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$\mathbb{C}$-FUCHSIAN SUBGROUPS OF SOME NON-ARITHMETIC LATTICES

Li-Jie Sun

Abstract. We give a general procedure to analyze the structure for certain $\mathbb{C}$-Fuchsian subgroups of some non-arithmetic lattices. We also show their presentations and describe their fundamental domains which lie in a complex geodesic, a set homeomorphic to the unit disk.

1. Introduction

Suppose that H is a Hermitian form of signature $(2, 1)$ on $\mathbb{C}^3$. Then the projective unitary Lie group $\mathrm{PU}(2, 1)$ of H contains two conjugacy classes of connected Lie subgroups, each of which is locally isomorphic to $\mathrm{PSL}(2, \mathbb{R})$. The subgroups in one class are conjugate to $\mathrm{PSU}(1, 1)$, and preserve a complex line for the projective action of $\mathrm{PU}(2, 1)$ on the projective plane $\mathbf{P}^2_{\mathbb{C}}$. The subgroups in the other class are conjugate to $\mathrm{PO}(2, 1)$, and preserve a totally real Lagrangian plane. If Γ is a discrete subgroup of $\mathrm{PU}(2, 1)$, the intersections of Γ with the connected Lie subgroups locally isomorphic to $\mathrm{PSL}(2, \mathbb{R})$ are its Fuchsian subgroups. The Fuchsian subgroups fixing a complex line are called $\mathbb{C}$-Fuchsian subgroups. See Section 2 for more details.

Fuchsian subgroups have remarkable geometrical properties and they are interesting on their own, see for instance [11, 12]. They also play an important role in complex hyperbolic space. Deraux [2] proved that the discrete deformation of some $\mathbb{R}$-Fuchsian triangle group in $\mathrm{PU}(2, 1)$ is a cocompact arithmetic lattice (a lattice in $\mathrm{PU}(2, 1)$ is a discrete group with finite covolume). There also have been significant developments on $\mathbb{C}$-Fuchsian subgroups. To this direction, let S be a hyperbolic surface. Gusevskii-Parker [7] studied the deformation space of a $\mathbb{C}$-Fuchsian representation $\pi_1(S) \to \mathbf{Isom}(\mathbf{H}^2_{\mathbb{C}})$ by formulating and proving Poincaré's polyhedron theorem for one special class of polyhedra in complex hyperbolic plane. Furthermore, Stover [14] proved that if Γ is a complex hyperbolic lattice containing a complex reflection, then Γ contains a $\mathbb{C}$-Fuchsian subgroup stabilising the complex geodesic fixed by the reflection. However, it is usually difficult to get an explicit description of such $\mathbb{C}$-Fuchsian subgroups from the complex hyperbolic lattice. In the present paper, we wish to identify the structures of the $\mathbb{C}$-Fuchsian subgroups (arising as stabilisers of the complex geodesics fixed by

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reflections) in complex hyperbolic lattices, mainly by applying Poincaré's polygon theorem. In this way, we also illustrate their actions on the fixing complex geodesic L. In other words, we get a more explicit version of Stover's result. The study on the structure of the stabilisers of the complex lines is useful in the study of complex hyperbolic lattices using algebraic geometry (see for example [3]), and is useful for considering lattices from the point of view of hybrids (see [16]).

In [4,5], Deraux, Parker and Paupert considered a family of groups which produce all currently known examples of non-arithmetic lattices in $\mathrm{PU}(2,1)$. Each of such groups is a complex hyperbolic triangle group generated by three complex reflections of the same order p ($p \geq 2$). They prove the discreteness by constructing an explicit fundamental domain for each group, and show that the geometric realisation gives an embedding of the combinatorial fundamental domain into the topological closure of complex hyperbolic plane $\overline{\mathbf{H}^2_{\mathbb{C}}}$. In particular, the authors listed the side (codimension-1) representatives of the fundamental domains for the sporadic triangle groups (see Section 3.1) and Thompson triangle groups (see Section 3.2), also gave the natural representation for each group.

In this paper, our goal is to identify the $\mathbb{C}$-Fuchsian subgroups of the sporadic triangle groups (subgroups of equilateral triangle groups) and Thompson triangle groups (subgroups of non-equilateral triangle groups), which appeared in [5]. We consider the equilateral triangle groups which are generated by three complex reflections R_1, R_2, R_3 with the property that there exists a complex hyperbolic isometry J of order 3 such that $R_{j+1} = JR_jJ^{-1}$ (the indices taken by mod 3). The equilateral triangle groups then can be parameterised by the order p of generators and the complex parameter

$$\tau = \mathrm{tr}(R_1 J).$$

We denote the sporadic triangle groups by $\mathcal{S}(p,\tau)$. See details in Section 3.1.

Our main theorem is the following:

THEOREM 1.1. *Let R_1, R_2, R_3 be three complex reflections of order p in* $\mathrm{SU}(2,1)$ *so that R_i fixes a complex geodesic L_i, $i = 1,2,3$. Suppose that R_1, R_2, R_3 is the generating set for $\mathcal{S}(p,\tau)$. Then there exist $\mathbb{C}$-Fuchsian subgroups fixing complex geodesic L_1 which have the following structure according to (τ, p) :*

(i) $\tau = -1 + i\sqrt{2}, p = 3,4,6$:

$$\left\langle (1\bar{3}23)^2, (13)^3, (12)^3, (123\bar{2})^2, (1232\bar{3}\bar{2})^3(123\bar{2})^2(12)^3 \right\rangle;$$

(ii) $\tau = -\frac{1+i\sqrt{7}}{2}, p = 3,4,5,6,8,12$:

$$\left\langle (12)^2, (13)^2, 23\bar{2}P^2 \right\rangle,$$

where $P = R_1 J$;

(iii) $\tau = \frac{1+\sqrt{5}}{2}, p = 3,4,5,10$:

$$\left\langle 1\bar{3}\bar{2}323, 1312\bar{1}\bar{3}, (1\bar{3}23)^3 1\bar{3}\bar{2}323 \right\rangle.$$

Here we just write $1\bar{3}23$ to denote $R_1R_3^{-1}R_2R_3$ (see Section 2.3), etc. Throughout this paper, we always investigate the $\mathbb{C}$-Fuchsian subgroups fixing a complex geodesic L_1. One should note that there naturally exist $\mathbb{C}$-Fuchsian subgroups fixing other complex geodesics in the complex hyperbolic lattice under consideration. For example, we could get a $\mathbb{C}$-Fuchsian subgroup in $\mathcal{S}(3,-1+i\sqrt{2})$ stabilising the complex geodesic L_2 fixed by the complex reflection R_2 which is identified with JR_1J^{-1}:

$$\langle J(1\bar{3}23)^2J^{-1}, J(13)^3J^{-1}, J(12)^3J^{-1}, J(123\bar{2})^2J^{-1}, J(1232\bar{3}\bar{2})^3(123\bar{2})^2(12)^3J^{-1}\rangle.$$

In [5], the authors build blocks of the fundamental domains bounded by spherical shells that surround the fixed point of $P = R_1J$ for the lattices of equilateral triangle groups type or $Q = R_1R_2R_3$ in the non-equilateral case. A spherical shell here means that the corresponding cell complex is an embedded copy of S^3, which bounds a well-defined 4-ball. Surrounding a point just means that the point is in the ball component of the complement of that copy of S^3. The basic building blocks for their fundamental domains are pyramids (for example, see Figure 1) in bisectors. They finally list side (codimension-1) representatives for each P-orbit of sides (or Q-orbit in the non-equilateral case), and one side for each pair of opposite sides which means paired in the sense of the Poincaré polyhedron theorem, see Appendix in [5]. In the present paper, our general procedure to distinguish $\mathbb{C}$-Fuchsian subgroups is as follows: We firstly focus on the pyramids of the side representatives of the fundamental domain for the non-arithmetic lattices; secondly, for each lattice, we force the side representatives to have the same base L_1 and obtain a polygon lying in the complex geodesic L_1; finally we prove that the polygon is a fundamental domain of some subgroup of the lattice. Actually the polygon can be matched by side pairing transformations, which are exactly the generators of the $\mathbb{C}$-Fuchsian subgroups as showed in Theorem 1.1. We also give the natural presentation for each $\mathbb{C}$-Fuchsian subgroup among the proof.

The paper is arranged as follows. Section 2 contains background material about complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$, totally geodesic subspaces and complex reflection. In Section 3 we recall the normalisation of two kinds of complex hyperbolic triangle groups in $\mathrm{PU}(2,1)$: equilateral triangle groups and non-equilateral triangle groups, in which we will clarify the $\mathbb{C}$-Fuchsian subgroups. In Section 4, we mainly state and prove our theorems, including describing the fundamental domains of certain $\mathbb{C}-$Fuchsian subgroups.

2. Preliminaries

The material for this section is standard. The reader may refer to [6] for more details.

2.1. Complex hyperbolic plane. We use $\mathbb{C}^{2,1}$ to denote $\mathbb{C}^3$ equipped with a Hermitian form of signature $(2,1)$. If we assume that $\mathbf{P}$ is the canonical projectivisation from $\mathbb{C}^{2,1}$ to $\mathbf{P}^2_{\mathbb{C}}$ and suppose that the Hermitian form of signature $(2,1)$ to be H, then the *complex hyperbolic plane* $\mathbf{H}^2_{\mathbb{C}}$ can be defined as follows:

$$\mathbf{H}^2_{\mathbb{C}} := \mathbf{P}\{\mathbf{z} \in \mathbb{C}^{2,1} : \langle \mathbf{z}, \mathbf{z}\rangle = \bar{\mathbf{z}}^{\mathbf{t}}\mathrm{H}\mathbf{z} < 0\}.$$

Correspondingly, the boundary $\partial\mathbf{H}^2_{\mathbb{C}}$ of complex hyperbolic plane is

$$\partial\mathbf{H}^2_{\mathbb{C}} := \mathbf{P}\{\mathbf{z} \in \mathbb{C}^3 : \langle \mathbf{z}, \mathbf{z}\rangle = \bar{\mathbf{z}}^t\mathrm{H}\mathbf{z} = 0\}.$$

There exists a natural action of the unitary group $\mathrm{U}(2,1)$ of the Hermitian from on $\mathbf{H}^2_{\mathbb{C}}$. The automorphism group of $\mathbf{H}^2_{\mathbb{C}}$ is then $\mathrm{PU}(2,1)$, the projectivisation of $\mathrm{U}(2,1)$. In particular, $\mathrm{SU}(2,1)$ is the subgroup of $\mathrm{U}(2,1)$ with the determinant of each element being 1, which is the three fold cover of the projection group $\mathrm{PU}(2,1)$.

