Nonlinear Functional Analysis and Applications
Vol. 27, No. 2 (2022), pp. 405-432
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2022.27.02.13
http://nfaa.kyungnam.ac.kr/journal-nfaa

EQUATIONS OF MOTION FOR CRACKED BEAMS AND SHALLOW ARCHES

Semion Gutman[1], Junhong Ha[2] and Sudeok Shon[3]

[1]Department of Mathematics, University of Oklahoma
Norman, Oklahoma 73019, USA
e-mail: sgutman@ou.edu

[2]School of Liberal Arts, Korea University of Technology and Education
Cheonan 31253, South Korea
e-mail: hjh@koreatech.ac.kr

[3]Department of Architectural Engineering, Korea University of Technology and Education
Cheonan 31253, South Korea
e-mail: sdshon@koreatech.ac.kr

Abstract. Cracks in beams and shallow arches are modeled by massless rotational springs. First, we introduce a specially designed linear operator that "absorbs" the boundary conditions at the cracks. Then the equations of motion are derived from the first principles using the Extended Hamilton's Principle, accounting for non-conservative forces. The variational formulation of the equations is stated in terms of the subdifferentials of the bending and axial potential energies. The equations are given in their abstract (weak), as well as in classical forms.

1. Introduction

Modeling dynamic behavior of cracked beams and arches has important engineering applications. The main goal of this paper is to develop a rigorous mathematical framework for such problems.

The theory of uniform beams and shallow arches is well developed. An early exposition can be found in [1]. More general models in the multidimensional

[0]Received December 1, 2021. Revised January 10, 2022. Accepted January 17, 2022.
[0]2020 Mathematics Subject Classification: 47J35, 35Q74, 35D30, 70G75.
[0]Keywords: Shallow arch, Beam, Subdifferential, Cracks, Hamilton's principle.
[0]Corresponding author: S. Shon(sdshon@koreatech.ac.kr).

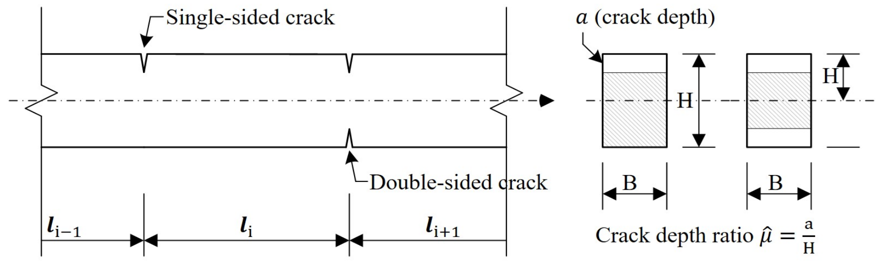

FIGURE 1. Crack parameters.

setting, and a literature survey are presented in [11]. A review for vibrating beams is given in [16]. Motion of uniform arches and a related parameter estimation problem are studied in [14]. These results are extended to point loads in [12]. The existence of a compact, uniform attractor is established in [13].

For a theory of cracked Bernoulli-Euler beams see [9]. A significant effort has been directed at the vibration analysis of cracked beams. Representation of a crack by a rotational spring has been proven to be accurate, and it is often used, see [4, 5] and the extensive bibliography there. Determination of the beam natural frequencies is discussed in [18, 19, 20]. S. Caddemi and his colleagues have further developed the theory using energy functions in [5]. Substantial reviews of cracked elements are presented in [6, 8, 10].

The transverse motion of a beam or an arch is described by the function $y(x,t)$, $x \in [0,\pi]$, $t \geq 0$, which represents the deformation of the beam/arch measured from the x-axis. For definiteness, the boundary conditions are of the hinged type

$$y(0,t) = y''(0,t) = 0, \quad y(\pi,t) = y''(\pi,t) = 0, \quad t \in (0,T). \tag{1.1}$$

Other types of boundary conditions, can be treated similarly.

A crack is fully described by its position along the axis, and the crack depth ratio $\hat{\mu}$, as shown in Figure 1. According to the common practice in the field, see [7], a crack is modeled by a massless rotational spring. The spring flexibility $\theta = \theta(\hat{\mu})$ depends on the crack depth ratio $\hat{\mu}$, and on whether the crack is one-sided or two-sided, open or closed, and so on. The flexibility θ is equal to 0 if there is no crack, and it increases with the crack depth. Explicit expressions for the functions $\theta(\hat{\mu})$ are provided in Section 4.

