\}$. In all cases, $|S_i| = \lceil \frac{k}{3} \rceil$, and each is a dominating set in $G^2$ so that $\gamma(G^2) \leq \lceil \frac{k}{3} \rceil$. If we remove one vertex from $S_i$, then it is evident that $S_i$ is not a dominating set of $G^2$. Hence $\gamma(G^2) = \lceil \frac{k}{3} \rceil$. Proof of (ii) is similar. □

THEOREM 3.7. *If $G$ is any connected graph of even order $n \geq 6$, then $\gamma(G)+\gamma(G^2) = n-2$ if and only if $G$ is isomorphic to $P_3 \circ K_1$, $C_3 \circ K_1$, $P_4 \circ K_1$, $C_4 \circ K_1$, $P_6$ or $C_6$.*

*Proof.* Assume that $\gamma(G) + \gamma(G^2) = n - 2$. Since $G$ is of even order, by Theorem 2.1 and Eqs.(3.1), (3.2),

$$\gamma(G) = \tfrac{n}{2} \text{ and } \gamma(G^2) = \tfrac{n-4}{2} \tag{3.4}$$

$$\text{(or) } \gamma(G) = \tfrac{n-2}{2} \text{ and } \gamma(G^2) = \tfrac{n-2}{2}. \tag{3.5}$$

When (3.4) is satisfied, by Theorem 2.2 and hypothesis, $G \cong H \circ K_1$. By Lemma 3.5, $H$ is either a path or a cycle. We claim that $|V(H)|$ is 3 or 4. If $|V(H)| \geq 5$, then by Proposition 3.6, it is easy to see that $\gamma((H \circ K_1)^2) \neq \frac{n-4}{2}$ and hence $H$ is either $P_3, P_4, C_3$ or $C_4$. When (3.5) is satisfied, by Theorem 3.2, $G$ is either $P_6$ or $C_6$. The converse can be easily verified. □

THEOREM 3.8. *If $G$ is any connected graph of odd order $n \geq 5$, then $\gamma(G)+\gamma(G^2) = n-2$ if and only if $G \cong P_5$, $P_7$, $C_7$ or any one of the graphs in $\mathcal{B} \setminus \{C_3\}$ and Figure 4.*

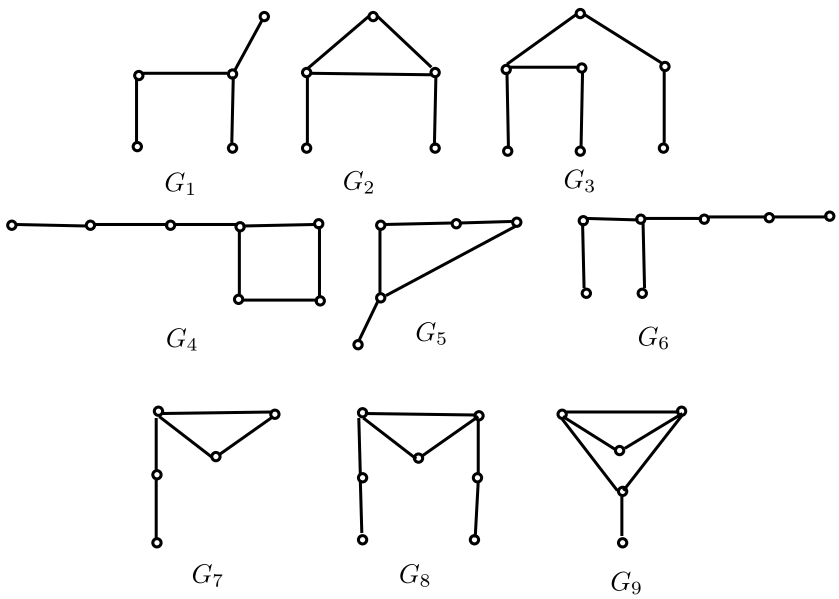


FIGURE 4. Graphs satisfying $\gamma(G) + \gamma(G^2) = n - 2$.

