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A FIXED POINT THEOREM IN HILBERT C^*-MODULES

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Abstract. Fixed point theory has many useful applications in applied sciences. The object of this paper is to obtain fixed point for continuous self mappings in Hilbert C^*-module with rational conditions.

1. Introduction

Let $\mathcal{A}$ be a C^*-algebra. A (right) inner-product $\mathcal{A}$-module is a linear space E, which is a right $\mathcal{A}$-module and $\lambda(xa) = (\lambda x)a = x(\lambda a)$ for all $x \in E$, $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$, together with an inner product $\langle\cdot,\cdot\rangle : E \times E \to \mathcal{A}$ satisfying the following conditions:
(i) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$,
(ii) $\langle x, \lambda y + z\rangle = \lambda\langle x, y\rangle + \langle x, z\rangle$,
(iii) $\langle x, ya\rangle = \langle x, y\rangle a$,
(iv) $\langle x, y\rangle^* = \langle y, x\rangle$,
for all $x, y, z \in E$, $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. A Hilbert $\mathcal{A}$-module (Hilbert C^*-module) is an inner product $\mathcal{A}$-module E which is complete under the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$. Let E be a Hilbert $\mathcal{A}$-module. A map $T : E \to E$ is called adjointable if there is a map $T^* : E \to E$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in E$. It is easy to see that T must be $\mathcal{A}$-linear (i.e., $T(xa) = T(x)a$ for all $x \in E$ and $a \in \mathcal{A}$) and bounded [9, P 8]. The set of all adjointable maps is denoted by $\mathcal{L}(E)$, which is a C^*-algebra. For every pair of vectors $x, y \in E$, we use $\theta_{x,y}$ to denote the rank 1 linear operator on E, defined by $\theta_{x,y}(z) = x\langle y, z\rangle$ for any $z \in E$. The closed linear subspace of $\mathcal{L}(E)$ spanned by $\{\theta_{x,y} : x, y \in E\}$ is denoted by $\mathcal{K}(E)$. In fact, $\mathcal{K}(E)$ is a closed ideal of $\mathcal{L}(E)$ and is called the algebra of "compact" operators.
In mathematics, a fixed point (sometimes shortened to fixpoint, also known as an invariant point) of a function is an element of the function's domain that is mapped to itself by the function. That is to say, c is a fixed point of the function f if $f(c) = c$. A topological space X is said to have the fixed point property (briefly FPP) if for any continuous function $f : X \to X$ there exists $x \in X$ such that $f(x) = x$. Banach space is said to have the fixed point property (briefly, FPP) if every nonexpansive a self mapping defined on a nonempty closed convex bounded subset has a fixed point. In 1965, Browder presented a fundamental fixed point theorem that states every Hilbert space has FPP [1]. In the same year, Browder [2] and Göhde [4] independently showed even more that every uniformly convex Banach space has FPP and Kirk established

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this property for a larger class of reflexive Banach spaces with normal structure [7]. As pointed out, the first known Banach spaces with the FPP were the Hilbert spaces [1]. On the other hand, Hilbert C-modules as a powerful tool in operator and operator algebra theory generalize the notion of a Hilbert space. Thus, a natural question raises that which classes of Hilbert C*-modules have FPP. In this paper, we aim to investigate FPP for Hilbert modules.

2. Preliminaries

The main result of this paper is that every Hilbert C*-module over C*-algebra has FPP. First we see some fundamental results:

THEOREM 2.1. [5] *If f is self mapping of a complete metric space X into itself satisfying:*

$$d(x,y) \leq \alpha(d(Tx,x) + d(Ty,y)),$$

for all $x, y \in X$ and $\alpha \in [0, 1/2)$, then f has a unique fixed point in X.