Let z and w be points in $\mathbf{H}^2_{\mathbb{C}}$ corresponding to vectors $\mathbf{z}, \mathbf{w} \in \mathbb{C}^{2,1}$. Then the *Bergman metric* ρ on $\mathbf{H}^2_{\mathbb{C}}$ is given by the following distance formula:

$$\cosh^2\left(\frac{\rho(z,w)}{2}\right) = \frac{\langle \mathbf{z}, \mathbf{w}\rangle\langle \mathbf{w}, \mathbf{z}\rangle}{\langle \mathbf{z}, \mathbf{z}\rangle\langle \mathbf{w}, \mathbf{w}\rangle}.$$

If we choose the Hermitian form of signature $(2,1)$ as follows

$$\langle \mathbf{z}, \mathbf{w}\rangle = z_1\overline{w_1} + z_2\overline{w_2} - z_3\overline{w_3},$$

with $\mathbf{z} = [z_1, z_2, z_3]^t$, $\mathbf{w} = [w_1, w_2, w_3]^t$, then the complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$ can be described in the affine chart $z_3 \neq 0$ as the unit ball in $\mathbb{C}^2$ endowed with the unique Kähler metric invariant under all biholomorphisms of the ball. The metric is symmetric and has non-constant negative real sectional curvature but pinched between -1 and $-1/4$. We normalise its holomorphic sectional curvature to be -1.

An automorphisms of $\mathbf{H}^2_{\mathbb{C}}$ is said to be elliptic if it fixes at least one point of $\mathbf{H}^2_{\mathbb{C}}$, parabolic if it fixes exactly one point of $\partial\mathbf{H}^2_{\mathbb{C}}$, loxodromic if it fixes exactly two points of $\partial\mathbf{H}^2_{\mathbb{C}}$. Throughout this paper, we freely use the classification of automorphisms of $\mathbf{H}^2_{\mathbb{C}}$ into regular elliptic, complex reflection, ellipto-parabolic, unipotent parabolic and loxodromic, e.g., an automorphism is regular elliptic if and only if it has a fixed point in $\mathbf{H}^2_{\mathbb{C}}$ and has distinct eigenvalues. We refer to Section 6.2 of [6] for the details.

2.2. Totally geodesic subspaces. Given two points z and w in $\overline{\mathbf{H}^2_{\mathbb{C}}} := \mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$, with lifts $\mathbf{z}, \mathbf{w}$ to $\mathbb{C}^{2,1}$ respectively, the complex span of $\mathbf{z}$ and $\mathbf{w}$ projects to a complex projective line in $\mathbf{P}^2_{\mathbb{C}}$. The intersection of a complex projective line with $\mathbf{H}^2_{\mathbb{C}}$ is called a *complex geodesic* L (homeomorphic to an open 2-dimensional disk), which can be simply obtained by taking the intersection of orthogonal complement of a positive vector $\mathbf{n}$ with $\mathbf{H}^2_{\mathbb{C}}$, i.e.,

$$L = \mathbf{P}\{\mathbf{z} \in \mathbb{C}^{2,1} : \langle \mathbf{z}, \mathbf{n}\rangle = 0\} \cap \mathbf{H}^2_{\mathbb{C}}.$$

We refer to $\mathbf{n}$ as a *polar vector* to L.

A maximal totally geodesic subspace in $\mathbf{H}^2_{\mathbb{C}}$ can only be one of the following:

(i) A complex geodesic, which is an isometrically embedded copy of $\mathbf{H}^1_{\mathbb{C}}$. It has the Poincaré model of hyperbolic geometry with constant curvature -1;

(ii) A totally real Lagrangian plane, which is an isometrically embedded copy of $\mathbf{H}^2_{\mathbb{R}}$. It has the Beltrami-Klein projective model with constant curvature $-1/4$.

2.3. Complex reflection. Suppose that the polar vector of a complex geodesic L_1 is $\mathbf{n_1}$. We consider the complex reflection R_1 in the complex geodesic L_1 which is of order p, i.e., complex reflection R_1 in $\mathrm{U}(2,1)$ maps $\mathbf{n_1}$ to $e^{i\phi}\mathbf{n_1}$, where $\phi = 2\pi/p$. Throughout this paper, we assume that $p \in \mathbb{Z}$ and $p \geq 2$. We take one lift of R_1 to a matrix in $\mathrm{SU}(2,1)$ and write the map here with the same symbol:

$$R_1(\mathbf{z}) = e^{-\frac{i\phi}{3}}\mathbf{z} + (e^{\frac{2i\phi}{3}} - e^{-\frac{i\phi}{3}})\frac{\langle \mathbf{z}, \mathbf{n_1}\rangle}{\langle \mathbf{n_1}, \mathbf{n_1}\rangle}\mathbf{n_1}. \tag{2.1}$$

In what follows, if g is a complex reflection and a complex geodesic L is pointwise fixed by g, we will always say that L is the mirror of g. A basic fact is that any complex reflection is an element of $\mathrm{PU}(2,1)$. We will restrict to the complex hyperbolic triangle groups generated by three complex reflections with the same order p ($p \geq 2$). In order to avoid tedious notation, we denote the three generators R_1, R_2, R_3 of complex hyperbolic triangle groups simply by 1, 2, 3. Unless otherwise stated, in what follows we also denote their inverse by $\bar{1}$, $\bar{2}$, $\bar{3}$. In this way, we just write $1\bar{3}23$ to denote $R_1R_3^{-1}R_2R_3$, etc.

We recall the definition for *braid relation* between group elements (see Section 2.2 of [8]). Let G be a group and $a, b \in G$. Then we will say that a, b satisfy a braid relation of length $n \in \mathbb{Z}_+$ if

$$(ab)^{n/2} = (ba)^{n/2},$$

where powers mean that the corresponding alternating product of a and b should have n factors. We denote the braid length n of a, b by $\mathrm{br}_n(a, b)$. For example, $\mathrm{br}_3(a, b)$ means that $aba = bab$.

Let A and B be two complex reflections in distinct complex geodesics L_A and L_B respectively, which correspond to polar vectors $\mathbf{n}_A$ and $\mathbf{n}_B$. The cross-product $\mathbf{z} := \mathbf{n}_A \boxtimes \mathbf{n}_B$ is defined as

$$\mathbf{z} = (\overline{\mathbf{n}_A}^t \mathrm{H}) \times (\overline{\mathbf{n}_B}^t \mathrm{H}).$$

Then three possibilities arise (see Section 3.3.2 in Goldman [6]):

1. $\mathbf{z}$ is negative, namely $\langle \mathbf{z}, \mathbf{z} \rangle < \mathbf{0}$. In this case L_A and L_B intersect in $\mathbf{P}(\mathbf{z}) \in \mathbf{H}^2_{\mathbb{C}}$ corresponding to the negative vector $\mathbf{z}$;
2. $\mathbf{z}$ is null, namely $\langle \mathbf{z}, \mathbf{z} \rangle = \mathbf{0}$. In this case L_A and L_B are asymptotic at the point $\mathbf{P}(\mathbf{z}) \in \partial \mathbf{H}^2_{\mathbb{C}}$;
3. $\mathbf{z}$ is positive, namely $\langle \mathbf{z}, \mathbf{z} \rangle > \mathbf{0}$. In this case L_A and L_B are ultraparallel, that is they are disjoint and have a common orthogonal complex geodesic, which is polar to $\mathbf{z}$.

3. Sporadic triangle groups and Thompson triangle groups

In this section we review sporadic triangle groups (Section 3.1) and Thompson triangle groups (Section 3.2), which we will mainly study in Section 4. For these two kinds of complex hyperbolic triangle groups, we refer for instance to [5, 10, 15] for the details.

3.1. Equilateral triangle groups. Recall from the introduction that an equilateral triangle group can be generated by a complex reflections R_1 and a complex hyperbolic isometry J of order 3. Let

$$R_2 = JR_1J^{-1}, \quad R_3 = JR_2J^{-1}$$

The equilateral triangle groups then can be parameterised by the order p of generators and the complex parameter

$$\tau = \mathrm{tr}(R_1 J).$$

It is difficult to give the conditions of p with τ so that the equilateral triangle group is a lattice, or at least discrete. However, the pairwise product of generators should be non-loxodromic (see [13]). This shows that there are two continuous families satisfying that R_1J and R_1R_2 are elliptic

$$\tau = -e^{i\psi/3}, \qquad \tau = e^{i\psi/6} \cdot 2\cos(\psi/2),$$

where ψ are rational multiples of π. These two families correspond to Mostow groups or certain subgroups of Mostow groups. For such groups, the list of lattices can be obtained from the work of Deligne-Mostow (see [9, 10]). There are still lattice candidates not lying on these two families. In [5] the authors show that the equilateral triangle groups for some values of $\tau = \mathrm{tr}(R_1 J)$ indeed contain lattices, of which the explicit values of τ and p are in Table 1. They are called *sporadic triangle groups*. We denote the corresponding group by $\mathcal{S}(p,\tau)$.