*Proof.* Assume that $\gamma(G) + \gamma(G^2) = n - 2$. Since $G$ is of odd order, from Eqs.(3.1),(3.2) and Theorem 2.1,

$$\gamma(G) = \tfrac{n-1}{2} \text{ and } \gamma(G^2) = \tfrac{n-3}{2} \tag{3.6}$$

and from Theorem 2.3, $G \in \mathcal{G} = \bigcup_{i=2}^{6} \mathcal{G}_i$. Then we have the following cases.

**Case 1:** $G \in \mathcal{G}_2$

For every graph $G \in \mathcal{A} \setminus \{C_4, C_7\}$, $\gamma(G^2) = 1 < 2 = \frac{n-3}{2}$. Further, it is easy to verify that for every graph $G \in \mathcal{B} \setminus \{C_3\}$, $\gamma(G^2) = 1 = \frac{n-3}{2}$.

**Case 2:** $G \in \mathcal{G}_3$

Let $n = 2k + 1$. Since $n \geq 5$, $|V(H)| = k \geq 2$. Let $H_1, H_2, \ldots, H_s$ be the components of $H$ such that $|V(H_i)| = k_i$, $1 \leq i \leq s$. Clearly, $\sum |V(H_i)| = k$, $1 \leq i \leq s$. We claim that $diam(H_i) \leq 1$. Suppose $H_i$ contains a $P_3$ as an induced subgraph (say $v_1v_2v_3$). Let $v_1', v_2', v_3'$ be the pendant vertices corresponding to $v_1, v_2, v_3$ respectively. Let $H_i' = G\langle\{v_1, v_2, v_3, v_1', v_2', v_3'\}\rangle$. Then clearly $\gamma(H_i'^2) = 1$ and by Proposition 3.1, $\gamma(G^2) \leq 1 + \frac{n-6}{2} = \frac{n-4}{2} < \frac{n-3}{2}$ which is a contradiction. Hence $diam(H_i) \leq 1$. Now we show that

$|V(H)| \leq 3$. Suppose $H$ contains at least 4 vertices, say $u_1, u_2, u_3, u_4$. Let $u'_1, u'_2, u'_3, u'_4$ be their pendant vertices. Let $H' = G\langle\{u_1, u_2, u_3, u_4, u'_1, u'_2, u'_3, u'_4, x\}\rangle$. Then clearly $\gamma(H'^2) \leq 2$. Then by Proposition 3.1, $\gamma(G^2) \leq 2 + \frac{n-9}{2} = \frac{n-5}{2}$, a contradiction. Hence $|V(H)| \leq 3$. If $diam(H_i) = 0$, then $H$ is a totally disconnected graph. Further by definition of $\mathcal{G}_3$ and hypothesis, $k$ must be 2. Hence $G \cong P_5$. If $diam(H_i) = 1$, then clearly $|V(H)| = 2$. Hence $G \cong G_1$ or $G_2$. Suppose $H$ contains $p$ copies of $K_1$ and $q$ copies of $K_2$. Since $|V(H)| \leq 3$, $p = q = 1$. Hence $H = K_1 \cup K_2$. If $x$ is adjacent to every vertex of $H$, then $\gamma(G^2) = 1 \neq 2 = \frac{n-3}{2}$. If $x$ is adjacent to exactly one vertex of each copy of $H$, then $\gamma(G^2) = 2 = \frac{n-3}{2}$. Hence $G \cong G_3$.

**Case 3:** $G \in \mathcal{G}_4$

Let $n = 2k+1$. Let $H_1, H_2, \ldots, H_s$ be the components of $H$. By definition of $\mathcal{G}_4$, $|V(H)| = \frac{2k-4}{2}$. We claim that $diam(H_i) = 0$ for every $1 \leq i \leq s$. Suppose $H_i$ contains a $P_2$ as an induced subgraph (say $u_1u_2$). Let $u'_1, u'_2$ be the pendant vertices corresponding to $u_1, u_2$ respectively. Let $H' = G\langle\{u_1, u_2, u'_1, u'_2\} \cup \{x\} \cup \{C_4\}\rangle$. Then clearly $\gamma(H'^2) = 2$ and by Proposition 3.1, $\gamma(G^2) \leq 2 + \frac{n-9}{2} = \frac{n-5}{2} < \frac{n-3}{2}$ which is a contradiction. Hence $H$ is a totally disconnected graph. Further by definition of $\mathcal{G}_4$ and hypothesis, $k$ must be 3. Hence $G \cong G_4$.