THEOREM 2.2. [3] *If $d(x,y) \leq \alpha(d(Tx,x) + d(Ty,y)) + \beta d(x,y)$, for all $x, y \in X$ and $\alpha, \beta \in [0, 1/2)$, then f has a unique fixed point in X.*

THEOREM 2.3. [8] *If T is a self mapping on a closed subset S of a Hilbert Space H with Kannan type condition*

$$\|Tx - Ty\|^2 \leq \alpha(\|Tx - x\|^2 + \|Ty - y\|^2),$$

for all $x, y \in S$ and $\alpha \in [0, 1/2)$, then T has a unique fixed point in S.

Sharma et.al [11] have proved the common fixed point theorems in Hilbert space with following condition

$$\|Tx - Ty\| \leq \alpha\left(\frac{\|Tx - x\|^2 + \|Ty - y\|^2}{\|Tx - x\| + \|Ty - y\|}\right) + \beta\|x - y\|,$$

for all $x, y \in S$ and $x \neq y, \alpha \in [0, 1/2), \beta \geq 0$ and $2\alpha + \beta < 1$.

3. main theorem

Now we express the main theorem in the following

THEOREM 3.1. *Let S be a non empty closed subset of a Hilbert module Space E. Let T be self mapping on S satisfying the condition,*

$$\begin{aligned}
\|Tx - Ty\| &\leq a_1\left(\frac{\|Tx - x\|^2 + \|Ty - y\|^2 + \|Ty - x\|^2 + \|Tx - y\|^2}{\|Tx - x\| + \|Ty - y\| + \|Ty - x\| + \|Tx - y\|}\right) \\
&+ a_2\left(\frac{\|Tx - x\|^2 + \|Ty - y\|^2}{\|Tx - x\| + \|Ty - y\|}\right) \\
&+ a_3\left(\frac{\|Ty - x\|^2 + \|Tx - y\|^2}{\|Ty - x\| + \|Tx - y\|}\right) + a_4\|x - y\|,
\end{aligned}$$

for all $x, y \in S$ and $a_1, a_2, a_3, a_4 \geq 0$ and $4a_1 + 2a_2 + 2a_3 + a_4 < 1$, then T has a unique fixed point in S.

Proof. Let S be a non empty closed subset of a Hilbert module Space E. Let T be a self mapping on S. Let x_0 be any arbitrary point in S. We define a sequence $\{x_n\}_{n=1}^{\infty}$ in S by

$$x_{n+1} = Tx_n = T^{n+1}x_0, \text{ for } n = 0, 1, 2, ...$$

Suppose that $x_{n+1} \neq x_n$, for $n = 0, 1, 2....$
It follows from assumptions that for any integer $n \geq 1$

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|Tx_n - Tx_{n-1}\| \\
&\leq a_1(\frac{\|x_n - Tx_n\|^2 + \|x_{n-1} - Tx_{n-1}\|^2 + \|x_n - Tx_{n-1}\|^2 + \|x_{n-1} - Tx_n\|^2}{\|x_n - Tx_n\| + \|x_{n-1} - Tx_{n-1}\| + \|x_n - Tx_{n-1}\| + \|x_{n-1} - Tx_n\|}) \\
&+ a_2(\frac{\|x_n - Tx_n\|^2 + \|x_{n-1} - Tx_{n-1}\|^2}{\|x_n - Tx_n\| + \|x_{n-1} - Tx_{n-1}\|}) \\
&+ a_3(\frac{\|x_n - Tx_{n-1}\|^2 + \|x_{n-1} - Tx_n\|^2}{\|x_n - Tx_{n-1}\| + \|x_{n-1} - Tx_n\|}) + a_4\|x_{n+1} - x_n\| \\
&\leq a_1(\frac{\|x_n - x_{n+1}\|^2 + \|x_{n-1} - x_n\|^2 + \|x_n - x_n\|^2 + \|x_{n-1} - x_{n+1}\|^2}{\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\| + \|x_n - x_n\| + \|x_{n-1} - x_{n+1}\|}) \\
&+ a_2(\frac{\|x_n - x_{n+1}\|^2 + \|x_{n-1} - x_n\|^2}{\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|}) \\
&+ a_3(\frac{\|x_n - x_n\|^2 + \|x_{n-1} - x_{n+1}\|^2}{\|x_n - x_n\| + \|x_{n-1} - x_{n+1}\|}) + a_4\|x_n - x_{n-1}\| \\
&\leq a_1(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\| + \|x_{n-1} - x_{n+1}\|) \\
&+ a_2(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|) \\
&+ a_3(\|x_{n-1} - x_{n+1}\|) + a_4\|x_n - x_{n-1}\| \\
&\leq a_1(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\| + \|x_{n-1} - x_n\| + \|x_n - x_{n+1}\|) \\
&+ a_2(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|) \\
&+ a_3(\|x_{n-1} - x_n\| + \|x_n - x_{n+1}\|) + a_4\|x_n - x_{n-1}\|,
\end{aligned}$$

