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ON STUDY OF f-APPROXIMATION PROBLEMS AND σ-INVOLUTORY VARIATIONAL INEQUALITY PROBLEMS

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Abstract. The purpose of the paper is to define f-projection operator to develop the f-projection method. The existence of a variational inequality problem is studied using fixed point theorem which establishes the existence of f-projection method. The concept of ρ-projective operator and σ-involutory operator are defined with suitable examples. The relation in between ρ-projective operator and σ-involutory operator are shown. The concept of σ-involutory variational inequality problem is defined and its existence theorem is also established.

1. Introduction

In a Banach space X with dual X^*, we say that an operator $\mathcal{A} : X \to X$ is ρ-projective if its minimal polynomial is $\rho^{-1}x^2 - x$ for some $\rho \in \mathbb{R}$ and an operator $\mathcal{B} : X \to X$ is σ-involutory if its minimal polynomial equation is $\sigma^{-1}x^2 - 1$ for some $\sigma \in \mathbb{R}$. Now for any two Banach spaces X and Y if $f : X \to Y$ and $g : Y \to X$, the $g \circ f : X \to X$ and $f \circ g : Y \to Y$. If there exist maps $\tilde{f} : X \to E \subset \mathbb{R}$ and $p_X : E \to Y$ such that $f = p_X \circ \tilde{f}$ is continuous, then $\tilde{f}$

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is called the lifting map of f and p_X is called the covering map. Similarly, if there exist maps $\tilde{g} : Y \to E \subset \mathbb{R}$ and $p_Y : E \to X$ such that $g = p_Y \circ \tilde{g}$ is continuous, then $\tilde{g}$ is called the lifting map of g and p_Y is called the covering map. Hence if $R(\tilde{f}) \cap R(\tilde{g}) \neq \varnothing$, $D(\tilde{f}) \cap R(\tilde{p_Y}) \neq \varnothing$ and $D(\tilde{g}) \cap R(\tilde{p_X}) \neq \varnothing$ where $D(f)$ represents the domain of f, then there exist at least one bijective map $h : \mathcal{E} \to \mathcal{W}$ where $\mathcal{E} = R(\tilde{f}) \cap R(\tilde{g})$ and $\mathcal{W} = D(\tilde{f}) \cap R(\tilde{p_Y}) \cap D(\tilde{g}) \cap R(\tilde{p_X})$ are finite and have same dimension.

In the recent decades, theory of variational inequalities is used to solve various types of inequality and equilibrium problems that arise in the branches of engineering, physical sciences, applied mathematics, finance, medical and so on. In fact these problems can be expressed in the form of variational inequality problems (VIP) which is introduced by Stampacchia [13] in the year 1964. Later the authors have defined the variational inequality problems in vector spaces, Hausdorff topological spaces and H-spaces. For reference, we refer Gianessi [9], Behera and Panda [3], Bardaro and Ceppitelli [1] and the references therein.

Various authors have studied the theory of variational inequalities using the projection methods. Solodov and Svaiter ([12], 1997) have developed the improved projection method to solve the variational inequalities. Nagurney and Zhang [10] have discussed the equilibrium solution of a projected dynamical system using the projective operator.

Let $F : K \subset X \to X^*$ be a linear mapping and the pairing $\langle f, x\rangle$ denotes the value of $f \in X^*$ at $x \in K$. The positive orthant K^+ and normal cone N are defined by

$$K^+ = \{f \in X^* : \langle f, x\rangle \geq 0, x \in K\} \text{ and } N = \{g \in X^* : \langle g, x\rangle \leq 0, x \in K\}.$$

The variational inequality problems (VIP) is to find $x^* \in K$ such that

$$\langle F(x^*), x - x^*\rangle \geq 0 \text{ for all } x \in K, \tag{VIP}$$

where the directed feasible set $K^+(x)$ of the solution of VIP is defined by

$$K^+(x) = \{x^* \in K : \langle F(x^*), x - x^*\rangle \geq 0 \text{ for all } x \in K\}.$$

The dual variational inequality problems (DVIP) is to find $x^* \in K$ such that

$$\langle F(x), x - x^*\rangle \geq 0 \text{ for all } x \in K, \tag{DVIP}$$

where directed feasible set $K^-(x)$ of the solution of DVIP is defined by

$$K^-(x) = \{x^* \in K : \langle F(x), x - x^*\rangle \leq 0 \text{ for all } x \in K\}.$$

In section 2, f-projection method is developed and a basic result is shown. In section 3, σ-involutory and ρ-projective operators are defined with suitable example. The equivalence between the operators are also established. The σ-involutory variational inequality problem is defined and its existence theorem of the problem is shown under certain conditions.