Note that the list here is given up to complex conjugation and multiplication by a cube root of unity. In Section 4, we will give the analysis on $\mathbb{C}$-Fuchsian subgroups of complex hyperbolic lattices $\mathcal{S}(p,\tau)$ for $\tau = \tau_1,\ \tau_2,\ \tau_4$.

τ	p
$\tau_1 = -1 + i\sqrt{2}$	$3, 4, 6$
$\tau_2 = -(1 + i\sqrt{7})/2$	$3, 4, 5, 6, 8, 12$
$\tau_3 = e^{-\pi i/9}(-e^{-2\pi i/3} - (1-\sqrt{5})/2)$	$2, 3, 4$
$\tau_4 = (1+\sqrt{5})/2$	$3, 4, 5, 10$

TABLE 1. Values of p, τ such that $\mathcal{S}(p,\tau)$ are lattices.

3.2. Non-equilateral triangle groups. In this section, we review notation for the non-equilateral triangle groups which come from Thompson's thesis [15]. They can be parameterised by a triple of complex numbers ρ, σ, τ. The three numbers will be all equal to τ as above in the case of equilateral triangle. In the same fashion, we assume that the generators are of order p, $u = e^{2\pi i/3p}$ and the Hermitian form is

$$\mathrm{H} = \begin{pmatrix} \alpha & \beta_1 & \bar{\beta}_3 \\ \bar{\beta}_1 & \alpha & \beta_2 \\ \beta_3 & \bar{\beta}_2 & \alpha \end{pmatrix},$$

where $\alpha = 2 - u^3 - \bar{u}^3$, $\beta_1 = (\bar{u}^2 - u)\rho$, $\beta_2 = (\bar{u}^2 - u)\sigma$, $\beta_3 = (\bar{u}^2 - u)\tau$ and

$$\rho = (u^2 - \bar{u})\frac{\langle \mathbf{n_2}, \mathbf{n_1}\rangle}{\|\mathbf{n_2}\|\|\mathbf{n_1}\|}, \quad \sigma = (u^2 - \bar{u})\frac{\langle \mathbf{n_3}, \mathbf{n_2}\rangle}{\|\mathbf{n_3}\|\|\mathbf{n_2}\|}, \quad \tau = (u^2 - \bar{u})\frac{\langle \mathbf{n_1}, \mathbf{n_3}\rangle}{\|\mathbf{n_1}\|\|\mathbf{n_3}\|}.$$

The generators which preserve the above Hermitian form H are given by
(3.2)

$$R_1 = \begin{pmatrix} u^2 & \rho & -u\bar{\tau} \\ 0 & \bar{u} & 0 \\ 0 & 0 & \bar{u} \end{pmatrix}, \quad R_2 = \begin{pmatrix} \bar{u} & 0 & 0 \\ -u\bar{\rho} & u^2 & \sigma \\ 0 & 0 & \bar{u} \end{pmatrix}, \quad R_3 = \begin{pmatrix} \bar{u} & 0 & 0 \\ 0 & \bar{u} & 0 \\ \tau & -u\bar{\sigma} & u^2 \end{pmatrix}.$$

The elements R_1, R_2, R_3 are determined up to conjugacy by $|\rho|, |\sigma|, |\tau|$ and $\arg(\rho\sigma\tau)$, see [4, 10]. Suppose that the order of 23, 31, 12 and $1\bar{3}23$ are a, b, c, d respectively.

	a	b	c	d	$o(123)$	ρ	σ	τ	lattices for p
S_2	3	3	4	5	5	$1 + \frac{1+\sqrt{5}}{2}e^{2\pi i/3}$	1	1	$3, 4, 5$
E_2	3	4	4	4	6	$\sqrt{2}$	$e^{-2\pi i/3}$	$\sqrt{2}$	$3, 4, 6, 12$
H_1	3	3	4	7	42	$\frac{-1+i\sqrt{7}}{2}$	$e^{-4\pi i/7}$	$e^{-4\pi i/7}$	$2, -7$
H_2	3	3	5	5	15	$-1 - e^{-2\pi i/5}$	$e^{4\pi i/5}$	$e^{4\pi i/5}$	$2, 3, 5, 10, -5$

TABLE 2. Lists of parameters of some lattices in Thompson triangle groups. The negative values of p correspond to the conjugate values of parameters of Thompson triangle groups.

We write $(a, b, c; d)$ for the groups generated by complex reflections in a triangle with angles $\pi/a, \pi/b, \pi/c$ satisfying that the order of $1\bar{3}23$ is d. Note that here a, b, $c \geq 3$ because the $(2, b, c)$ triangle groups are rigid in $\mathrm{PU}(2,1)$. In Table 2, we list only some values of ρ, σ, τ which correspond to lattices. For the construction of the fundamental domain of these lattices, we refer to [5] for further details. We give the explicit structures of the ℂ-Fuchsian subgroups stabilising the complex geodesic L_1 in Thompson triangle groups S_2 and E_2 after Remark 4.4.

It is plausible to consider that one could also identify the ℂ-Fuchsian subgroups for $\mathcal{S}(p, \tau_3)$ $(p = 2, 3, 4)$, H_1, H_2; however, it has not been achieved by our present method. The main difficulty is to find an appropriate polygon and the transformations which pair the sides of the polygons lying the complex geodesics under consideration.

4. ℂ-Fuchsian subgroups and their explicit Fundamental domains

Let us firstly recall the Poincaré polygon theorem in hyperbolic plane (see [1]), which is the tool for us to elaborate the structure of certain ℂ-Fuchsian subgroups in complex hyperbolic triangle lattices, then give the proof of Theorem 1.1.

Theorem 4.1. *Let D be a polygon in the hyperbolic plane satisfying the following conditions and denote $D \cup \partial D$ by $\bar{D}$.*
(i) *For each side s of D, there is a side s' and an element g_s (of the isometries of the hyperbolic plane) such that $g_s(s) = s'$, we call each g_s the side pairing transformation.*
(ii) $g_{s'} = g_s^{-1}$. *Observe that if there is a side s, with $s' = s$, then it implies that $g_s^2 = Id$. If this occurs, the relation $g_s^2 = Id$ is called a reflection relation. Now let G be the group generated by the g_s's.*
(iii) $g_s(D) \cap D = \varnothing$.
(iv) *For each vertex x of D, there are vertices $x_0(= x), x_1, \cdots, x_n$ of D and elements $f_0(= Id), f_1, \cdots, f_n$ of G such that the sets $f_j(N_j)$ $(N_j = \{y \in \bar{D} : d(y, x_j) < \epsilon\})$ are non-overlapping sets whose union is $B(x, \epsilon)$ (the ball centered at x with radius ϵ) and such that each f_{j+1} is of the form $f_j g_s$ for some s $(j = 1, \cdots, n; f_{n+1} = Id)$.*
(v) *The ϵ in the above condition can be chosen independently of x in $\bar{D}$.*
Then the group G generated by the side pairing transformations is discrete, and D is a fundamental polygon for G.

Before we give the proof of Theorem 1.1, we state two propositions for giving the whole presentation of the Fuchsian groups below.

Proposition 4.2. *Suppose that a hyperbolic triangle have sides a, b and c and opposite angles α, β, and γ. Then the following formula holds*

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma. \tag{4.3}$$

For the details, see Section 7.12 in [1].

Proposition 4.3 (Propostion 2.5 of [5]). *Suppose that A and B are complex reflections of order p $(p \geq 2)$. If $\mathrm{br}(A, B) = q$ for some integer $q > 1$, then:*

1. *if q is odd, then the center of $\langle A, B\rangle$ is $(AB)^q$ which is a complex reflection with rotation angle $\frac{|(q-2)p-2q|}{p}\pi$;*
2. *if q is even, then the center of $\langle A, B\rangle$ is $(AB)^{q/2}$ which is a complex reflection with rotation angle $\frac{|(q-2)p-2q|}{2p}\pi$.*

In particular, the mirror of $(AB)^q$ (respectively $(AB)^{q/2}$) is a complex geodesic orthogonal to the mirror of A, and so no power of AB equals A.

On the other hand, suppose that A and B are complex reflections of order p ($p \geq 2$) and $\mathrm{br}(A, B) = 1$. If there exist integers $1 \leq m < \mathrm{ord}(A)$ and $2 \leq n < \mathrm{ord}(B)$ for which $B^n = A^{\pm m}$, then B^n commutes with all elements stabilising the mirror L_A of A. In fact, for any element g which fixes the complex geodesic L_A, the action of $A^{\pm m} g A^{\mp m}$ on L_A is the same with g. Therefore, one can get that $B^n g B^{-n} = g$ which yields to $B^n g = g B^n$.