**Case 4:** $G \in \mathcal{G}_5$

Let $n = 2k+1$. Let $H_1, H_2, \ldots, H_s$ be the components of $H$. By definition of $\mathcal{G}_5$, let $u, v, w$ be a sequence of path $P_3$ in $G$. We claim that $diam(H_i) \leq 1$. Suppose $H_i$ contains a $P_3$ as an induced subgraph (say $w_1w_2w_3$). Let $w'_1, w'_2, w'_3$ be the pendant vertices corresponding to $w_1, w_2, w_3$ respectively. Let $H'_i = G\langle\{u, v, w\} \cup \{w_1, w_2, w_3, w'_1, w'_2, w'_3\}\rangle$. Then clearly $\gamma(H_i'^2) \leq 2$ and by Proposition 3.1, $\gamma(G^2) \leq 2+\frac{n-9}{2} = \frac{n-5}{2} < \frac{n-3}{2}$ which is a contradiction. Hence $diam(H_i) \leq 1$. Now we show that $|V(H)| \leq 2$. Suppose $|V(H)| \geq 3$. Then by a similar argument, $\gamma(G^2) \leq 2+\frac{n-9}{2} = \frac{n-5}{2}$, a contradiction. Hence $|V(H)| \leq 2$. By hypothesis, $|V(G)| = 5$ or 7. From definition of $\mathcal{G}_5$, $\langle\{u, v, w\}\rangle$ is either $P_3$ or $C_3$ in $G$.

**Subcase 4.1:** $|V(G)| = 5$

Then $H$ must be $K_1$. If $\langle\{u, v, w\}\rangle \cong P_3$, then $G \cong P_5$ or $G_5$. If $\langle\{u, v, w\}\rangle \cong C_3$, then $G \cong G_7$ or $G_9$.

**Subcase 4.2:** $|V(G)| = 7$

Then $H$ must be either $K_2$ or $K_1 \cup K_1$. Suppose $\langle\{u, v, w\}\rangle \cong C_3$. If $H = K_2$, then $\gamma(G^2) = 1 \neq \frac{n-3}{2}$. If $H = K_1 \cup K_1$, then $G \cong G_8$. Suppose $\langle\{u, v, w\}\rangle \cong P_3$. If $H = K_2$, then $G \cong G_6$. If $H = K_1 \cup K_1$, then $G \cong P_7$.

**Case 5:** $G \in \mathcal{G}_6$

Let $H_1, H_2, \ldots, H_s$ be the components of $H$. We claim that $diam(H_i) = 0$ for every $1 \leq i \leq s$. Suppose $H_i$ contains a $P_2$ as an induced subgraph (say $x_1x_2$). Let $x'_1, x'_2$ be the pendant vertices corresponding to $x_1, x_2$ respectively. Let $X \in \mathcal{B}$ (See Figure. 2) and $H' = G\langle\{x_1, x_2, x'_1, x'_2\} \cup V(X)\rangle$. Then clearly $\gamma(H'^2) \leq 2$ and by Proposition 3.1, $\gamma(G^2) \leq \frac{n-5}{2}$, a contradiction. Hence $H$ is a totally disconnected graph.

**Subcase 5.1:** $X = B_1$

We claim that $|V(H)| \leq 2$. If $H$ has three vertices, let $H' = G\langle H \circ K_1 \cup V(X)\rangle$. Then clearly $\gamma(H'^2) \leq 2$ and by Proposition 3.1, $\gamma(G^2) \leq 2 + \frac{n-9}{2} = \frac{n-5}{2}$, a contradiction. Hence $|V(H)| \leq 2$. If $|V(H)| = 1$, then by definition of $\mathcal{G}_6$, $G \cong G_7$ or $G_9$. Suppose $|V(H)| = 2$. If $H = K_1 \cup K_1$, then $G \cong G_8$.