2. Projection operator and its application

Throughout of this section X is considered as a Hilbert space. A map $P : X \to X$ is a projection if it satisfies $P^2 = P$, that is, $P(x) = 0$ or $(I - P)(x) = 0$ for all $x \in X$. The zero of P is defined by $Z(P) = \{x \in X : P(x) = 0\}$, and the range of P is defined by $R(P) = \{x \in X : P(x) = x\}$. In fact, $X = Z(P) \oplus R(P)$, that is, $X = Z(P) \cup R(P)$ and $Z(P) \cap R(P) = \varnothing$.

Let X be a Hilbert space and K be a nonempty bounded closed convex subset of X. For our study we denote $f : X \to X$ is a fixed point function if $f(x) = x$ for at least one $x \in X$. Let $\mathcal{F}$ be the set of fixed point functions.

Definition 2.1. ([14]) A mapping $\Gamma : K \to K$ is said to be nonexpansive, if

$$\|\Gamma(x) - \Gamma(y)\| \leq \|x - y\|$$

for all $x, y \in K$.

It is well known that every nonexpansive mapping defined on a nonempty bounded closed convex subset of X has a fixed point (see [14]).

The approximation problem (AP) is to minimize the function $g : X \to \mathbb{R}$ defined by

$$g(x) = \|x - z\|$$

for all $z \in K$ and for each $x \in K$. If for each $x \in K$, there exists a unique $y \in K$ such that

$$\|x - y\| = \min_{z \in X} \|x - z\|, \tag{2.1}$$

then the point $y \in K$ is called the projection of x on K and written as

$$y = \mathbb{P}_K(x).$$

The projection operator $\mathbb{P}_K : K \to K$ is nonexpansive, that is,

$$\|\mathbb{P}_K(x_1) - \mathbb{P}_K(x_2)\| \leq \|x_1 - x_2\|$$

for all $x_1, x_2 \in K$. $\mathbb{P}_K \in \mathcal{F}$, implying $\mathbb{P}_K$ is continuous on K and has a fixed point in K.

It is obvious that $\mathbb{P}_K(x) = x$ for all $x \in K$ and holds the following result. If K is a closed set in X, then for each $x \in K$, there exists a unique $y \in K$ such that $y = \mathbb{P}_K(x)$, that is,

$$\|x - y\| = \min_{z \in X} \|x - z\|, \tag{2.2}$$

which can be written as a problem to find a unique $y \in K$ such that

$$\langle y, z - y \rangle \geq \langle x, z - y \rangle$$

for all $z \in X$. This is a particular case of the multilinear variational inequality problem of finding $y \in K$ such that

$$\langle H(x, y), z - y \rangle \geq 0$$

for all $z \in X$ and for each $x \in K$ where $H(x, y) = y - x$ studied by Nayak and Das [11].

2.1. ***f*-Projection Method.** Behera and Das [2] have studied the existence of variational inequality problem using fixed point theorems of homology theory. Later, Das and coauthors have extended the study of various variational inequality problems using fixed point theorems and homotopy map. For reference see Das [4, 5], Das and Behera [7], Das and Mohanta [8].