therefore

$$\|x_{n+1} - x_n\| \leq (2a_1 + a_2 + a_3 + a_4)\|x_n - x_{n-1}\| + (2a_1 + a_2 + a_3)\|x_{n+1} - x_n\|,$$

so we have

$$\|x_{n+1} - x_n\| \leq k\|x_n - x_{n-1}\|,$$

where $k = \frac{2a_1+a_2+a_3+a_4}{1-2a_1-a_2-a_3}$ and according to the assumption of the theorem $4a_1 + 2a_2 + 2a_3 + a_4 \leq 1$. Similarly to above proof we have $\|x_n - x_{n-1}\| \leq k\|x_{n-1} - x_{n-2}\|, ..., \|x_2 - x_1\| \leq k\|x_1 - x_0\|$, therefore

$$\|x_{n+1} - x_n\| \leq k\|x_n - x_{n-1}\| \leq k^2\|x_{n-1} - x_{n-2}\| \leq k^3\|x_{n-2} - x_{n-3}\| \leq ... \leq k^n\|x_1 - x_0\|.$$

Now for any positive integer $p \geq n \geq 1$ we have

$$\begin{aligned}
\|x_n - x_p\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + ... + \|x_{p-1} - x_p\| \\
&\leq k^n\|x_1 - x_0\| + k^{n+1}\|x_1 - x_0\| + ... + k^{p-1}\|x_1 - x_0\| \\
&\leq (k^n + k^{n+1} + ... + k^{p-1})\|x_1 - x_0\| \\
&\leq k^n(1 + k^n + ... + k^{p-n-1})\|x_1 - x_0\|,
\end{aligned}$$

thus

$$\|x_n - x_p\| \leq \frac{k^n}{1-k}\|x_1 - x_0\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

So $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since S is closed subset of Hilbert module E, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to point $\nu \in S$. Now we show that ν is a fixed point of self mapping T. Let $T\nu \neq \nu$. Consider

$$\begin{aligned}
\|\nu - T\nu\| &= \|\nu - x_n + x_n - T\nu\| \leq \|\nu - x_n\| + \|x_n - T\nu\| \leq \|x_n - T\nu\| \\
&\leq \|Tx_{n-1} - T\nu\| \qquad (\text{as} \quad n \longrightarrow \infty, x \longrightarrow \nu) \\
&\leq a_1(\frac{\|x_{n-1} - Tx_{n-1}\|^2 + \|\nu - T\nu\|^2 + \|x_{n-1} - T\nu\|^2 + \|\nu - Tx_{n-1}\|^2}{\|x_{n-1} - Tx_{n-1}\| + \|\nu - T\nu\| + \|x_{n-1} - T\nu\| + \|\nu - Tx_{n-1}\|}) \\
&+ a_2(\frac{\|x_{n-1} - Tx_{n-1}\|^2 + \|\nu - T\nu\|^2}{\|x_{n-1} - Tx_{n-1}\| + \|\nu - T\nu\|}) \\
&+ a_3(\frac{\|x_{n-1} - T\nu\|^2 + \|\nu - Tx_{n-1}\|^2}{\|x_{n-1} - T\nu\| + \|\nu - Tx_{n-1}\|}) + a_4\|x_{n-1} - \nu\| \\
&\leq a_1(\frac{\|x_{n-1} - x_n\|^2 + \|\nu - T\nu\|^2 + \|x_{n-1} - T\nu\|^2 + \|\nu - x_n\|^2}{\|x_{n-1} - x_n\|^2 + \|\nu - T\nu\|^2 + \|x_{n-1} - T\nu\|^2 + \|\nu - x_n\|^2}) \\
&+ a_2(\frac{\|x_{n-1} - x_n\|^2 + \|\nu - T\nu\|^2}{\|x_{n-1} - x_n\| + \|\nu - T\nu\|}) + a_3(\frac{\|x_{n-1} - T\nu\|^2 + \|\nu - x_n\|^2}{\|x_{n-1} - T\nu\| + \|\nu - x_n\|}) \\
&+ a_4\|x_{n-1} - \nu\|
\end{aligned}$$