**Subcase 5.2:** $X \in \mathcal{B} \setminus \{C_3\}$

If $H$ is non-empty, then $|V(H)| \geq 1$, say $z$. Let $z'$ be the pendant vertex corresponding to $z$. By definition of $\mathcal{G}_6$, note that at least one vertex of $X$ is adjacent to a vertex $z$ in $H$. Let $H' = G\langle\{z, z'\} \cup X\rangle$. Since $X^2 = K_5$, $\gamma(X^2) = 1$. Then clearly, $\gamma(H'^2) = 1$ and by

Proposition 3.1, $\gamma(G^2) \leq 1 + \frac{n-7}{2} = \frac{n-5}{2}$. Hence no graph exists in this case. Converse can be easily verified. □

## 4. Bounds in terms of order and size

In [2], C. Berge gave the lower bound for the domination number of a graph $G$ in terms of its order $n$ and size $m$ and noted that $\gamma(G^2)$ is also a lower bound for the domination number of a graph. By this motivation, we have the following

THEOREM 4.1. *Let $G$ be a connected graph of order $n$ and size $m$. Then $\gamma(G^2) \geq n - m$ and the equality holds if and only if $G$ is a tree with diameter at most 4.*

*Proof.* Since $n - m \leq 1$, by definition of domination number $\gamma(G^2) \geq 1 \geq n - m$. Assume that $\gamma(G^2) = n - m$. We claim that $G$ is a tree. Suppose $G$ contains a cycle. Then $m \geq n$ and hence by the assumption, $\gamma(G^2) \leq 0$, which is a contradiction. Hence $G$ is a tree. Next we claim that $diam(G) \leq 4$. Suppose $diam(G) \geq 5$. Then $\gamma(G^2) \geq 2$ and hence, $n - m \geq 2$ which implies $G$ is disconnected, a contradiction. Hence $diam(G) \leq 4$.

Conversely, assume that $G$ is a tree with diameter at most 4. Then by Property 4 mentioned in section 3, $\gamma(G^2) = 1 = n - (n-1) = n - m$. □

THEOREM 4.2. *Let $G$ be a connected graph of order $n$ and size $m$. Then $\gamma(G) + \gamma(G^2) \geq 2(n - m)$ and the equality holds if and only if $G$ is a star.*

*Proof.* It follows from Theorem 4.1 and Theorem 2.4. □

THEOREM 4.3. *Let $G$ be a connected graph of order $n$ and size $m$. Then $\gamma(G) + \gamma(G^2) = 2n - 2m + 1$ if and only if $G$ is either a bistar or $P_3(S)$.*

*Proof.* Assume that $\gamma(G) + \gamma(G^2) = 2n - 2m + 1$. Then by Eq.(3.1),

$$\gamma(G) = n - m + 1 \text{ and } \gamma(G^2) = n - m. \tag{4.1}$$

By Theorem 4.1, $G$ is a tree with diameter at most 4. Then by Eq.(4.1), $\gamma(G) = 2$ and $\gamma(G^2) = 1$. Hence the required graphs follows from Lemma 2.11. Converse can be easily verified. □

THEOREM 4.4. *Let $G$ be a connected graph of order $n$ and size $m$. Then $\gamma(G) + \gamma(G^2) = 2n - 2m + 2$ if and only if $G$ is either $C_3$ or $P_4(S)$ or any one of the graphs given in Figure 5.*

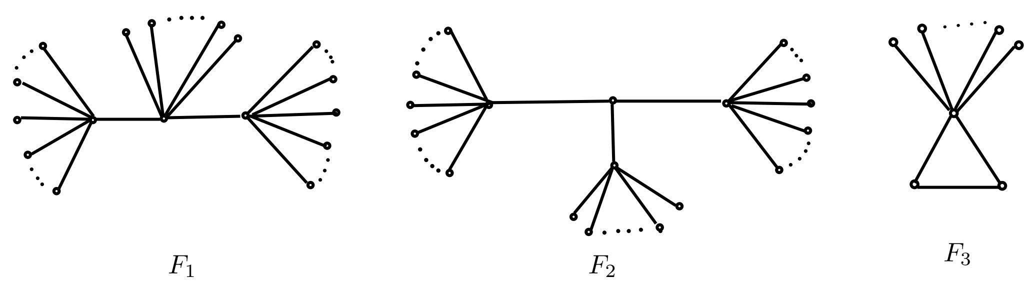


FIGURE 5. Graphs satisfying $\gamma(G) + \gamma(G^2) = 2n - 2m + 2$.