Proof of Theorem 1.1. Suppose that $\mathbf{n}_i$ is the polar vector of R_i ($i = 1, 2, 3$) and $u = e^{i\phi/3} = e^{2ip/3}$. By the trace formula of $\mathrm{tr}(R_1 J)$ in [9], we may write τ as

$$\tau = \mathrm{tr}(R_1 J) = (u^2 - \bar{u}) \frac{\langle \mathbf{n_{j+1}}, \mathbf{n_j} \rangle}{\|\mathbf{n_{j+1}}\| \|\mathbf{n_j}\|}.$$

We normalise $\mathbf{n}_i$ so that $\langle \mathbf{n}_i, \mathbf{n}_i \rangle = 2 - u^3 - \bar{u}^3$. Then one can get that $\langle \mathbf{n}_{i+1}, \mathbf{n}_i \rangle = (\bar{u}^2 - u)\tau$. We now choose the polar vectors $\mathbf{n_i}$ of the complex geodesics R_i ($i = 1, 2, 3$) to be the normal basis of $\mathbb{C}^3$, i.e.,

$$\mathbf{n}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{n}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{n}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.4}$$

Therefore the corresponding matrix representation of complex hyperbolic isometry J and the Hermitian form H are given respectively by

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathrm{H} = \begin{pmatrix} \alpha & \beta & \bar{\beta} \\ \bar{\beta} & \alpha & \beta \\ \beta & \bar{\beta} & \alpha \end{pmatrix}. \tag{4.5}$$

where $\alpha = 2 - u^3 - \bar{u}^3$, $\beta = (\bar{u}^2 - u)\tau$. Then the Hermitian form is of signature $(2, 1)$ if and only if

$$\det(\mathrm{H}) = \alpha^3 + 2\mathrm{Re}(\beta^3) - 3\alpha|\beta|^2 < 0.$$

All the lattices we will consider below satisfy the above inequality. We can get the matrix representation of R_1 in $\mathrm{SU}(2, 1)$ by the formula (2.1)

$$R_1 = \begin{pmatrix} u^2 & \tau & -u\bar{\tau} \\ 0 & \bar{u} & 0 \\ 0 & 0 & \bar{u} \end{pmatrix}. \tag{4.6}$$

Correspondingly, one can get the matrix forms of R_2, R_3 by the relations

$$R_2 = J R_1 J^{-1}, \; R_3 = J R_2 J^{-1}.$$

Let $\mathbf{v_i}$ ($i = 1, 2, 3$) denotes a lift of the three vertices of the triangle, i.e., $\mathbf{v}_i = \mathbf{n}_{i+1} \boxtimes \mathbf{n}_{i+2}$. A direct computation yields

$$\mathbf{v}_1 = \begin{bmatrix} \alpha^2 - |\beta|^2 \\ \beta^2 - \alpha\bar{\beta} \\ -\alpha\beta + \bar{\beta}^2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -\alpha\beta + \bar{\beta}^2 \\ \alpha^2 - |\beta|^2 \\ \beta^2 - \alpha\bar{\beta} \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} \beta^2 - \alpha\bar{\beta} \\ -\alpha\beta + \bar{\beta}^2 \\ \alpha^2 - |\beta|^2 \end{bmatrix}.$$

In what follows, we investigate the subgroups (which fix complex geodesic L_1) of triangle lattices $\mathcal{S}(p, \tau)$ (see Table 1) for τ_1, τ_2, τ_4. We refer to Appendix in [5] for the details of the explicit presentations of triangle lattices $\mathcal{S}(p,\tau)$ and the combinatorial invariant shells. We wish to emphasize that the invariant shells are the side representatives of the fundamental domains for the complex hyperbolic lattices.

(i) $\tau = -1 + i\sqrt{2}$.

The triangle lattice Γ is generated by R_1, R_2, R_3, J, explicitly

(4.7)
$$\langle R_1, R_2, R_3, J : R_1^p, J^3, (R_1J)^8, R_3 = JR_2J^{-1} = J^{-1}R_1J, (R_1R_2)^{|\frac{3p}{p-3}|}, br_3(R_1, R_2R_3R_2R_3^{-1}R_2^{-1}),$$
$$br_6(R_1,R_2), br_4(R_1, R_2R_3R_2^{-1}), (R_1R_2R_3R_2^{-1})^{|\frac{4p}{p-4}|}, br_3(R_1, R_3^{-1}R_2^{-1}R_3R_2R_3)\rangle$$

Throughout the paper, relations involving infinite exponents shall be removed from the presentation. In the form of a list of side representatives of Γ's fundamental domain, the rough structure of the invariant shells is given by

(4.8)
$$[6]\,1;2,3;\ [4]\,2;1,23\bar{2};\ [3]\,23\bar{2};1,2323\bar{3}\bar{2};\ [3]\,2323\bar{3}\bar{2};1,\bar{3}\bar{2}323,$$

where $[k]\,a;b,c$ denotes a k-gon pyramid with base L_a (which is fixed by element a).

In Figure 1, we give a rough picture of $[6]\,1;2,3$, where each vertex $\mathbf{z}_i$ is the intersection point of the lateral edge with the base edge L_1, therefore usually the formula of the vertices can be written in this form: $\mathbf{z}_1 = \mathbf{n}_1 \boxtimes \mathbf{n}_2$, $\mathbf{z}_2 = \mathbf{n}_1 \boxtimes R_2(\mathbf{n}_3)$ and so on. However, one should note that the form of each vertex of such a pyramid depends on p; for example, the vertex $\mathbf{z}_2$ (the intersection point of $R_2(L_3)$ with L_1) will be slightly changed when $p = 6$. One can check that $\mathbf{n}_1 \boxtimes R_2(\mathbf{n}_3)$ is a positive vector which is also a polar vector of the common perpendicular complex geodesic $L_{123\bar{2}}$ to L_1 and $R_2(L_3)$. Actually, the point $\mathbf{z}_2$ will be $\mathbf{n}_1 \boxtimes (\mathbf{n}_1 \boxtimes 2(\mathbf{n}_3))$.

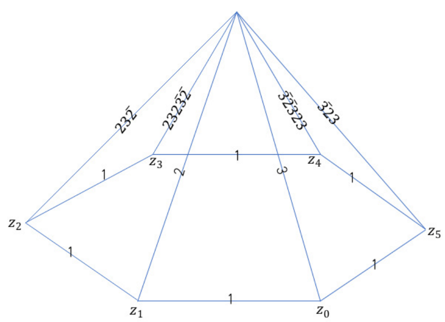

FIGURE 1. Pyramid corresponding to $[6]\,1;2,3$ with the base L_1 fixed by the complex reflection R_1. Note that $\bar{3}\bar{2}\bar{3}2323 = 2323\bar{3}\bar{2}$ is a consequence of the braid relation $br_6(R_1,R_2)$.

We take elements of the triangle lattice such that each of the four shells in (4.8) to a pyramid with the same base L_1. We leave $[6]\,1;2,3$ invariant and do minor surgeries to the other three pyramids such that each of them has base L_1. Firstly, we consider the action of the element J^{-1} on the pyramid $[4]\,2;1,23\bar{2}$ with base L_2. Then one can get a new pyramid with base $J^{-1}(L_2)$ fixed by $J^{-1}2J$. The new pyramid is identified with $[4]\,J^{-1}2J; J^{-1}1J, J^{-1}23\bar{2}J$ which can be written as $[4]\,1;3,12\bar{1}$ due to $R_{j+1} = JR_jJ^{-1}$.

Similarly, we deform the other two pyramids in (4.8) to be with the same base L_1 :

$$[4]\,2;1,23\bar{2} \xrightarrow{J^{-1}} [4]\,1;3,12\bar{1}$$
$$[3]\,23\bar{2};1,232\bar{3}\bar{2} \xrightarrow{J} [3]\,31\bar{3};2,313\bar{1}\bar{3} \xrightarrow{\bar{3}} [3]\,1;\bar{3}23,13\bar{1}$$
$$[3]\,232\bar{3}\bar{2};1,\bar{3}\bar{2}323 \xrightarrow{J^{-1}} [3]\,121\bar{2}\bar{1};3,\bar{2}\bar{1}212 \xrightarrow{\bar{2}\bar{1}} [3]\,1;\bar{2}\bar{1}312,12\bar{1}.$$

Along the same base L_1, we paste the four pyramids

$$[6]\,1;2,3;\,[4]\,1;3,12\bar{1};\,[3]\,1;\bar{3}23,13\bar{1};\,[3]\,1;\bar{2}\bar{1}312,12\bar{1}, \tag{4.9}$$

and mainly focus on the obtained decagon which lies in the closure of complex geodesic L_1 (homeomorphic to $\overline{\mathbf{H}^1_{\mathbb{C}}}$). Its vertices are the intersection points of complex geodesic L_1 with other complex geodesics. Generally, the decagon F (see Figure 2) has vertices $x_j = \mathbf{P}(\mathbf{n}_1 \boxtimes \mathbf{a}_j)$ $(j = 0, 1, \cdots, 9)$ where

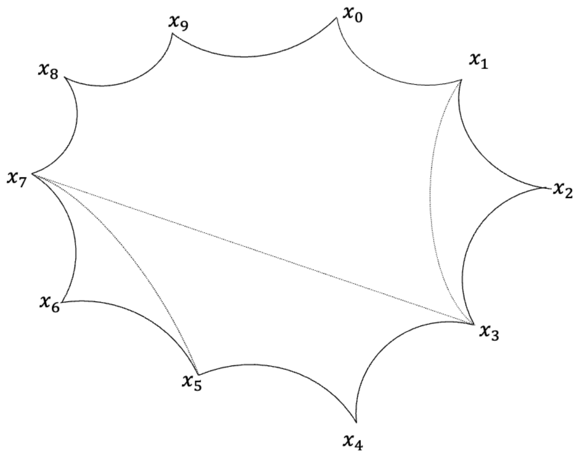

FIGURE 2. The decagon F

$$\mathbf{a}_0 = \bar{3}\bar{2}\mathbf{n}_3,\ \mathbf{a}_1 = \bar{3}\mathbf{n}_2,\quad \mathbf{a}_2 = \bar{3}231\mathbf{n}_3,\ \mathbf{a}_3 = \mathbf{n}_3,\ \mathbf{a}_4 = 31\mathbf{n}_2,$$
$$\mathbf{a}_5 = \bar{2}\bar{1}\mathbf{n}_3,\ \mathbf{a}_6 = \bar{2}\bar{1}3121\mathbf{n}_2,\ \mathbf{a}_7 = \mathbf{n}_2,\ \mathbf{a}_8 = 2\mathbf{n}_3,\ \mathbf{a}_9 = 23\mathbf{n}_2.$$

Here x_0 just denotes the intersection point of L_1 with $R_3^{-1}R_2^{-1}(L_3)$ (the point fixed by the complex reflection $\bar{3}\bar{2}323$), i.e., x_0 is the fixed point of $1\bar{3}\bar{2}323$. Just as we stated previously, one should note that the formulas of the vertices above depend on p. For example, the formula for the point x_7 (the intersection point of L_1 with L_2) will be $\mathbf{P}(\mathbf{n}_1 \boxtimes (\mathbf{n}_1 \boxtimes \mathbf{n}_2))$ when $p = 4$, which lies in the complex geodesic L_1 and is fixed by 12.