where we have used the assumptions, hence

$$\begin{aligned}
\|\nu - T\nu\| &\leq a_1(\|x_{n-1} - x_n\| + \|\nu - T\nu\| + \|x_{n-1} - T\nu\| + \|\nu - x_n\|) \\
&+ a_2(\|x_{n-1} - x_n\| + \|\nu - T\nu\|) + a_3(\|x_{n-1} - T\nu\| + \|\nu - x_n\| \\
&+ a_4\|x_{n-1} - \nu\|,
\end{aligned}$$

so

$$\begin{aligned}
\|\nu - T\nu\| &\leq a_1(\|\nu - T\nu\| + \|\nu - T\nu\|) + a_2\|\nu - T\nu\| \\
&+ a_3\|\nu - T\nu\| \qquad x_{n-1}, x_n \longrightarrow \nu \quad \text{as} \quad n \longrightarrow \infty
\end{aligned}$$

thus

$$\begin{aligned}
\|\nu - T\nu\| &\leq (2a_1 + a_2 + a_3)\|\nu - T\nu\| \\
&\rightarrow (1 - 2a_1 - a_2 - a_3)\|\nu - T\nu\| \leq 0 \\
&\rightarrow \nu = T\nu.
\end{aligned}$$

Hence ν is fixed point for the self mapping T.
Uniqueness: Let $\nu \neq w$ such that $Tw = w$, take

$$\begin{aligned}
\|\nu - w\| = \|T\nu - Tw\| &\leq a_1(\frac{\|\nu - T\nu\|^2 + \|w - Tw\|^2 + \|\nu - Tw\|^2 + \|w - T\nu\|^2}{\|\nu - T\nu\| + \|w - Tw\| + \|\nu - Tw\| + \|w - T\nu\|}) \\
&+ a_2(\frac{\|\nu - T\nu\|^2 + \|w - Tw\|^2}{\|\nu - T\nu\| + \|w - Tw\|}) + a_3(\frac{\|\nu - Tw\|^2 + \|w - T\nu\|^2}{\|\nu - Tw\| + \|w - T\nu\|}) \\
&+ a_4\|\nu - w\|,
\end{aligned}$$

so we have

$$\|\nu - w\| \leq (2a_1 + a_3 + a_4)\|\nu - w\| \Rightarrow \|\nu - w\| = 0 \Rightarrow \nu = w.$$

