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INEQUALITIES FOR A POLYNOMIAL WHOSE ZEROS ARE WITHIN OR OUTSIDE A GIVEN DISK

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Abstract. In this paper we prove some results by using a simple but elegant techniques to improve and strengthen some generalizations and refinements of two widely known polynomial inequalities and thereby deduce some useful corollaries.

1. Introduction

Let $\mathbb{P}_n$ be the space of complex polynomials $P(z) := \sum_{j=1}^{n} c_j z^j$ of degree at most n. For each real number $k > 0$, we define the following:

$$D_k := \{z \in \mathbb{C} : |z| = k\}$$
$$D_k^- := \{z \in \mathbb{C} : |z| < k\}$$
$$D_k^+ := \{z \in \mathbb{C} : |z| > k\}$$

To be brief, we shall denote D_1, D_1^-, D_1^+ simply by D, D^-, D^+ respectively.
For every $P \in \mathbb{P}_n$ and P' as its derivative one form of the classical Bernstein inequality [2] for polynomials can be

$$\max_{z\in D} |P'(z)| \leq n \max_{z\in D} |P(z)|. \tag{1}$$

An improved form of this inequality due to Frappier, Rahman and Rusheweyh [3] states that, if $P(z)$ is a polynomial of degree n, then

$$\max_{z\in D} |P'(z)| \leq n \max_{1\leq k\leq 2n} |P(e^{\frac{ik\pi}{n}})|. \tag{2}$$

Clearly (2) represents a refinement of (1), since the maximum of $|P(z)|$ for $z \in D$ may be larger than the maximum of $|P(z)|$ taken over the $(2n)^{th}$ roots of unity, as is shown by the simpler example $P(z) = z^n + ia,\ a > 0$. Its worth mentioning that equality holds in (1) if and only if P has all its zeros at the origin. Dependence of inequalities on location of zeros made it prerequisite to learn the behaviour of inequality (1) while

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restricting ourselves to the class of polynomials having zeros in a given region. Among various forms, we mention following two results of Malik [7] which stand out in terms of their impact in the journey carried out in this direction :
If $P \in \mathbb{P}_n$ is such that it does not vanish in the open disk D_k^-, then for $k \geq 1$

$$\max_{z \in D} |P'(z)| \leq \frac{n}{1+k} \max_{z \in D} |P(z)| \tag{3}$$

and in case it does not vanish in the open disk D_k^+, then for $k \leq 1$

$$\max_{z \in D} |P'(z)| \geq \frac{n}{1+k} \max_{z \in D} |P(z)|. \tag{4}$$

For $k = 1$, inequality (3) reduces to a result conjectured by Erdös and latter proved by Lax [6] , whereas inequality (4) reduces to a result proved by Turán [8]. In this direction the following result analogous to inequality (2) was proved by Aziz [1].

Theorem 1.1. *If $P(z)$ is a polynomial of degree n having no zeros in the disk D^-, then for every real α*

$$\max_{z \in D} |P'(z)| \leq \frac{n}{2} \{M_\alpha^2 + M_{\alpha+\pi}^2\}^{\frac{1}{2}} \tag{5}$$

where

$$M_\alpha = \max_{1 \leq k \leq n} |P(e^{\frac{i(\alpha+2k\pi)}{n}})| \tag{6}$$

and $M_{\alpha+\pi}$ is obtained from (6) by replacing α by $\alpha + \pi$.

It was Dubinin [4] who improved on Tuŕan's result [8] and proved the following:

Theorem 1.2. *If $P(z) := \sum_{j=0}^{n} c_j z^j = c_n \prod_{j=1}^{n} (z - z_j)$, $|z_j| \leq 1$, $j = 1, 2, \ldots, n$ is a polynomial of degree n, then the following inequality holds at each point z on the circle D such that $P(z) \neq 0$,*

$$\max_{z \in D} |P'(z)| \geq \left[\frac{n}{2} + \frac{1}{2} \frac{|c_n| - |c_0|}{|c_n| + |c_0|}\right] \max_{z \in D} P(z). \tag{7}$$

The following Lemma which is due to Aziz [1]:

Lemma 1.3. *If $P(z)$ is a polynomial of degree n and $P^*(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, then for $|z| = 1$ and for every real α,*

$$|P'(z)|^2 + |(P^*(z))'|^2 \leq \frac{n^2}{2} (M_\alpha^2 + M_{\alpha+\pi}^2), \tag{8}$$

where M_α is defined by (6).