From the combinatorics of the four pyramids (4.9), we know that the decagon composes of a hexagon P_1 with vertices x_0, x_1, x_3, x_7, x_8, x_9, a quadrilateral P_2 with vertices x_3, x_4, x_5, x_7, a triangle P_3 with vertices x_1, x_2, x_3 and a triangle P_4 with vertices x_5, x_6, x_7, i.e., it comprises ten sides $l_1, l_2, \cdots l_{10}$, where

$$l_i = \mathbf{P}(\mathrm{Span}_{\mathbb{C}}\{x_{i-1}, x_i\}) \cap L_1. \tag{4.10}$$

In order to find the explicit structure of the Fuchsian group stabilising L_1, we start from the side pairing transformations for the decagon. One can separately consider the transformations which convert vertices from the construction of above four polygons. Note that the element $\bar{3}23$ transfers the complex geodesic $L_{\bar{3}\bar{2}323}$ fixed by $\bar{3}\bar{2}323$ to the complex geodesic L_3 fixed by R_3 when focusing on the hexagon P_1. Also, the element $\bar{3}23$ transfers the complex geodesic L_3 to the complex geodesic $L_{\bar{3}2313\bar{1}\bar{3}\bar{2}3}$ when focusing on the triangle P_3. Then we consider the element $(1\bar{3}23)^2$ which fixes the vertex x_1 and obtain that $(1\bar{3}23)^2(l_1) = l_2$. In the same manner, one can get that $(13)^3$ fixes x_3 and maps l_3 to l_4; $(12)^3$ fixes x_7 and maps l_7 to l_8. Now, let $g_1 = (1\bar{3}23)^2$, $g_2 = (13)^3$, $g_3 = (12)^3$. One can know that

$$g_1(l_1) = l_2,\ g_2(l_3) = l_4,\ g_3(l_7) = l_8.$$

The difficulty here is pairing the remaining four sides l_5, l_6, l_9, l_{10}. We have that $(12)^3(x_6) = x_8$, and pay attention to the stabiliser $123\bar{2}$ of x_8. Denote $(123\bar{2})^2(12)^3$ by g_4, one can immediately get that $g_4(x_6) = x_8$. We claim that $g_4(x_5) = x_9$. Note that $(123)^3 = 1J$, because the order of $1J$ is 8 and $(1J)^3 = 123$. Using $\mathrm{br}_6(1, 2)$, we have

$$\begin{aligned}
&\bar{1}^2(123\bar{2})^2(12)^3(\mathbf{a}_5)\\
=&\bar{1}^2(123\bar{2})^2(212121)\bar{1}(\bar{2}\bar{1}\,\mathbf{n}_3)\\
=&\bar{1}23\bar{2}123123\,\mathbf{n}_3\\
=&\bar{1}23\bar{2}\bar{3}\bar{2}J\,\mathbf{n}_3\\
=&\bar{1}23\bar{2}\bar{3}\bar{2}\,\mathbf{n}_1.
\end{aligned}$$

Due to $\mathrm{br}_3(1, 232\bar{3}\bar{2})$, we see at once that $\bar{1}232\bar{3}\bar{2}1232\bar{3}\bar{2}1 = 232\bar{3}\bar{2}$, which means $\bar{1}23\bar{2}\bar{3}\bar{2}\,\mathbf{n}_1 = 23\,\mathbf{n}_2 = \mathbf{a}_9$. Therefore, we obtain that $g_4(l_6) = l_9$.

Let $g_5 = (1232\bar{3}\bar{2})^3 \circ g_4 = (1232\bar{3}\bar{2})^3(123\bar{2})^2(12)^3$. It follows immediately that $g_5(x_5) = x_9$, because $g_4(x_5) = x_9$ and $1232\bar{3}\bar{2}$ fixes x_9. We claim that $g_5(x_4) = x_0$ by checking $\bar{1}^3 g_5(\bar{1}\mathbf{a}_4) = 1\mathbf{a}_0$. Considering the braid relations in the group presentation (4.7), we have

$$\begin{aligned}
&\bar{1}^3(1232\bar{3}\bar{2})^3(123\bar{2})^2(12)^3(\bar{1}\mathbf{a}_4)\\
=&\bar{1}^3(1232\bar{3}\bar{2})^3(23\bar{2}1)^2(21)^3(\bar{1}31\mathbf{n}_2)\\
=&\bar{1}232\bar{3}\bar{2}1(123)^2\bar{2}(12)^3(31\mathbf{n}_2)\\
=&\bar{1}232\bar{3}\bar{2}1(\bar{3}\bar{2}J)\bar{2}(21)^3(31\mathbf{n}_2)\\
=&\bar{1}232\bar{3}\bar{2}1\bar{3}\bar{2}J1212131\mathbf{n}_2\\
=&\bar{1}(232\bar{3}\bar{2}1232\bar{3}\bar{2})2312J\mathbf{n}_2\\
=&232\bar{3}\bar{2}(123)^2\mathbf{n}_3\\
=&232\bar{3}\bar{2}\bar{3}\bar{2}J\mathbf{n}_3\\
=&\bar{3}\bar{2}\bar{3}(23)^3(\bar{3}\bar{2})^2\mathbf{n}_1\\
=&\bar{3}\bar{2}\bar{3}23\mathbf{n}_1.
\end{aligned}$$

It is easy to know that $\bar{3}\bar{2}\bar{3}23\mathbf{n}_1 = 1\bar{3}\bar{2}\mathbf{n}_3 = 1\mathbf{a}_0$, because of $(\bar{3}\bar{2}\bar{3}23)1(\bar{3}\bar{2}323) = 1\bar{3}\bar{2}323\bar{1}$ from $\mathrm{br}_3(1, \bar{3}\bar{2}323)$. Therefore $g_5(x_4) = x_0$, i.e., $g_5(l_5) = l_{10}$.

We consider the ℂ-Fuchsian subgroup Γ_0 generated by g_1, g_2, g_3, g_4, g_5, and only need to check the local tiling condition near every vertex of the decagon F in Figure 2 to show that it is indeed the fundamental domain of the ℂ-Fuchsian subgroup Γ_0.

It is sufficient to consider the three cycles: $\{x_0,\ x_2,\ x_4\}$, $\{x_5,\ x_9\}$, $\{x_6,\ x_8\}$. We would like to take the vertex x_8 when $p=3$ for example. It is easily seen that the order of the stabiliser $g_3\circ g_4^{-1}$ in Γ_0 of the vertex x_8 is 6 by using Proposition 4.3. Because the invariant shells (4.8) are the side representatives of the fundamental domain of the complex hyperbolic lattice Γ, the domains $(g_3\circ g_4^{-1})^m(F)$ do not intersect with each other for $m=1,\ 2,\ 3,\ 4,\ 5$. Let θ_1, θ_2 be the internal angles of x_6, x_8 respectively. Using the Cosine Rule (4.3) for the triangle with vertices $g_3(x_5)$, x_8, x_7 lying in the complex geodesic L_1 (an embedded copy of $\mathbf{H}^1_{\mathbb{C}}$), one can get that $\theta_1+\theta_2$ is exactly $2\pi/6$. We conclude, therefore, that the local tiling condition is satisfied for the vertex x_8 when $p=3$. In the same way, one can check the local tiling condition of all vertices for $p=3,\ 4,\ 6$ by considering the relation of the sum of angles at all elliptic vertices belonging to an elliptic cycle with the order of that cycle. In particular, by computing the the angle of the elliptic cycle, we could get the orders of the three cycle transformations at x_0, x_5, x_8 respectively. Finally we obtain that the decagon F is the fundamental domain of the $\mathbb{C}$-Fuchsian subgroup Γ_0 by Theorem 4.1.

From the presentation (4.7) of the triangle lattice. We see at once that both g_2 and g_3 is of order $|\frac{p}{p-3}|$. Recalling the element $P^2=1J1J$, we have

$$P^2 1 P^{-2} = 1J1J1J^{-1}\bar{1}J^{-1}\bar{1} = 1(23\bar{2})\bar{1},$$
$$P^2(\bar{3}23)P^{-2} = 1J1J\bar{3}23J^{-1}\bar{1}J^{-1}\bar{1} = 1,$$

which yields that the order of $g_1=(1\bar{3}23)^2$ is $|\frac{2p}{p-4}|$. One can obtain the normal representations of the $\mathbb{C}$-Fuchsian subgroup Γ_0 for $p=3,4,6$:

- $p=3,4,$

$$\langle g_1,g_2,g_3,g_4,g_5 : g_1^{|\frac{2p}{p-4}|},\ g_2^{|\frac{p}{p-3}|},\ g_3^{|\frac{p}{p-3}|}, (g_5\circ g_2\circ g_1)^{|\frac{2p}{p-2}|},\ (g_5\circ g_4^{-1})^{|\frac{2p}{p-6}|},\ (g_4\circ g_3^{-1})^{|\frac{2p}{p-4}|}\rangle;$$

- $p=6,$

$$\langle g_1,g_2,g_3,g_4,g_5 : g_1^6,\ g_2^2,\ g_3^2,\ (g_4^{-1}\circ g_3)^6\rangle.$$

where $g_5\circ g_2\circ g_1$ and $g_5\circ g_4^{-1}$ are ellipto-parabolic element. In this case x_0, x_5 lie in the boundary of the disk.