Hence ν is a unique fixed point. □

THEOREM 3.2. *Let S be a non empty closed subset of a Hilbert module space E. Let T_1 and T_2 are self mapping on S satisfying the condition,*

$$\begin{aligned} \|T_1x - T_2y\| &\leq a_1\Big(\frac{\|x-T_1x\|^2+\|y-T_2y\|^2+\|x-T_2y\|^2+\|y-T_1x\|^2}{\|x-T_1x\|+\|y-T_2y\|+\|x-T_2y\|+\|y-T_1x\|}\Big) \\ &+ a_2\Big(\frac{\|x-T_1x\|^2+\|y-T_2y\|^2}{\|x-T_1x\|+\|y-T_2y\|}\Big) \\ &+ a_3\Big(\frac{\|x-T_2y\|^2+\|y-T_1x\|^2}{\|x-T_2y\|+\|y-T_1x\|}\Big) + a_4\|x-y\|, \end{aligned}$$

for all $x, y \in S$ and $a_1, a_2, a_3, a_4 \geq 0$ and $4a_1 + 2a_2 + 2a_3 + a_4 < 1$, then T_1 and T_2 have a unique common fixed point in S.

Proof. Let S be a non empty closed subset of a Hilbert module Space E. Let T_1 and T_2 are self mappings on S. Let x_o be any arbitrary point in S. We define a sequence $\{x_n\}_{n=1}^{\infty}$ in S by $x_{2n+1} = T_1x_{2n}$ and $x_{2n+2} = T_2x_{2n+1}$ for $n = 0, 1, 2\ldots.$ Suppose that for some n, $x_{n+2} \neq x_{n+1} \neq x_n$, for $n = 0, 1, 2\ldots.$ By the assumptions, for any integer $n \geq 1$

$$\begin{aligned} &\|x_{2n+1} - x_{2n}\| \\ &= \|T_1x_{2n} - T_2x_{2n-1}\| \\ &\leq a_1\Big(\frac{\|x_{2n}-T_1x_{2n}\|^2+\|x_{2n-1}-T_2x_{2n-1}\|^2+\|x_{2n}-T_2x_{2n-1}\|^2+\|x_{2n-1}-T_1x_{2n}\|^2}{\|x_{2n}-T_1x_{2n}\|+\|x_{2n-1}-T_2x_{2n-1}\|+\|x_{2n}-T_2x_{2n-1}\|+\|x_{2n-1}-T_1x_{2n}\|}\Big) \\ &+ a_2\Big(\frac{\|x_{2n}-T_1x_{2n}\|^2+\|x_{2n-1}-T_2x_{2n-1}\|^2}{\|x_{2n}-T_1x_{2n}\|+\|x_{2n-1}-T_2x_{2n-1}\|}\Big) \\ &+ a_3\Big(\frac{\|x_{2n}-T_2x_{2n-1}\|^2+\|x_{2n-1}-T_1x_{2n}\|^2}{\|x_{2n}-T_2x_{2n-1}\|+\|x_{2n-1}-T_1x_{2n}\|}\Big) + a_4\|x_{2n}-x_{2n-1}\| \\ &\leq a_1\Big(\frac{\|x_{2n}-x_{2n+1}\|^2+\|x_{2n-1}-x_{2n}\|^2+\|x_{2n}-x_{2n}\|^2+\|x_{2n-1}-x_{2n+1}\|^2}{\|x_{2n}-x_{2n+1}\|+\|x_{2n-1}-x_{2n}\|+\|x_{2n}-x_{2n}\|+\|x_{2n-1}-x_{2n+1}\|}\Big) \\ &+ a_2\Big(\frac{\|x_{2n}-x_{2n+1}\|^2+\|x_{2n-1}-x_{2n}\|^2}{\|x_{2n}-x_{2n+1}\|+\|x_{2n-1}-x_{2n}\|}\Big) \\ &+ a_3\Big(\frac{\|x_{2n}-x_{2n}\|^2+\|x_{2n-1}-x_{2n+1}\|^2}{\|x_{2n}-x_{2n}\|+\|x_{2n-1}-x_{2n+1}\|}\Big) + a_4\|x_{2n}-x_{2n-1}\| \\ &\leq a_1(\|x_{2n}-x_{2n+1}\|+\|x_{2n-1}-x_{2n}\|+\|x_{2n-1}-x_{2n+1}\|) \\ &+ a_2(\|x_{2n}-x_{2n+1}\|+\|x_{2n-1}-x_{2n}\|) + a_3\|x_{2n-1}-x_{2n+1}\| \\ &+ a_4\|x_{2n}-x_{2n-1}\| \\ &\leq a_1(\|x_{2n}-x_{2n+1}\|+\|x_{2n-1}-x_{2n}\|+\|x_{2n-1}-x_{2n}\|+\|x_{2n}-x_{2n+1}\|) \\ &+ a_2(\|x_{2n}-x_{2n+1}\|+\|x_{2n-1}-x_{2n}\|) + a_3(\|x_{2n-1}-x_{2n}\|+\|x_{2n}-x_{2n+1}\|) \\ &+ a_4\|x_{2n}-x_{2n-1}\|, \end{aligned}$$