2. Main Results

In this paper we prove some results which besides the above two theorems refine some other polynomial inequalities. In fact we prove :

THEOREM 2.1. *If $P(z) := \sum_{j=0}^{n} c_j z^j = c_n \prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree n having no zeros in the disk D_k^-, $k \geq 1$, then for each point z on D_k such that $P(z) \neq 0$ and for every given real α,*

$$\max_{z\in D}|P'(z)| \leq$$

$$\frac{1}{2}\left[n^2 - \frac{2n^2(k-1)}{k+1}\frac{|P(z)|^2}{M_\alpha^2 + M_{\alpha+\pi}^2} - \frac{4n|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)(1+k)}\left\{n + \frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|}\right\}\right]^{\frac{1}{2}}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}$$

where M_α and $M_{\alpha+\pi}$ are defined by (6).

Proof. Suppose that $P(z) \neq 0$ for $z \in D_k$. Since $P(z) = c_n \sum_{j=1}^{n}(z - z_j)$, therefore

$$Re\frac{zP'(z)}{P(z)} = Re\sum_{j=1}^{n}\frac{z}{z - z_j}, |z_j| \geq k \geq 1.$$

Now for $z \neq z_j$ we have

$$\begin{aligned}
Re\frac{z}{z - z_j} &= Re\frac{e^{i\theta}}{e^{i\theta} - r_j e^{i\theta_j}},\ |r_j| \geq k \geq 1, \forall j = 1, 2, \ldots, n \\
&= Re\frac{1 - r_j e^{i(\theta-\theta_j)}}{1 - 2r_j cos(\theta - \theta_j) + r_j^2} \\
&= \frac{1 - r_j cos(\theta - \theta_j)}{1 - 2r_j cos(\theta - \theta_j) + r_j^2} \\
&\leq \frac{1}{1 + r_j} \\
&= \frac{1}{1 + |z_j|}.
\end{aligned}$$

Therefore,

$$Re\frac{zP'(z)}{P(z)} \leq \sum_{j=1}^{n}\frac{1}{1 + |z_j|}. \tag{9}$$

Also if $P^*(z) = z^n\overline{P(\frac{1}{\bar{z}})}$, then we have for $z \in D$,

$$|(P^*(z))'| = |nP(z) - zP'(z)|.$$

This gives for $z \in D$

$$\begin{aligned}
\left|\frac{z(P^*(z))'}{P(z)}\right|^2 &= \left|n - z\frac{P'(z)}{P(z)}\right|^2 \\
&= n^2 + \left|\frac{zP'(z)}{P(z)}\right|^2 - 2nRe\left(\frac{zP'(z)}{P(z)}\right). \\
&\geq n^2 + \left|z\frac{P'(z)}{P(z)}\right|^2 - 2n\left(\sum_{j=1}^{n}\frac{1}{1 + |z_j|}\right).
\end{aligned} \tag{10}$$

This gives

$$|(P^*(z))'|^2 \geq n^2|P(z)|^2 + |zP'(z)|^2 - 2n|P(z)|^2\left(\sum_{j=1}^{n}\frac{1}{1+|z_j|}\right).$$

Equivalently for $|z|=1$

$$|P'(z)|^2 \leq |(P^*(z))'|^2 - n^2|P(z)|^2 + 2n|P(z)|^2\left(\sum_{j=1}^{n}\frac{1}{1+|z_j|}\right).$$

Therefore

$$2|P'(z)|^2 \leq |P'(z)|^2 + |(P^*(z))'|^2 - n\left\{n - 2\sum_{j=1}^{n}\frac{1}{1+|z_j|}\right\}|P(z)|^2. \tag{11}$$

Now using Lemma 1.3 in (11), we get

$$2|P'(z)|^2 \leq \frac{n^2}{2}\left\{(M_\alpha^2 + M_{\alpha+\pi}^2)\right\} - n\left(n - 2\sum_{j=1}^{n}\frac{1}{1+|z_j|}\right)|P(z)|^2,$$

which gives,

(12)