(ii) $\tau=-\frac{1+i\sqrt{7}}{2}$.

The triangle lattice Γ is generated by R_1, R_2, R_3, J, explicitly

$$\langle R_1,R_2,R_3,J : R_1^p, J^3, (R_1J)^7, R_3=JR_2J^{-1}=J^{-1}R_1J, (R_1R_2)^{|\frac{4p}{p-4}|}, br_4(R_1,R_2), (R_1R_2R_3R_2^{-1})^{|\frac{6p}{p-6}|}, br_3(R_1,R_2R_3R_2^{-1})\rangle \tag{4.11}$$

The rough structure of the invariant shell is given by

$$[4]\,1;\ 2,\ 3;\quad [3]\,2;\ 1,\ 23\bar{2}.$$

The element J^{-1} maps the shell $[3]\,2;\ 1,\ 23\bar{2}$ to $[3]\,1;\ 3,\ 12\bar{1}$. Pasting the two shells

$$[4]\,1;\ 2,\ 3;\quad [3]\,1;\ 3,\ 12\bar{1} \tag{4.12}$$

along the bases in L_1, we get a pentagon (see Figure 3) with vertices $x_j=\mathbf{P}(\mathbf{n}_1\boxtimes\mathbf{a}_j)$ where

$$\mathbf{a}_1=2\mathbf{n}_3,\quad \mathbf{a}_2=\bar{3}\mathbf{n}_2,\quad \mathbf{a}_3=\mathbf{n}_3,\quad \mathbf{a}_4=31\mathbf{n}_2,\quad \mathbf{a}_5=\mathbf{n}_2.$$

The pentagon composes of a tetragon P_1 with vertices x_1,x_2,x_3,x_5 and a triangle P_2 with vertices x_3,x_4,x_5.

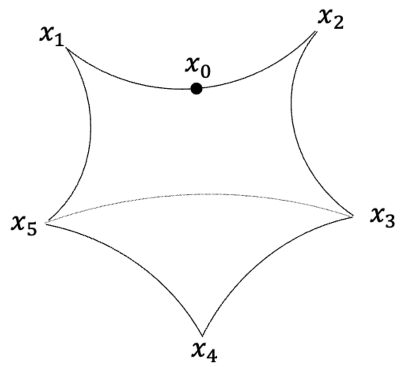

FIGURE 3. The polygon F

We restrict to the singular point x_0 fixed by $23\bar{2}P^2$ (where $P = R_1J$). A direct computation yields that x_0 lies in the geodesic spanned by x_1, x_2 and $23\bar{2}P^2$ maps x_1 to x_2. Indeed, using $(23\bar{2}121)2(\bar{1}\bar{2}\bar{1}23\bar{2}) = (23)2(\bar{3}\bar{2})$, we have

$$23\bar{2}P^2(\mathbf{a}_1) = 23\bar{2}1J1J(2\mathbf{n}_3) = 23\bar{2}121\mathbf{n}_2 = 23\mathbf{n}_2 = \bar{3}\mathbf{n}_2 = \mathbf{a}_2.$$

Thence we consider a hexagon comprising the following sides

$$l_i = \begin{cases} \mathbf{P}(\mathrm{Span}_{\mathbb{C}}\{x_i, x_0\}) \cap L_1, & i = 1, 2. \\ \mathbf{P}(\mathrm{Span}_{\mathbb{C}}\{x_{i-1}, x_i\}) \cap L_1, & i = 3, 4, 5, 6. \end{cases} \tag{4.13}$$

By the composition of this hexagon, one could check that the second power of the element 12 maps x_4 to x_1. Then we get that $(12)^2(l_5) = l_6$ because of $(12)^2(x_4) = x_1$ and $(12)^2(x_5) = x_5$. Similarly, it is easy to check that $(13)^2(l_3) = l_4$. Furthermore, it follows from $23\bar{2}P^2(x_0) = x_0$, $23\bar{2}P^2(x_1) = x_2$ that $23\bar{2}P^2(l_1) = l_2$. Now we have paired the sides of the pentagon drawn above. For each vertex of the cycle $\{x_1, x_2, x_4\}$, one can verify that it satisfies local tiling condition by considering the sum of angle of all vertices and the order of its stabiliser.

By Theorem 4.1, we obtain the fundamental domain in L_1 (which is the hexagon F in Figure 3) of the $\mathbb{C}$-Fuchsian subgroup generated by g_1, g_2, g_3, where

$$g_1 = (12)^2, \quad g_2 = 23\bar{2}P^2, \quad g_3 = (13)^2.$$

We are now in a position to show the presentation of this $\mathbb{C}$-Fuchsian subgroup for all values of p. Because the order of $1J$ is 7, it follows from $P = 1J$ and $(1J)^3 = 123$ that $P^2 = P^{-5} = P^{-2}\bar{3}\bar{2}\bar{1} = J^{-1}\bar{1}J^{-1}\bar{1}\bar{3}\bar{2}\bar{1}$. We have that

$$g_2^2 = (23\bar{2}1J1J)(23\bar{2}J^{-1}\bar{1}J^{-1}\bar{1}\bar{3}\bar{2}\bar{1}) = 23\bar{2}1212\bar{1}\bar{2}\bar{1}\bar{3}\bar{2}\bar{1} = \bar{1},$$

which shows that the order of g_2 is $2p$ and g_1 commutes with g_1 and g_3 by recalling the statement after Proposition 4.3. From the braid relation $\mathrm{br}_4(1,2)$, one can easily see that $\mathrm{br}_4(3,1)$. Then we get that the order of g_3 is $|\frac{2p}{p-4}|$ because the elements 12, 23, and 31 have the same order $|\frac{4p}{p-4}|$. What is left is to consider the order of the elliptic cycle. Note that $P^2 = (1J)^{-12} = (123)^{-4} = (\bar{3}\bar{2}\bar{1})^4$. Using the relations in the presentation (4.11), we have

$$\begin{aligned} g_1\circ g_3\circ g_2 =& 1^2(12)^3\bar{1}^2(13)^2(23\bar{2}P^2)\\ =& 1^3212\bar{1}31(323\bar{2}\bar{3}\bar{2})\bar{1}(\bar{3}\bar{2}\bar{1})^3\\ =& 1^3212\bar{1}31\bar{2}(3\bar{1}\bar{3})\bar{2}\bar{1}(\bar{3}1\bar{1}\bar{2}\bar{1}\bar{3}\bar{2}\bar{1})\\ =& 1^32(12\bar{1}31\bar{2}\bar{1}\bar{3})(\bar{1}31\bar{2}\bar{1}\bar{3}1)\bar{1}\bar{2}\bar{1}\bar{3}\bar{2}\bar{1}\\ =& 1^32\bar{3}(12\bar{1}\bar{2}\bar{1})\bar{3}(12\bar{1}\bar{2}\bar{1})\bar{3}\bar{2}\bar{1}\\ =& 1^3(2\bar{3}\bar{2}\bar{1})^3, \end{aligned}$$

which indicates the order of $g_1\circ g_3\circ g_2$ is $|\frac{2p}{p-6}|$, since the order of $123\bar{2}$ is $|\frac{6p}{p-6}|$ and $1^3(2\bar{3}\bar{2}\bar{1})^3=(2\bar{3}\bar{2}\bar{1})^3 1^3$. We get the following presentation of the $\mathbb{C}$-Fuchsian subgroup for $\tau=-\frac{1+i\sqrt{7}}{2}$:

$$\langle g_1,\, g_2,\, g_3\,:\, g_1^{|\frac{2p}{p-4}|},\, g_2^{2p},\, g_3^{|\frac{2p}{p-4}|},\, (g_1\circ g_3\circ g_2)^{|\frac{2p}{p-6}|},\, [g_1,g_2^2],\, [g_3,g_2^2]\rangle.$$

(iii) $\tau=\frac{1+\sqrt{5}}{2}$.

The triangle lattice Γ is generated by R_1,R_2,R_3,J, explicitly

$$\begin{aligned} \langle R_1,R_2,R_3,J: R_1^p,\, J^3,\, (R_1J)^5,\, R_3=JR_2J^{-1}=J^{-1}R_1J,\, \mathrm{br}_5(R_1,R_2),\\ (R_1R_2)^{|\frac{10p}{3p-10}|},\, \mathrm{br}_3(R_1,R_2R_3R_2^{-1}),\, (R_1R_2R_3R_2^{-1})^{|\frac{6p}{p-6}|}\rangle. \end{aligned}$$

We consider the combinatorics of the fundamental domain for this triangle lattice which comprises the following two invariant shells

$$[5]\,1;\,2,\,3;\quad [3]\,2;\,1,\,23\bar{2}.$$

Following the process of the previous cases, we firstly map the pyramid $[3]\,2;\,1,\,23\bar{2}$ to $[3]\,1;\,3,\,12\bar{1}$ by the action of the element J^{-1}. We paste the two pyramids

$$[5]\,1;\,2,\,3;\quad [3]\,1;\,3,\,12\bar{1} \tag{4.14}$$

along the bases in L_1. Then we get a hexagon in Figure 4 with the vertices $x_j=\mathbf{P}(\mathbf{n_1}\boxtimes\mathbf{a}_j)$ $(j=0,1,\cdots 5)$, where