therefore

$$\|x_{2n+1} - x_{2n}\| \leq (2a_1 + a_2 + a_3 + a_4)\|x_{2n} - x_{2n-1}\| + (2a_1 + a_2 + a_3)\|x_{2n+1} - x_{2n}\|,$$

so we have $\|x_{2n+1} - x_{2n}\| \leq k\|x_{2n} - x_{2n-1}\|$ where $k = \frac{2a_1+a_2+a_3+a_4}{1-2a_1-a_2-a_3}$ and $4a_1 + 2a_2 + 2a_3 + a_4 < 1$. In general

$$\begin{aligned}\|x_{n+1} - x_n\| \leq k\|x_n - x_{n-1}\| &\leq k^2\|x_{n-1} - x_{n-2}\| \leq k^3\|x_{n-2} - x_{n-3}\| \leq ... \leq k^n\|x_1 - x_0\| \\ &\Longrightarrow \|x_{n+1} - x_n\| \leq k^n\|x_1 - x_0\|.\end{aligned}$$

Now for any positive integer $p \geq n \geq 1$

$$\begin{aligned}\|x_n - x_p\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + ... + \|x_{p-1} - x_p\| \\ &\leq k^n\|x_1 - x_0\| + k^{n+1}\|x_1 - x_0\| + ... + k^{p-1}\|x_1 - x_0\| \\ &\leq (k^n + k^{n+1} + ... + k^{p-1})\|x_1 - x_0\| \\ &\leq k^n(1 + k^n + ... + k^{p-n-1})\|x_1 - x_0\| \\ &\leq \frac{k^n}{1-k}\|x_1 - x_0\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.\end{aligned}$$

$\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since S is closed subset of a Hilbert module space E, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a point ν in S. Now we show that ν is a common fixed point of T_1 and T_2. First we see that ν is fixed point of T_1. Let $T_1\nu \neq \nu$. Consider