$$\begin{aligned}
&4|P'(z)|^2 \leq n^2(M_\alpha^2 + M_{\alpha+\pi}^2) + 4n|P(z)|^2\sum_{j=1}^{n}\frac{1}{1+|z_j|} - 2n^2|P(z)|^2\\
&= \left[n^2 + \frac{4n|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)}\sum_{j=1}^{n}\frac{1}{1+|z_j|} - \frac{2n^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)}|P(z)|^2\right](M_\alpha^2 + M_{\alpha+\pi}^2)\\
&= \left[n^2 - \frac{2n^2(k-1)|P(z)|^2}{(k+1)(M_\alpha^2 + M_{\alpha+\pi}^2)} - \frac{4n^2|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)(k+1)} + \frac{4n|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)}\sum_{j=1}^{n}\frac{1}{1+|z_j|}\right\}\right](M_\alpha^2 + M_{\alpha+\pi}^2)\\
&= \left[n^2 - \frac{2n^2(k-1)}{k+1}\frac{|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)} - \frac{4n|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)}\left\{\frac{n}{1+k} - \frac{1}{1+k}\sum_{j=1}^{n}\frac{k+1}{1+|z_j|}\right\}\right](M_\alpha^2 + M_{\alpha+\pi}^2)\\
&\leq \left[n^2 - \frac{2n^2(k-1)}{k+1}\frac{|(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)} - \frac{4n|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)}\left\{\frac{n}{1+k} - \frac{1}{1+k}\sum_{j=1}^{n}\frac{k-|z_j|}{k+|z_j|}\right\}\right](M_\alpha^2 + M_{\alpha+\pi}^2)\\
&= \left[n^2 - \frac{2n^2(k-1)}{k+1}\frac{|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)} - \frac{4n|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)}\left\{\frac{n}{1+k} - \frac{1}{1+k}\sum_{j=1}^{n}\frac{1-\frac{|z_j|}{k}}{1+\frac{|z_j|}{k}}\right\}\right](M_\alpha^2 + M_{\alpha+\pi}^2).
\end{aligned}$$

We have by a simple application of principle mathematical induction,

$$\sum_{j=1}^{n}\frac{1-c_j}{1+c_j} \leq \frac{1-\prod_{j=1}^{n}c_j}{1+\prod_{j=1}^{n}c_j}\ \forall\ n\in\mathbb{N} \text{ and } c_j \geq 1,\ j=1,2,\ldots,n.$$

Using this fact in (12), as $\frac{|z_j|}{k} \geq 1$, and then using Vitali's formula, we get

$$|P'(z)|$$

$$\leq \frac{1}{2}\left[n^2 - \frac{2n^2(k-1)}{k+1}\frac{|P(z)|^2}{(M_\alpha^2+M_{\alpha+\pi}^2)} - \frac{4n|P(z)|^2}{(M_\alpha^2+M_{\alpha+\pi}^2)(1+k)}\left\{n - \frac{1-\prod_{j=1}^n \frac{|z_j|}{k}}{1+\prod_{j=1}^n \frac{|z_j|}{k}}\right\}\right]^{\frac{1}{2}} (M_\alpha^2+M_{\alpha+\pi}^2)^{\frac{1}{2}}$$

$$= \frac{1}{2}\left[n^2 - \frac{2n^2(k-1)}{k+1}\frac{|P(z)|^2}{(M_\alpha^2+M_{\alpha+\pi}^2)} - \frac{4n|P(z)|^2}{(M_\alpha^2+M_{\alpha+\pi}^2)(1+k)}\left\{n - \frac{k^n|c_n|-|c_0|}{k^n|c_n|+|c_0|}\right\}\right]^{\frac{1}{2}} (M_\alpha^2+M_{\alpha+\pi}^2)^{\frac{1}{2}}$$

$$= \frac{1}{2}\left[n^2 - \frac{2n^2(k-1)}{k+1}\frac{|P(z)|^2}{(M_\alpha^2+M_{\alpha+\pi}^2)} - \frac{4n|P(z)|^2}{(M_\alpha^2+M_{\alpha+\pi}^2)(1+k)}\left\{n + \frac{|c_0|-k^n|c_n|}{|c_0|+k^n|c_n|}\right\}\right]^{\frac{1}{2}} (M_\alpha^2+M_{\alpha+\pi}^2)^{\frac{1}{2}}.$$

This completes the proof of theorem. □

For $k=1$, Theorem 2.1 reduces to the following:

Corollary 2.2. *If $P(z) := c_n\prod_{j=1}^n(z-z_j),\ |z_j| \geq 1,\ j=1,2,\ldots,n$ is a polynomial of degree n then for each point z on D such that $P(z) \neq 0$ and every given real α*

$$\max_{z\in D}|P'(z)| \leq \frac{1}{2}\left[n^2 - \frac{2n|P(z)|^2}{(M_\alpha^2+M_{\alpha+\pi}^2)}\left\{n + \frac{|c_0|-|c_n|}{|c_0|+|c_n|}\right\}\right]^{\frac{1}{2}} (M_\alpha^2+M_{\alpha+\pi}^2)^{\frac{1}{2}}, \tag{13}$$

where M_α and $M_{\alpha+\pi}$ are defined by (6).