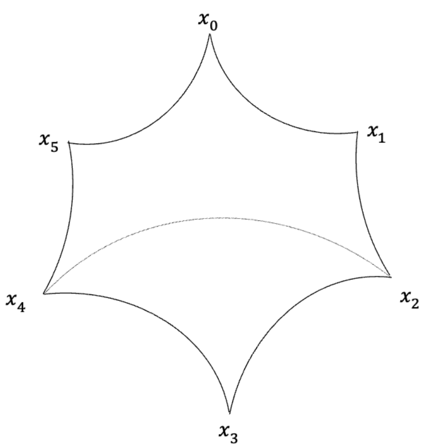

FIGURE 4. The hexagon F_2

$$\mathbf{a}_0 = \bar{3}\bar{2}\mathbf{n_3}, \quad \mathbf{a}_1 = \bar{3}\mathbf{n_2}, \quad \mathbf{a}_2 = \mathbf{n}_3, \quad \mathbf{a}_3 = 31\mathbf{n_2}, \quad \mathbf{a}_4 = \mathbf{n}_2, \quad \mathbf{a}_5 = 2\mathbf{n_3}.$$

The hexagon composes of a pentagon P_1 with vertices x_0, x_1, x_2, x_4, x_5 and a triangle P_2 with vertices x_2, x_3, x_4. We write the sides of the hexagon F_2 as follows

$$l_i = \mathbf{P}(\mathrm{Span}_{\mathbb{C}}\{x_{i-1}, x_i\}) \cap L_1, \quad i = 1, 2, \cdots, 6. \tag{4.15}$$

Let $g_1 = 1\bar{3}\bar{2}323$, $g_2 = 1312\bar{1}\bar{3}$. A trivial verification shows that

$$g_1(l_6) = l_1, \quad g_2(l_3) = l_4,$$

since $g_1(x_0) = x_0$, $g_1(x_5) = x_1$, $g_2(x_3) = x_3$, $g_2(x_2) = x_4$. About the sides l_5 and l_2, we firstly apply the element g_1 to map l_5 to a geodesic l, one of whose endpoint is x_1 because of $g_1(x_5) = x_1$. Due to the construction of the fundamental domain for the triangle lattice Γ and $1\bar{3}23$ fixes x_1, a direct calculation yields that $(1\bar{3}23)^3 \cdot g_1$ maps x_5, x_4 to x_2, x_1 respectively. Therefore, we know that $g_3(l_5) = l_2$, where $g_3 = (1\bar{3}23)^3 \circ g_1$. One could finally check the local tiling condition for the vertices of the two cycles: $\{x_1, x_5\}$, $\{x_2, x_4\}$ which follows by the same method as in the first case $\tau = -1 + i\sqrt{2}$.

We claim that for any holomorphic isometry g fixing the complex geodesic L_1, g commutes with R_1. Indeed, gR_1g^{-1} has the same action with R_1 on L_1, therefore, $gR_1g^{-1} = R_1$. i.e., $gR_1 = R_1g$. Similarly, g also commutes with R_1^{-1}. Note that $1J = (1J)^6 = 123123$ implies that $J = 23123 = 31231 = 12312$, also $(1J)^2 = (1J)^{-3} = \bar{3}\bar{2}\bar{1}$. Now we check the order of elliptic cycles at x_2 and x_5:

$$\begin{aligned}
g_3 \circ g_2 &= (1\bar{3}23)^3(1\bar{3}\bar{2}323)(1312\bar{1}\bar{3}) \\
&= 1^3(1\bar{3}231\bar{3}231\bar{3}23)(\bar{3}\bar{2}323)(312\bar{1}\bar{3}\bar{1}) \\
&= 1^5\bar{1}\bar{3}\bar{1}\bar{3}(31231)\bar{3}(23123)(31231)\bar{1}\bar{3}\bar{1}\bar{3}\bar{1} \\
&= 1^5(\bar{1}\bar{3})^2 J\bar{3}J^2(\bar{3}\bar{1}\bar{3}\bar{1}\bar{3}) \\
&= 1^5(\bar{1}\bar{3})^5, \\
g_1^{-1} \circ g_3 &= (1\bar{3}\bar{2}323)^{-1}(1\bar{3}23)^3(1\bar{3}\bar{2}323) \\
&= 1^2(\bar{3}\bar{2}\bar{3}23)\bar{1}\bar{3}231\bar{3}23123 \\
&= 1^2 23\bar{2}(\bar{3}\bar{2}\bar{1})\bar{3}231\bar{3}(23123) \\
&= 1^2 23\bar{2}1J1J^{-1}J^{-1}\bar{3}231\bar{3}J \\
&= 1(123\bar{2}1)123\bar{2} \\
&= (123\bar{2})^3,
\end{aligned}$$

which yield that the order of $g_3 \circ g_2$ is $|\frac{2p}{3p-10}|$, and the order of $g_1^{-1} \circ g_3$ is $|\frac{2p}{p-6}|$. By Theorem 4.1, we obtain the fundamental domain in L_1 (which is a hexagon) of the ℂ-Fuchsian group generated by g_1, g_2, g_3. We list the explicit presentation for all values of $p = 3, 4, 5, 10$:

$$\langle g_1,\, g_2,\, g_3 : g_1^p,\, g_2^p,\, (g_3 \circ g_2)^{|\frac{2p}{3p-10}|},\, (g_3^{-1} \circ g_1)^{|\frac{2p}{p-6}|} \rangle.$$

Now the proof is complete. □

Remark 4.4. By an almost similar method, one can also consider the structure of ℂ-Fuchsian subgroups in Thompson triangle groups for S_2 and E_2 in Table 2. Recall the matrix normalisation (3.2) for Thompson triangle groups in section 3.2. We describe these two cases roughly in what follows. We stress that calculations can be done in the same manner with the proof above and the situation is improving significantly when one uses Mathematica for example.

(i) Thompson triangle group S_2

$$\langle R_1, R_2, R_3 : R_1^p, R_2^p, R_3^p, (R_1R_2R_3)^5, \mathrm{br}_3(R_1, R_3), \mathrm{br}_3(R_2, R_3), \mathrm{br}_4(R_1, R_2), (R_1R_2)^{|\frac{4p}{p-4}|}, \mathrm{br}_5(R_1, R_2R_3R_2^{-1}), (R_1R_2R_3R_2^{-1})^{|\frac{10p}{3p-10}|}\rangle.$$

It has the following pyramids of the side representatives of its fundamental domain

$$[3]1;2,3, \quad [5]\,2;1,23\bar{2}, \quad [4]\,3;1,2, \quad [3]\,23\bar{2};1,3.$$

We force them to having the same base L_1 :

$$[5]\,2;1,23\bar{2} \xrightarrow{23} [5]\,3;231\bar{3}\bar{2},2 \xrightarrow{\bar{3}\bar{1}} [5]\,1;\bar{3}\bar{1}2313\bar{3}\bar{2}13,\bar{3}\bar{1}213$$

$$[4]\,3;1,2 \xrightarrow{\bar{3}\bar{1}} [4]\,1;\bar{3}13,\bar{3}\bar{1}213$$

$$[3]\,23\bar{2};1,3 \xrightarrow{\bar{2}} [3]\,3;\bar{2}12;\bar{2}32 \xrightarrow{\bar{3}\bar{1}} [3]\,1;\bar{3}\bar{1}\bar{2}1213,\bar{3}\bar{1}\bar{2}3213.$$

We pay attention to the nonagon F with vertices $x_j = \mathbf{P}(\mathbf{n}_1 \boxtimes \mathbf{a}_j)$ $(j = 1,\cdots,9)$ as follows:

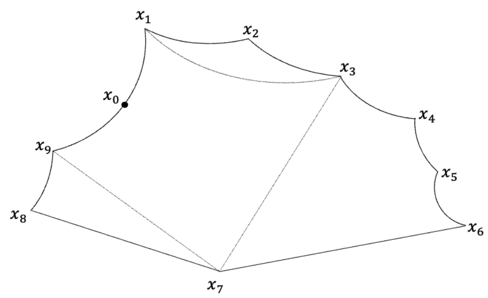

FIGURE 5. The nonagon F

$$\mathbf{a}_1 = \bar{3}\bar{1}\bar{2}\mathbf{n}_1,\ \mathbf{a}_2 = \bar{3}\bar{1}\bar{2}1\mathbf{n}_3,\ \mathbf{a}_3 = \bar{3}\bar{1}\mathbf{n}_2,\ \mathbf{a}_4 = \bar{3}\bar{1}23\mathbf{n}_1,\ \mathbf{a}_5 = \bar{3}\bar{1}2312\mathbf{n}_3,$$
$$\mathbf{a}_6 = (\bar{3}\bar{1})^2 231232\bar{2}\mathbf{n}_1,\ \mathbf{a}_7 = \mathbf{n}_3,\ \mathbf{a}_8 = \mathbf{n}_2,\ \mathbf{a}_9 = 2\mathbf{n}_3.$$

Note that the singular point x_0 fixed by $\bar{3}\bar{1}Q^{-1}2Q^3 13$ (where $Q = R_1R_2R_3$), lies in the geodesic spanned by x_1, x_9. A direct computation yields that $\bar{3}\bar{1}Q^{-1}2Q^3 13$ maps x_1 to x_9. Thence we consider the decagon comprising the following sides

$$l_i = \begin{cases} \mathbf{P}(\mathrm{Span}_{\mathbb{C}}\{x_{i-1}, x_i\}) \cap L_1, & i = 1,2,\cdots 9. \\ \mathbf{P}(\mathrm{Span}_{\mathbb{C}}\{x_9, x_0\}) \cap L_1, & i = 10. \end{cases} \tag{4.16}$$

In the same manner with the proof of Theorem 1.1, one can check that the following side pairing transformations

$g_1 = (13)^3$, $g_2 = (1\bar{3}\bar{1}213)^3$, $g_3 = (1\bar{3}\bar{1}2313\bar{3}\bar{2}13)^2 \circ g_2$, $g_4 = (12)^2 \circ g_1$, $g_5 = \bar{3}\bar{1}Q^{-1}2Q^3 13$

satisfy

$$g_1(l_7) = l_8,\ g_2(l_3) = l_4,\ g_3(l_2) = l_5,\ g_4(l_6) = l_9,\ g_5(l_{10}) = l_1.$$