*Proof.* Assume that $\gamma(G) + \gamma(G^2) = 2n - 2m + 2$. Then by Eq.(3.1)

$$\gamma(G) = n - m + 2 \text{ and } \gamma(G^2) = n - m \tag{4.2}$$

$$\text{(or) } \gamma(G) = n - m + 1 \text{ and } \gamma(G^2) = n - m + 1. \tag{4.3}$$

From Eq.(4.2) and Lemma 4.1, $G$ is a tree with diameter at most 4. Then $\gamma(G) = 3$ and

$\gamma(G^2) = 1$. Hence $diam(G) \neq 2, 3$. If $diam(G) = 4$, let $v_1v_2v_3v_4v_5$ be a diametral path in $G$ with $d(v_1) = d(v_5) = 1$ and $v_2$, $v_4$ are support vertices. Clearly $d(v_3) \geq 3$. If $v_3$ is a support vertex, then $G \cong F_1$. Otherwise, it is adjacent to a support vertex in which case $G \cong F_2$. From Eq.(4.3), $m$ is either $n-1$ or $n$. If $m = n-1$, then $G$ is a tree with $\gamma(G) = \gamma(G^2) = 2$. By Lemma 2.11, $G \cong P_4(S)$. If $m = n$, then $G$ contains a unique cycle $C = (v_1v_2 \dots v_kv_1)$ in $G$ with $\gamma(G) = \gamma(G^2) = 1$. Since $\gamma(G) = 1$, $\Delta(G) = n - 1$ and hence $k = 3$. If $G$ has no pendant vertices, then $G \cong C_3$. Otherwise, $G \cong F_3$. Converse can be easily verified. □

## 5. Bounds for planar graphs

DEFINITION 5.1. A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends.

In [9], MacGillivray and Seyffarth established the following Results.

THEOREM 5.2. [9] *If $G$ is a planar graph with $diam(G) = 2$, then $\gamma(G) \leq 3$.*

THEOREM 5.3. [6] *If $G$ is a planar graph with $diam(G) = 2$, then $\gamma(G) \leq 2$ or $G$ is isomorphic to $P$ where $P$ is the graph of Figure 6.*

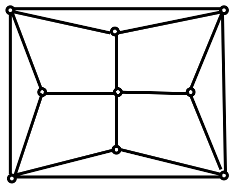

FIGURE 6. A planar graph $P$ of diameter 2 with domination number 3.

COROLLARY 5.4. *If $G$ is a planar graph with $diam(G) = 2$, then $\gamma(G) + \gamma(G^2) \leq 4$ and the equality holds if and only if $G \cong P$ given in Figure 6.*

THEOREM 5.5. *If $G \neq P$ is a planar graph with $diam(G) = 2$, then $\gamma(G) + \gamma(G^2) \leq 3$ and the equality holds for regular graphs if and only if $G \cong C_4$, $C_5$, $K_2 \times K_3$ or any one of the graphs given in Figure 3.*

*Proof.* The required upper bound follows from Theorems 5.3 and Corollary 5.4. If $\gamma(G) + \gamma(G^2) = 3$, then $\gamma(G) = 2$ and $\gamma(G^2) = 1$. Since $G$ is planar and regular, $\Delta(G) \leq 5$. If $\Delta(G) = 2$, then $G \cong C_n$ and by hypothesis, $G \cong C_4$ or $C_5$. If $\Delta(G) = 3$, then by Theorem 2.7, $G$ is isomorphic to the Cartesian product $K_2 \times K_3$. If $\Delta(G) = 4$, then by Theorem 2.8, $G$ is isomorphic to one of the three graphs given in Figure 3. If $\Delta(G) = 5$, then by Theorem 2.9, no graph exists. Converse can be easily verified. □

## 6. Bounds in terms of order and maximum degree

In this section, we obtain the upper bound for the sum $\gamma(G) + \gamma(G^2)$ in terms of the order and the maximum degree $\Delta(G)$ of a graph $G$ and characterize the extremal graphs.