Remark 2.3. Since $\frac{|c_0|-|c_n|}{|c_0|+|c_n|} \geq 0$, therefore Corollary 2.2 is an improvement over Theorem 1.1.

Remark 2.4. We have

$$\left(1-\sqrt{\left|\frac{k^n c_n}{c_0}\right|}\right)^2 \geq 0,$$

therefore

$$\sqrt{\left|\frac{k^n c_n}{c_0}\right|} + \left|\frac{k^n c_n}{c_0}\right|^{\frac{3}{2}} \geq 2\left|\frac{k^n c_n}{c_0}\right|.$$

Equivalently

$$1-\left|\frac{k^n c_n}{c_0}\right| \geq 1+\left|\frac{k^n c_n}{c_0}\right| - \sqrt{\left|\frac{k^n c_n}{c_0}\right|} - \left|\frac{k^n c_n}{c_0}\right|^{\frac{3}{2}},$$

or

$$\frac{1-|\frac{k^n c_n}{c_0}|}{1+|\frac{k^n c_n}{c_0}|} \geq \frac{\sqrt{|c_0|}-\sqrt{k^n|c_n|}}{\sqrt{|c_0|}}$$

which gives,

$$\frac{|c_0|-k^n|c_n|}{|c_0|+k^n|c_n|} \geq \frac{\sqrt{|c_0|}-\sqrt{k^n|c_n|}}{\sqrt{|c_0|}}$$

Therefore from Theorem 2.1, we get:

Corollary 2.5. *If $P(z) := \sum_{j=0}^{n} c_j z^j = c_n \prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree n having no zeros in the disk D_k^-, $k \geq 1$, then for each point z on D_k such that $P(z) \neq 0$ and for every given real α,*

$$\max_{z\in D}|P'(z)| \leq$$

$$\frac{1}{2}\left[n^2 - \frac{2n^2(k-1)}{k+1}\frac{|P(z)|^2}{M_\alpha^2 + M_{\alpha+\pi}^2} - \frac{4n|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)(1+k)}\left\{n + \frac{\sqrt{|c_0|} - \sqrt{k^n|c_n|}}{\sqrt{|c_0|}}\right\}\right]^{\frac{1}{2}}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}$$

where M_α and $M_{\alpha+\pi}$ are defined by (6).

We next prove the following result which is a generalization of Theorem 1.2.

Theorem 2.6. *Suppose $P(z) := \sum\limits_{j=0}^{n} c_j z^j = c_n \prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree n having no zeros in the disk D_k^+, $k \leq 1$, then*

$$\max_{z\in D}|P'(z)| \geq \left[\frac{n}{1+k} + \frac{k}{1+k}\left\{\frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\right\}\right]\max_{z\in D}|P(z)|.$$

The result is sharp and equality holds for the polynomial $P(z) = \left(\frac{z+k}{1+k}\right)^n$.

Proof. Since $P(z)$ has no zeros in D_k^+, therefore, we can write $P(z) := \sum\limits_{j=1}^{n} c_j z^j = c_n \sum\limits_{j=1}^{n}(z - z_j)$, where $|z_j| \leq k \leq 1, \forall j = 1, 2, \ldots, n$. This gives, for the points $z \in D_k$, such that $P(z) \neq 0$

$$Re\left(\frac{zP'(z)}{P(z)}\right) = Re\sum_{j=1}^{n}\frac{z}{z - z_j}.$$

Hence for $z \in D$, we have

$$\begin{aligned}\left|\frac{P'(z)}{P(z)}\right| &\geq Re\left(\frac{zP'(z)}{P(z)}\right)\\ &= Re\sum_{j=1}^{n}\frac{z}{z-z_j}\\ &\geq \frac{1}{1+|z_j|}\\ &= \frac{n}{1+k} - \sum_{j=1}^{n}\left(-\frac{1}{k+1} - \frac{1}{1+|z_j|}\right)\\ &= \frac{n}{1+k} + \sum_{j=1}^{n}\frac{k-|z_j|}{(k+1)(1+|z_j|)}\\ &\geq \frac{n}{1+k} + \frac{1}{1+k}\sum_{j=1}^{n}\frac{k-|z_j|}{k+|z_j|}.\end{aligned} \tag{14}$$