Let $f : K \to K$ be a contraction map on $K \subset X$. For our need, we define the concept of f-approximation problem and f-projection operator. Let $K \subset X$ and $f : K \to K$ be a map. The f-approximation problem is to find a unique $y \in K$ such that

$$\|f(x) - y\| = \min_{z \in X} \|f(x) - z\| \qquad (f\text{-}AP)$$

for each $x \in K$. The point $y \in K$ is called the *f-projection* of x on K and written as

$$y = \mathbb{P}_K(f(x)) = \mathbb{P}_K^f(x).$$

It is obvious that $\mathbb{P}_K^f(x) = f(x)$. Now $\mathbb{P}_K^f(x) = x$ if $x \in K$ is the fixed point of f.

Note: If $y \in K$ is a f-projection of x on K, that is, $y = \mathbb{P}_K^f(x)$, then

$$\|f(x) - y\| = \min_{z \in X} \|f(x) - z\|$$

for each $x \in K$, that is,

$$\|f(x) - y\| \leq \|f(x) - z\|$$

which can be written as

$$\langle y, z - y \rangle \geq \langle f(x), z - y \rangle$$

for all $z \in K$ and for each $x \in K$. It is the particular case of the multilinear variational inequality problem of finding $y \in K \subset X$ such that

$$\langle H(x, y), z - y \rangle \geq 0$$

for all $z \in K$ and for each $x \in K$, where $H(x, y) = y - f(x)$ [11].

Definition 2.2. *The projection functional operator $\mathbb{P}_K^f : ran(f) \to ran(f)$ is said to be nonexpansive on $f(K)$ if*

$$\left\|\mathbb{P}_K^f(x_1) - \mathbb{P}_K^f(x_2)\right\| \le \|f(x_1) - f(x_2)\|$$

for all $x_1, x_2 \in K$.

$\mathbb{P}_K^f$ is continuous on $ran(f) \subset K$ and has a fixed point in K.

Theorem 2.3. *Let $f : K \to K$ be a contraction map on the closed convex set $K \subset X$. Let $F : X \to X$ be any continuous map. For each $x \in K$, the set*

$$\{y \in X : f(x) = x + F(x) - y\}$$

is convex and compactly closed on K. Then

(1) *f and F have same fixed point,*
(2) *there exists a $y \in K$ such that $y = \mathbb{P}_K^f(x)$, that is, to find a $y \in K$ such that*

$$\langle y, z - y\rangle \ge \langle f(x), z - y\rangle \qquad (f\text{-VIP})$$

for all $z \in K$, where x corresponds to a unique y.

Proof. (1) Since $f : K \to K$ is a contraction map on K, $\|f(x) - f(y)\| \le c\|x - y\|$ for some rank $c \in (0, 1)$, so it has a fixed point on K. Again K is closed, for each $x \in K$ there exists a unique $y \in K$ such that $f(x) = x+F(x)-y \in K$, that is, $(1_K - f)(x) = x - f(x) = y - F(x)$ for each $x \in K$, where 1_K is the identity operator on K.

For $x = y$ the unique element, $(1_K - f)(y) = y - F(y) = (1_K - F)(y)$. A map $G : K \times I \to K$ is defined by

$$G(x,t) = \begin{cases} (1_K - f)(2ty + (1-2t)x), & \text{if } 0 \le t \le 1/2;\\ (1_K - F)(2(1-t)y + (2t-1)x), & \text{if } 1/2 \le t \le 1. \end{cases}$$

For $t = 0$, $G(x, 0) = (1_K - f)(y)$, for $t = 1$, $G(x, 1) = (1_K - F)(x)$ and for $t = 1/2$, $G(x, 1/2) = (1_K - f)(y) = (1_K - F)(y)$. Hence G is continuous by Pasting lemma, G is the homotopy in between $(1_K - f)$ and $(1_K - F)$, that is, the coincidence index set related to the maps $(1_K - f)$ and $(1_K - F)$ is nonempty. Since fixed point set of f is nonempty, fixed point set of F is also nonempty and the fixed point sets of f and F are equal on K, that is, $f(x^*) = x^* = F(x^*)$ on K. This proves (1).