One can check that $g_5^2 = \bar{1}$ and the cycle transformation for x_4 satisfying

$$(g_2 \circ g_3^{-1}) = 1^3(2\bar{3}\bar{2}\bar{1})^2.$$

Then we get the the presentation of ℂ-Fuchsian group fixing L_1 in Thompson group S_2 :

$$\langle g_1, g_2, g_3, g_4, g_5 : g_1^{|\frac{2p}{p-6}|},\ g_2^{|\frac{2p}{p-6}|},\ g_5^{2p},\ (g_2 \circ g_3^{-1})^{|\frac{2p}{p-4}|},\ (g_1 \circ g_4^{-1})^{|\frac{2p}{p-4}|},$$
$$[g_1, g_5^2],\ [g_2, g_5^2],\ [g_3, g_5^2],\ [g_4, g_5^2]\rangle.$$

(ii) Thompson triangle group E_2

$$\langle R_1, R_2, R_3 : R_1^p, R_2^p, R_3^p, (R_1R_2R_3)^6, \mathrm{br}_3(R_2, R_3), \mathrm{br}_4(R_3, R_1), (R_1R_3)^{|\frac{4p}{p-4}|}, \mathrm{br}_4(R_1, R_2),$$
$$(R_1R_2)^{|\frac{4p}{p-4}|}, \mathrm{br}_4(R_1, R_2R_3R_2^{-1}), (R_1R_2R_3R_2^{-1})^{|\frac{4p}{p-4}|}, \mathrm{br}_6(R_3, R_1R_2R_1^{-1}), (R_3R_1R_2R_1^{-1})^{|\frac{3p}{p-3}|}\rangle.$$

We restrict to the pyramids of the side representatives of its fundamental domain

$$[3]1; 2, 3, \quad [6]\,\bar{3}13; 12\bar{1}, 3, \quad [4]\,23\bar{2}; 1, 3, \quad [4]\,3; 1, 2, \quad [4]\,2; 23\bar{2}, 2\bar{3}\bar{2}123\bar{2},$$

and pay attention to $[6]\,\bar{3}13; 12\bar{1}, 3$. Let $Q = R_1R_2R_3$. It is easily seen that Q^3 acts as a complex reflection with order 2 mapping the opposite vertices to each other. The image of it under R_3 is $[6]\,1; 3, 12\bar{1}$. Now, we firstly consider the pentagon lying in L_1 comprising the triangle from $[3]\,1; 2, 3$ and a quadrilateral which is half of the hexagon from $[6]\,1; 3, 12\bar{1}$. It has vertices $x_j = \mathbf{P}(\mathbf{n}_1 \boxtimes \mathbf{a}_j)$ (see Figure 3), where

$$\mathbf{a}_1 = 31\mathbf{n}_2,\ \mathbf{a}_2 = \bar{2}\bar{1}\mathbf{n}_3,\ \mathbf{a}_3 = \mathbf{n}_2,\ \mathbf{a}_4 = 2\mathbf{n}_3,\ \mathbf{a}_5 = \mathbf{n}_3.$$

Note that $\mathbf{a}_0$ is the polar vector of $3Q^3\bar{3}$ which has fixed point lying in the geodesic spanned by x_1, x_2. It is a simple matter to check that the following side pairing transformations

$$g_1 = 3Q^3\bar{3} : \quad x_0 \longmapsto x_0, \quad x_1 \longmapsto x_2,$$
$$g_2 = (12)^2 : \quad x_3 \longmapsto x_3, \quad x_2 \longmapsto x_4,$$
$$g_2 = (13)^2 : \quad x_5 \longmapsto x_5, \quad x_4 \longmapsto x_1.$$

We get the ℂ-Fuchsian group fixing L_1 has the presentation

$$\langle g_1, g_2, g_3 : g_1^2, g_2^{|\frac{2p}{p-4}|}, g_3^{|\frac{2p}{p-4}|}, (g_2 \circ g_1^{-1} \circ g_3)^{|\frac{2p}{p-4}|}\rangle.$$

We claim that there exist ℂ-Fuchsian subgroups fixing complex geodesic L_3 which are obviously not conjugate to the ones stated above. We consider the three pyramids with quadrilateral bases and make each of them have the base in L_3 :

$$[4]\,3; 1, 2$$
$$[4]\,23\bar{2}; 1, 3 \xrightarrow{\bar{2}} [4]\,3; \bar{2}12, 32\bar{3}$$
$$[4]\,2; 23\bar{2}, 2\bar{3}\bar{2}123\bar{2} \xrightarrow{23} [4]\,3; 2, 31\bar{3}.$$

We glue the three quadrilaterals lying in L_3 and get an octagon (see Figure 6) with vertices $y_j = \mathbf{P}(\mathbf{n}_3 \boxtimes \mathbf{b}_j)$, where

$$\mathbf{b}_0 = \mathbf{n}_2,\ \mathbf{b}_1 = 23\mathbf{n}_1,\ \mathbf{b}_2 = \bar{1}\bar{3}\mathbf{n}_2,\ \mathbf{b}_3 = \mathbf{n}_1,$$
$$\mathbf{b}_4 = 1\mathbf{n}_2, \mathbf{b}_5 = \bar{2}\mathbf{n}_1,\ \mathbf{b}_6 = \bar{2}1\mathbf{n}_3,\ \mathbf{b}_7 = \bar{2}\bar{3}\bar{2}\mathbf{n}_1.$$

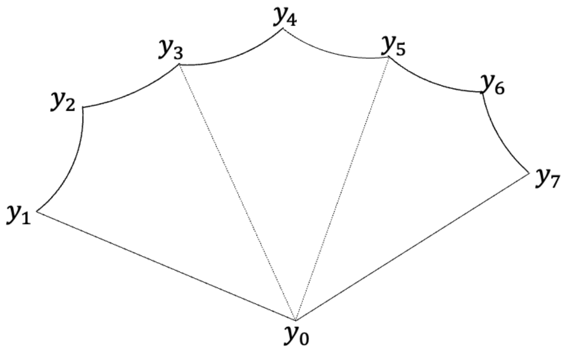

FIGURE 6. The octagon F

Let the sides denote by

$$l_i = \begin{cases} \mathbf{P}(\mathrm{Span}_{\mathbb{C}}\{y_{i-1}, y_i\}) \cap L_3, & i = 1, 2, \cdots 7. \\ \mathbf{P}(\mathrm{Span}_{\mathbb{C}}\{y_7, y_0\}) \cap L_3, & i = 8. \end{cases} \tag{4.17}$$

One can check that the side pairing transformations

$$h_1 = (23)^3, \quad h_2 = (13)^2, \quad h_3 = (\bar{2}123)^2, \quad h_4 = (231\bar{3}\bar{2}3)^2 \circ h_1$$

satisfy

$$h_1(l_8) = l_1, \quad h_2(l_3) = l_4, \quad h_3(l_5) = l_6, \quad h_4(l_7) = l_2.$$

In particular, we give an explanation for $h_4(l_7) = l_2$. Note that $131\mathbf{n}_3 = \mathbf{n}_3$ and $231\bar{3}\mathbf{n}_2 = 3\bar{1}\bar{3}\mathbf{n}_2$ hold due to $1313\bar{1}\bar{3}\bar{1} = 3$ and $\mathrm{br}_3(1, 23\bar{2})$. Then we have

$$\begin{aligned} (231\bar{3}\bar{2}3)^2(23)^3(\bar{2}1\mathbf{n}_3) &= 3^2(231\bar{3}\bar{2}3231\bar{3}\bar{2})(23232)(\bar{2}1\mathbf{n}_3) \\ &= 3^2 231(\bar{3}\bar{2}323)1231\mathbf{n}_3 \\ &= 3^2 23(1212)31\mathbf{n}_3 \\ &= 3^2 23212(131\mathbf{n}_3) \\ &= 3^3 231\bar{3}\mathbf{n}_2 \\ &= 3^3 3\bar{1}\bar{3}\mathbf{n}_2 \\ &= 3^4 \bar{1}\bar{3}\mathbf{n}_2, \end{aligned}$$

which indicates that $h_4(y_6) = y_2$. One can check the elliptic cycle at y_4 satisfies $h_2 \circ h_4 \circ h_3 = 3^6(\bar{3}1\bar{2}\bar{1})$, then obtain the presentation of $\mathbb{C}$-Fuchsian group fixing L_3 :

$$\langle h_1, h_2, h_3, h_4 : h_1^{|\frac{2p}{p-6}|}, h_2^{|\frac{2p}{p-4}|}, h_3^{|\frac{2p}{p-4}|}, (h_4 \circ h_1^{-1})^{|\frac{2p}{p-4}|}, (h_2 \circ h_4 \circ h_3)^{|\frac{p}{p-3}|} \rangle.$$

The Fuchsian groups we investigate above are subgroups of non-arithmetic lattices acting on the complex hyperbolic plane. One can immediately get that: if Γ is a lattice and $A \in \Gamma$ is a complex reflection fixing a complex geodesic L_A, then $\mathbf{Stab}_\Gamma(L_A)$ intersects $\mathbf{Stab}_{\mathrm{SU}(2,1)}(L_A)$ in a lattice. Then one can easily get the following proposition.

PROPOSITION 4.5. *There exist lattices in* $\mathbf{Stab}_{SU(2,1)}(L_1)$*, which could be embedded in* $SU(1,1)$*. They are subgroups of the complex hyperbolic triangle groups, which we considered in Theorem 1.1.*

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Li-Jie Sun
Department of applied science, Yamaguchi University,
2-16-1 Tokiwadai, Ube 755-8611, Japan.
E-mail: `ljsun@yamaguchi-u.ac.jp`