From (14), we get

$$\max_{z\in D}|P'(z)| \geq \left[\frac{n}{1+k} + \frac{1}{1+k}\sum_{j=1}^{n}\frac{k-|z_j|}{k+|z_j|}\right]\max_{z\in D}|P(z)| \tag{15}$$

$$= \left[\frac{n}{1+k} + \frac{1}{1+k}\sum_{j=1}^{n}\frac{1-\frac{|z_j|}{k}}{1+\frac{|z_j|}{k}}\right]\max_{z\in D}|P(z)|.$$

We have by a simple application of principle of mathematical induction, $\sum_{j=1}^{n}\frac{1-c_j}{1+c_j} \geq \frac{1-\prod_{j=1}^{n}c_j}{1+\prod_{j=1}^{n}c_j}$ $\forall\, n \in \mathbb{N}$ and $c_j \leq 1$.
Using this fact in (15), as $\frac{|z_j|}{k} \leq 1$, and then using Vitali's formula, we get

$$\max_{z\in D}|P'(z)| \geq \left[\frac{n}{1+k} + \frac{1}{1+k}\left\{\frac{1-\prod_{j=1}^{n}\frac{|z_j|}{k}}{1-\prod_{j=1}^{n}\frac{|z_j|}{k}}\right\}\right]\max_{z\in D}|P(z)|.$$

$$= \left[\frac{n}{1+k} + \frac{1}{1+k}\left\{\frac{k^n|c_n|-|c_0|}{k^n|c_n|+|c_0|}\right\}\right]\max_{z\in D}|P(z)|.$$

This completes the proof of theorem . □

Remark 2.7. Theorem 2.6 is in fact a refinement of the result due to Malik (inequality (4)) and also generalises a result due to Dubinin [4].

It is easy to verify that

$$\frac{k^n|c_n|-|c_0|}{k^n|c_n|+|c_0|} \geq \frac{\sqrt{k^n|c_n|}-\sqrt{|c_0|}}{\sqrt{k^n|c_n|}},$$

therefore, from Theorem 2.6 we have

Corollary 2.8. *Suppose $P(z) := \sum_{j=0}^{n} c_j z^j = c_n\prod_{j=1}^{n}(z-z_j)$ is a polynomial of degree n having no zeros in the disk D_k^+, $k \leq 1$ then*

$$\max_{z\in D}|P'(z)| \geq \left[\frac{n}{1+k} + \frac{k}{1+k}\left\{\frac{\sqrt{k^n|c_n|}-\sqrt{|c_0|}}{\sqrt{k^n|c_n|}}\right\}\right]\max_{z\in D}|P(z)|.$$

For $k = 1$, it reduces to a result due to Dubinin [5].

References

[1] Abdul Aziz, *A refinement of an inequality of S. Bernstein*, J. Math. Anal. Appl., **142** (1989), 1–10.

[2] S. Bernstein, *Sur la limitation des dérivées des polynomes*, C. R. Acad. Sci. Paris., **190** (1930), 338–340.

[3] C. Frappier, Q. I. Rahman and St. Rusheweyh, *New Inequalities for polynomials*, Trans. Amer. Math. Soc., **288** (1985), 69–99.

[4] V. N. Dubinin, *Applications of the Schwarz Lemma to inequalities for entire functions with constraints on zeros*, J. Math. Sci.,(N.Y) **143** (3) (2007), 3069–3076.

[5] V. N. Dubinin, *Distortion theorems for polynomials on the circle*, Sbornik: Mathematics, **191** (12) (2000), 1797–1807.

[6] P. D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc., **50** (1944), 509–513.
[7] M. A. Malik, *On the derivative of Polynomial*, J. London Math. Soc., **1** (1969), 57–60.
[8] P. Turán, *Uber die Ableitung von Polynomen*, Compositio Math., **7** (1939), 89–95.
[9] S. L. Wali and W. M. Shah, *Some applications of Dubinin's lemma to rational functions with prescribed poles*, J. Math. Anal. Appl., **450** (2017), 769–779.

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