(2) Again, since K is closed, for each $x \in K$, there exists an unique $y \in K$ closest to $F(x)$, that is, $\|y - F(x)\| \le \|z - F(x)\|$ for every $z \in K$. If $x^* \in \{x \in K : F(x) = x\}$, the fixed point set of F and y^* corresponds to x^*, then at $x = x^*$,

$$\|y^* - F(x^*)\| \le \|z - F(x^*)\|$$

for every $z \in K$. It implies

$$\|y^* - f(x^*)\| \leq \|z - f(x^*)\|$$

for every $z \in K$, that is,

$$\langle y^*, z - y^* \rangle \geq \langle f(x^*), z - y^* \rangle$$

for all $z \in K$. Hence $y^* = \mathbb{P}_K^f(x^*)$. □

3. σ-involutory variational inequality problems

Let X be a Banach space. We consider a class of map $A : X \to X$ satisfying the condition $A^3 = \sigma A$ for some $\sigma \in \mathbb{R}$. In this case, we have either $A(x) = 0$ or $(I - \sigma^{-1}A^2)(x) = 0$, that is, $(A^{-1} - \sigma^{-1}A)(x) = 0$ for all $x \in X$. Hence we have $(A^{-1} - \omega\sigma^{-1}A)(x) = 0$ for some $x \in X$ and $\omega \in (0,1)$. For simplicity we denote the operator $A_{\sigma,2} = A_2(\ ;\sigma\ ) = \sigma^{-1}A^2$ and $\mathcal{A}_{\sigma,2} = I - A_2(\ ;\sigma) = A^{-1} - \sigma^{-1}A$. Thus for each $x \in X$,

$$\mathcal{A}_{\sigma,2}(x) = \mathcal{A}_2(x;\sigma) = (I - A_2(x;\sigma))(x) = x - A_2(x;\sigma) = (A^{-1} - \sigma^{-1}A)(x).$$

For $\sigma = 1$, we have $A_{1,2} = A^2$. Therefore $A_{\sigma,2} = \sigma^{-1}\mathcal{A}_{1,2}$ and $\mathcal{A}_{\sigma,2} = I - A_{\sigma,2}$. The zero of A and the range of A are defined by $Z(A) = \{x \in X : A(x) = 0\}$, and $R(A) = \{y \in X : A_2(x;1) = \sigma y\} = \{y \in X : A_2(x;\sigma) = y\}$ respectively. It is obvious that $R(A) = \sigma y = Z(A_{1,2})$ but $Z(A_{1,2}) \subset Z(A_{\sigma,2})$. Thus X has a superclass partition as

(a) $X = Z(A) \oplus R(A_{\sigma,2})$, i.e., $X = Z(A) \cup R(A_{\sigma,2})$ and $Z(A) \cap R(A_{\sigma,2}) = \varnothing$ or
(b) $X = Z(A) \oplus Z(\mathcal{A}_{\sigma,2})$, i.e., $X = Z(A) \cup Z(\mathcal{A}_{\sigma,2})$ and $Z(A) \cap Z(\mathcal{A}_{\sigma,2}) = \varnothing$.

3.1. **σ-involutory operator and ρ-projective operator.** Let $\mathbf{M}_{mn}$ be the set of all rectangular matrices of order $m \times n$, $\mathbf{M}_n$ be the set of all square matrices of order n, $\mathbf{N}_n$ be the set of all nonsingular matrices of order n, $\mathbf{Inv}(\mathbf{N}_n)$ be the set of all involutory matrices of order n, $\mathbf{S}_n$ be the set of all singular matrices of order n and $\mathbb{I}_n$ be the set of all identity matrices of order n. For $\sigma, \rho > 0$, consider the class of sets:

$$\mathbf{Inv}(\mathbf{N}_n) = \left\{A \in \mathbf{N}_n : A^2 = I\right\} \text{ and } \mathbf{Inv}(\mathbf{N}_n;\sigma) = \left\{A \in \mathbf{N}_n : A^2 = \sigma I\right\},$$
$$\mathbf{Pr}(\mathbf{S}_n) = \left\{B \in \mathbf{S}_n : B^2 = B\right\} \text{ and } \mathbf{Pr}(\mathbf{M}_n;\rho) = \left\{B \in \mathbf{S}_n : B^2 = \rho B\right\}.$$

For $\sigma < 0$, we say A is skew σ-involutory (or skew σ-idempotent) and for $\rho < 0$, we say B is skew ρ-projective.