THEOREM 6.1. *If $G$ is a connected graph of order $n$ with maximum degree $\Delta(G)$, then $\gamma(G) + \gamma(G^2) \leq 2(n - \Delta(G))$.*

*Proof.* The required upper bound follows from Eq.(3.1) and Theorem 2.6. □

THEOREM 6.2. *If $G$ is a connected graph of order $n$, then $\gamma(G)+\gamma(G^2)=2n-4$ if and only if $G\cong P_3$ or $C_3$.*

*Proof.* If $\gamma(G)+\gamma(G^2)=2n-4$, then by Theorem 2.6 and Eq.(3.1),
$$\gamma(G)=\gamma(G^2)=n-2. \tag{6.1}$$
By Theorem 2.1, $n\leq 4$. If $n=3$, then $P_3$ and $C_3$ satisfy Eq.(6.1). If $n=4$, then by Property 3, $\gamma(G^2)=1=n-3\neq n-2$. Converse is obvious. □

THEOREM 6.3. *If $G$ is a connected graph of order $n$, then $\gamma(G)+\gamma(G^2)=2n-5$ if and only if $G\cong P_4$ or $C_4$.*

*Proof.* If $\gamma(G)+\gamma(G^2)=2n-5$, then by Theorem 2.6 and Eq.(3.1),
$$\gamma(G)=n-2 \text{ and } \gamma(G^2)=n-3. \tag{6.2}$$
By Theorems 2.1 and 6.2, $n=4$. Then by Property 3 and Theorem 2.2, $G\cong P_4$ or $C_4$ which satisfy Eq.(6.2). Converse is obvious. □

THEOREM 6.4. *If $G$ is a connected graph of order $n$, then $\gamma(G)+\gamma(G^2)=2n-6$ if and only if $G$ is either $K_4,\ K_4-e,\ K_{1,3}$ or $K_{1,3}+e$.*

*Proof.* If $\gamma(G)+\gamma(G^2)=2n-6$, then by Theorem 2.6 and Eq.(3.1),
$$\gamma(G)=n-2 \text{ and } \gamma(G^2)=n-4 \tag{6.3}$$
$$\text{(or) } \gamma(G)=\gamma(G^2)=n-3\ . \tag{6.4}$$
Eq.(6.3) is not possible by Theorem 2.1. From Eq. (6.4) and Theorem 2.1, $n\leq 6$. If $n=4$, then $\gamma(G)=\gamma(G^2)=1$ and hence $G$ is either $K_4-e$ or $K_4$ or $K_{1,3}$ or $K_{1,3}+e$. If $n=5$ or 6, then by Properties 3 and 5, no graph exists. Converse follows by verification. □

THEOREM 6.5. *If $G$ is a connected graph of order $n$, then $\gamma(G)+\gamma(G^2)=2n-7$ if and only if $G$ is either $P_5,\ C_5,\ G_1,\ G_2,\ G_5,\ G_7,\ G_9,\ B_3,\ B_4$ or $B_5$.*

*Proof.* If $\gamma(G)+\gamma(G^2)=2n-7$, then by Theorem 2.6 and Eq.(3.1),
$$\gamma(G)=n-2 \text{ and } \gamma(G^2)=n-5 \tag{6.5}$$
$$\text{(or) } \gamma(G)=n-3 \text{ and } \gamma(G^2)=n-4. \tag{6.6}$$
Eq.(6.5) is not possible by Theorem 2.1. From Eq.(6.6) and Theorem 2.1, $n\leq 6$. If $n=5$, then $\gamma(G^2)=1$ and $\gamma(G)=2$. Clearly $\Delta(G)\neq 4$. Hence $G\cong P_5,\ C_5,\ G_1,\ G_2,\ G_5,\ G_7,\ B_3, B_4$, $B_5$ or $G_9$. If $n=6$, then $\gamma(G)=3$ and $\gamma(G^2)=2$. By Theorem 2.2 and Property 4, no such graph exists. The converse follows by verification. □