$$\begin{aligned}\|\nu - T_1\nu\| &= \|\nu - x_{2n+2} + x_{2n+2} - T_1\nu\| \\ &\leq \|\nu - x_{2n+2}\| + \|x_{2n+2} - T_1\nu\| \text{ as } n \longrightarrow 0,\ x_{n+2} \longrightarrow \nu \\ &\leq \|T_2x_{2n+1} - T_1\nu\| + \|T_1\nu - T_2x_{2n+1}\| \\ &\leq a_1\left(\frac{\|\nu - T_1\nu\|^2 + \|x_{2n+1} - T_2x_{2n+1}\|^2 + \|\nu - T_2x_{2n+1}\|^2 + \|x_{2n+1} - T_1\nu\|^2}{\|\nu - T_1\nu\| + \|x_{2n+1} - T_2x_{2n+1}\| + \|\nu - T_2x_{2n+1}\| + \|x_{2n+1} - T_1\nu\|}\right) \\ &+ a_2\left(\frac{\|\nu - T_1\nu\|^2 + \|x_{2n+1} - T_2x_{2n+1}\|^2}{\|\nu - T_1\nu\| + \|x_{2n+1} - T_2x_{2n+1}\|}\right) \\ &+ a_3\left(\frac{\|\nu - T_2x_{2n+1}\|^2 + \|x_{2n+1} - T_1\nu\|^2}{\|\nu - T_2x_{2n+1}\| + \|x_{2n+1} - T_1\nu\|}\right) + a_4\|x_{2n+1} - \nu\| \\ &\leq a_1\left(\frac{\|\nu - T_1\nu\|^2 + \|x_{2n+1} - x_{2n+2}\|^2 + \|\nu - x_{2n+2}\|^2 + \|x_{2n+1} - T_1\nu\|^2}{\|\nu - T_1\nu\| + \|x_{2n+1} - x_{2n+2}\| + \|\nu - x_{2n+2}\| + \|x_{2n+1} - T_1\nu\|}\right) \\ &+ a_2\left(\frac{\|\nu - T_1\nu\|^2 + \|x_{2n+1} - x_{2n+2}\|^2}{\|\nu - T_1\nu\| + \|x_{2n+1} - x_{2n+2}\|}\right) + a_3\left(\frac{\|\nu - x_{2n+2}\|^2 + \|x_{2n+1} - T_1\nu\|^2}{\|\nu - x_{2n+2}\| + \|x_{2n+1} - T_1\nu\|}\right) \\ &+ a_4\|x_{2n+1} - \nu\|\end{aligned}$$

so we have

$$\begin{aligned}\|\nu - T_1\nu\| &\leq a_1(\|\nu - T_1\nu\| + \|\nu - T_1\nu\|) + a_2\|\nu - T_1\nu\| \\ &+ a_3\|\nu - T_1\nu\| + a_4\|\nu - x_{2n+1}\|.\end{aligned}$$

Therefore when $n \longrightarrow \infty$ then $x_{2n+1}, x_{2n+2} \longrightarrow \nu$, hence we conclude

$$\begin{aligned}\|\nu - T_1\nu\| &\leq (2a_1 + a_2 + a_3)\|\nu - T_1\nu\| \\ &\Rightarrow (1 - 2a_1 - a_2 - a_3)\|\nu - T_1\nu\| \leq 0 \\ &\Rightarrow \nu = T_1\nu.\end{aligned}$$

Hence ν is fixed point for the self mapping T_1. Similarly we can show that ν is a fixed point of T_2 . Hence ν is a common fixed point of T_1 and T_2.

Uniqueness: Let $w \neq \nu$ be another fixed point such that $T_2 w = w$.
Take

$$\begin{aligned} \|\nu - w\| &= \|T_1\nu - T_2 w\| \\ &\leq a_1\Big(\frac{\|\nu - T_1\nu\|^2 + \|w - T_2 w\|^2 + \|\nu - T_2 w\|^2 + \|w - T_1\nu\|^2}{\|\nu - T_1\nu\| + \|w - T_2 w\| + \|\nu - T_2 w\| + \|T_1 w - \nu\|}\Big) \\ &+ a_2\Big(\frac{\|\nu - T_1\nu\|^2 + \|w - T_2 w\|^2}{\|\nu - T_1\nu\| + \|w - T_2 w\|}\Big) + a_3\Big(\frac{\|\nu - T_2 w\|^2 + \|w - T_1\nu\|^2}{\|\nu - T_2 w\| + \|w - T_1\nu\|}\Big) \\ &+ a_4\|\nu - w\|. \end{aligned}$$

Thus we arrive to

$$\begin{aligned} \|\nu - w\| &\leq (2a_1 + a_2 + a_4)\|\nu - w\| \\ &\Rightarrow \|\nu - w\| = 0 \Rightarrow \nu = w. \end{aligned}$$

Hence ν is a unique common fixed point of T_1 and T_2. □

4. Conclusion

In this paper we obtained a unique fixed point for a continuous self mapping T as well as a unique common fixed point for two self mappings T_1 and T_2 satisfying rational conditions in Hilbert module space.

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