Example 3.1. Let $X \subset \mathbb{R}$ and the function $f : X \to X$ be any arbitrary function.

(i) The matrix $A(x) = \begin{pmatrix} x & 0 \\ f(x) & -x \end{pmatrix} \in \mathbf{Inv}(\mathbf{N}_2; x^2)$ and the transpose of $A(x)$ is also x^2-involutory for all $x \in X$.

(ii) The matrix $B(x,y) = \begin{pmatrix} x & x \\ y-x & x \end{pmatrix} \in \mathbf{Inv}(\mathbf{N}_2; xy)$ and the transpose of $B(x,y)$ is also xy-involutory for all $x \in X$.

(iii) The matrix $C(x,y) = \begin{pmatrix} x & 0 & 0 \\ x & -x & 0 \\ x & -2x & x \end{pmatrix} \in \mathbf{Inv}(\mathbf{N}_3; x^2)$ and the transpose of $C(x,y)$ is also x^2-involutory for all $x \in X$.

(iv) In general, the matrix $D(x,y) = (d_{ij})$ where

$$d_{ij} = \begin{cases} 0, & \text{if } i < j; \\ (-1)^{i+j}\binom{i}{j}x, & \text{if } i \geq j. \end{cases} \in \mathbf{Inv}(\mathbf{N}_3; x^n)$$

for $1 \leq i,j \leq n$ and the transpose of $D(x,y)$ is also x^n-involutory for all $x \in X$.

To prove the Theorem 3.7, we recall the result of Das and Baliarsingh [6] where the equivalence in between the ρ-projective and involutory difference operator is established.

Proposition 3.2. ([6], Proposition-2.8, p. 57) *Let D be any difference operator and A be any operator on $[X]$. Then*

(a) *$A = \frac{\rho}{2}(I+D)$ and $\frac{\rho}{2}(I-D)$ are ρ-projective on $[X]$, if D is involutory on $[X]$,*

(b) *$D = -\frac{1}{\rho}(\rho I - 2A)$ and $D = \frac{1}{\rho}(\rho I - 2A)$ are involutory on $[X]$, if A is ρ-projective on $[X]$.*

Theorem 3.3. *Let $A : X \to X$ and $B : X \to X$ be any two operators.*

(a) *If B is σ-involutory, then for any real $\rho > 0$, $A = \frac{\rho}{2}(1 \pm \sigma^{-1/2}B)$ are ρ-projective.*

(b) *If the A is ρ-projective, then for any real $\sigma > 0$, $B = \pm\frac{\sqrt{\sigma}}{\rho}(\rho I - 2A)$ are σ-involutory.*

Proof. (a) Since B is σ-involutory, $B^2 = \sigma I$, that is, $D^2 = I$ where $D = \sigma^{-1/2}B$, implying D is involutory. Therefore by Proposition 3.2(a), $A = \frac{\rho}{2}(I \pm D) = \frac{\rho}{2}(I \pm \sigma^{-1/2}B)$ are ρ-projective.

(b) Since A is ρ-projective, $A^2 = \rho A$. By Proposition 3.2(b), $M = \pm\frac{1}{\rho}(\rho I - 2A)$ are involutory, that is, $M^2 = I$. Therefore $B^2 = \frac{\sigma}{\rho^2}(\rho I - 2A)^2 = \sigma M^2 = \sigma I$, and hence B is σ-involutory. □

Example 3.4. For $X \subset \mathbb{R}$, $A : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $A(x) = \begin{pmatrix} x & 0 \\ f(x) & -x \end{pmatrix} \in$ $\mathbf{Inv}(\mathbf{N}_2; x^2)$, then $A(x)$ is x^2-involutory for all $x \in X$, that is, $A^2(x) = x^2 I$ for all x. By Theorem 3.3, the mapping $B : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $B(x) = \dfrac{\rho}{2}(I \pm x^{-1}A(x))$ are ρ-projective.