THEOREM 6.6. *If $G$ is a connected graph of order $n$, then $\gamma(G)+\gamma(G^2)=2n-8$ if and only if $G$ is either $P_3\circ K_1,\ C_3\circ K_1,\ P_6,\ C_6$ or a graph on 5 vertices having a vertex of degree 4.*

*Proof.* Assume that $\gamma(G)+\gamma(G^2)=2n-8$. Then by Theorem 2.6 and Eq.(3.1), we have three cases.
$$\gamma(G)=n-2 \text{ and } \gamma(G^2)=n-6 \tag{6.7}$$
$$\text{(or) } \gamma(G)=n-3 \text{ and } \gamma(G^2)=n-5 \tag{6.8}$$
$$\text{(or) } \gamma(G)=n-4 \text{ and } \gamma(G^2)=n-4. \tag{6.9}$$
Clearly Eq.(6.7) is not possible. From Eq.(6.8) and Theorem 2.1, $n=6$. Then $\gamma(G)=3$ and $\gamma(G^2)=1$. Hence by Theorem 2.2 and Property 4, $G\cong P_3\circ K_1,\ C_3\circ K_1$. From Eq.(6.9) and Theorem 2.1, $n\leq 8$. If $n=5$, then $\gamma(G)=\gamma(G^2)=1$. Hence $G$ is a graph on 5 vertices having a vertex of degree 4. If $n=6$, then $\gamma(G)=\gamma(G^2)=2$. By Theorem 3.2, $G\cong P_6$ or $C_6$. If $n=7$, then $\gamma(G)=\gamma(G^2)=3$. By Property 6, no graph exists. If $n=8$, then $\gamma(G)=\gamma(G^2)=4$. By Theorem 2.2, $G\cong H\circ K_1$ where $|V(H)|=4$ and $H$ is connected for which $\gamma(G^2)\leq 2$, a contradiction. The converse is obvious. □

Theorem 6.7. *For any connected graph $G$ of order $n$, $\gamma(G)+\gamma(G^2)=2n-9$ if and only if $G$ is either $C_7,\ P_7,\ G_3,\ G_4,\ G_6,\ G_8$ or a graph on 6 vertices having $\Delta(G)=3$ or 4 except $P_3\circ K_1,\ C_3\circ K_1$.*

*Proof.* Assume that $\gamma(G)+\gamma(G^2)=2n-9$. Clearly the two cases $\gamma(G)=n-2$ and $\gamma(G^2)=n-7$, $\gamma(G)=n-3$ and $\gamma(G^2)=n-6$ are not possible. Now we consider the remaining case

$$\gamma(G)=n-4 \text{ and } \gamma(G^2)=n-5. \tag{6.10}$$

By Theorem 2.1, $n\leq 8$. If $n=6$, then $\gamma(G)=2$ and $\gamma(G^2)=1$. Clearly $\Delta(G)\neq 2,5$. From our choice of $n$ and Theorem 2.2, $G\neq P_3\circ K_1,\ C_3\circ K_1$. Hence $G$ is a graph on 6 vertices having $\Delta(G)=3$ *or* 4 except $P_3\circ K_1,\ C_3\circ K_1$ which are satisfy Eq.(6.10). If $n=7$, then $\gamma(G)=3$ and $\gamma(G^2)=2$. By Theorem 3.8, $G\cong C_7,\ P_7,\ G_3,\ G_4,\ G_6,\ G_8$. If $n=8$, then $\gamma(G)=4$ and $\gamma(G^2)=3$. By Theorem 2.2, $G\cong H\circ K_1$ where $|V(H)|=4$ and by Theorem 3.2, no such graph exists. The converse can be easily verified. □

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