3.2. **σ-involutory problems.** Let X be a Hilbert space with dual space X^* and $K \subset X$ such that $K^* = K$. Let $A : K \to K^*$ be a continuous, σ-involutory map on K.

The variational inequality problem is to find $y \in K$ such that

$$\langle A(y), x - y \rangle \geq 0 \text{ for all } x \in K. \tag{VIP}$$

Replacing y by Ay in the problem (VIP) we get

$$\sigma \langle y, x - Ay \rangle \geq 0 \text{ for all } x \in K.$$

For $\sigma > 0$, we have

$$\langle y, x - Ay \rangle \geq 0 \text{ for all } x \in K.$$

The σ-involutory variational inequality problem (σ-IVIP) is to find $y \in K$ such that

$$\langle F_A(y), x - y \rangle \geq 0 \text{ for all } x \in K. \tag{σ-IVIP}$$

where $F_A = A^{-1} - A$ for $0 < \sigma < 1$. Replacing y by Ay in the problem (σ-IVIP) we get

$$\begin{aligned} \langle F_A(Ay), x - Ay \rangle &= \left\langle (A^{-1} - A)(Ay), x - Ay \right\rangle \\ &= \left\langle y - A^2 y, x - Ay \right\rangle = (1 - \sigma) \langle y, x - Ay \rangle \geq 0 \text{ for all } x \in K. \end{aligned}$$

The following theorem establish the existence of the solution of the problem (σ-IVIP).

Theorem 3.5. *Let K be a compact and convex set in a Hilbert space X with dual space $X^* = X$. If the map $A : K \to K^*$ is continuous, σ-involutory where $0 < \sigma < 1$ and satisfied $\langle A(x), v \rangle \geq 0$ for all $x \in K$ and $v \in X$, then there exists a $y \in K$ such that y solves (σ-IVIP).*

Proof. Since $\langle A(x), v \rangle \geq 0$ for all $x \in K$ and $v \in X$, we have $\langle A(y), x - y \rangle \geq 0$ for all $x, y \in K$, it implies

$$\langle A^{-1}(y), x - y \rangle = \sigma^{-1} \langle A(y), x - y \rangle \geq \langle A^{-1}(y), x - y \rangle,$$

that is,

$$\langle A^{-1}(y), x - y \rangle \geq \langle A(y), x - y \rangle$$

for all $x, y \in X$ and $\sigma \in (0, 1)$. Hence there exists a $y \in K$ such that y solves (σ-IVIP). This proves the theorem. □

From the above theorem, we can easily prove the following theorem.

Theorem 3.6. *Let K be a compact and convex set in a Hilbert space X with dual space $X^* = X$. If the map $A : K \to K^*$ is continuous, σ-involutory where $0 < \sigma < 1$ and $I - \pi A$ is nonexpansive, then there exists a $y \in K$ such that y solves* (VIP).

Theorem 3.7. *Let K be a compact and convex set in a Hilbert space X with dual space $X^* = X$. If the map $A : K \to X^*$ is continuous invertible map and $A^{-1} - A$ is nonexpansive, then there exists a $y \in K$ such that y solves* (σ-IVIP).

Proof. Since K is a compact and convex set in X with dual space $X^* = X$, for every continuous invertible map $A : K \to X^*$ and $F = A^{-1} - A$, the map $I - \pi F$ is continuous and nonexpansive maps, so it has a fixed point y in K, that is, $y = \mathbb{P}_K(I - \pi F)(y)$ which follows that y solves the VIP

$$\langle y, x - y\rangle \geq \langle (I - \pi F)(y), x - y\rangle$$

for all $x \in K$, that is,

$$\left\langle A^{-1}(y), x - y\right\rangle \geq \langle A(y), x - y\rangle$$

for all $x \in K$. This completes the proof. □

Conclusion: The concept of f-projection operator is defined. The existence of the solution of f-variational inequality problem equivalent to f-approximation problem is studied using fixed point theorem. A pair of operators such as σ-involutory and ρ-projective operators are defined with suitable example. The equivalence between the operators are also established. A class of new problem σ-involutory variational inequality problem is defined using the σ-involutory operator and studied its existence theorem.

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