Remark 1.1. The following discussion is applicable to both arches and beams, but to avoid repetitions we will refer just to arches.

Suppose that there are m cracks along the length of the arch, located at $0 < x_1 < \cdots < x_m < \pi$. For convenience, we denote $x_0 = 0$, and $x_{m+1} = \pi$. Consequently, the cracked arch is modeled as a collection of $m+1$ uniform arches over the intervals $l_i = (x_{i-1}, x_i)$, $i = 1, \ldots, m+1$, as shown in Figure 2(b).

We consider only the transverse motion of the arch, so its position can be described by the function $y = y(x,t)$, $0 \le x \le \pi$, $t \ge 0$. The boundary conditions at the cracks enforce the continuity of the displacement field y, the bending moment y'', and the shear force y'''. Condition $y'(x_i^+, t) - y'(x_i^-, t) = \theta_i y''(x_i^+, t)$ expresses the discontinuity of the arch slope at the i-th crack, where $\theta_i = \theta(\hat{\mu}_i)$, see Figure 2(b).

To simplify the statement of the boundary conditions at the cracks, we introduce the notion of the *jump* $J[u](x)$ of a function $u = u(x)$ at any $x \in (0, \pi)$, as follows

$$J[u](x) = u(x^+) - u(x^-). \tag{1.2}$$

With this notation the conditions at the cracks (joint conditions) are

$$J[y](x_i, t) = 0, \quad J[y''](x_i, t) = 0, \quad J[y'''](x_i, t) = 0 \tag{1.3}$$

and

$$J[y'](x_i, t) = \theta_i y''(x_i^+, t), \tag{1.4}$$

where $\theta_i = \theta(\hat{\mu}_i)$, $i = 1, 2, \ldots, m$ and $t \ge 0$. Note that $y''(x_i^+, t) = y''(x_i^-, t)$ by (1.3).

In Section 2 we review our recent results from [15] on the variational setting for cracked beams and arches. First, special Hilbert spaces V, H_0^1, H are defined satisfying

$$V \subset H_0^1 \subset H \subset (H_0^1)' \subset V' \tag{1.5}$$

with continuous and dense embeddings. These spaces are broad enough to contain continuous functions with discontinuous derivatives at the joint points. Then in Section 3 we introduce the operator $\mathcal{A} : V \to V'$, by

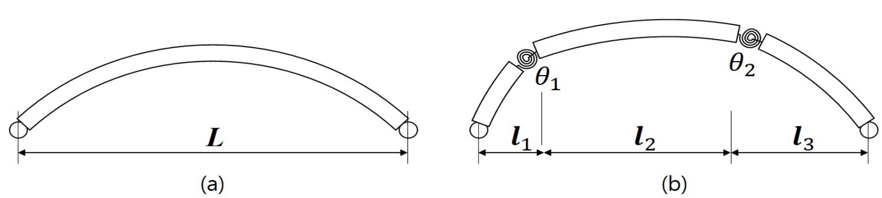

FIGURE 2. Beam or shallow arch: (a) uniform, (b) with two cracks.

$$\langle \mathcal{A}u, v\rangle_V = \sum_{i=1}^{m+1} (u'', v'')_i + \sum_{i=1}^{m} \frac{1}{\theta_i} J[u'](x_i) J[v'](x_i), \tag{1.6}$$

for any $u, v \in V$, where $J[u'](x) = u'(x^+) - u'(x^-)$.

Our main result is that the solution u of the equation $\mathcal{A}u = f$ in H satisfies the joint conditions, including (1.4). Thus the operator $\mathcal{A}$ "absorbs" the boundary conditions, as expected of the weak formulation of the steady state problem. This result allows us to mathematically justify the existence of the eigenvalues and the eigenfunctions of $\mathcal{A}$.

In Section 4 we consider relevant physical quantities, including the potential energy U_b, due to bending, and the potential energy U_a, due to the axial force. This is the only section in the paper, that contains physical variables. Their non-dimensional equivalents are used in all the other sections.

Typically, equations of motion are derived from the Hamilton's Principle $\delta I = 0$, which seeks the stationary paths of the action I. A closer examination of this statement in the framework of the Hilbert spaces reveals the importance of the subdifferential $\partial\phi$ of a convex lower-semicontinuous function ϕ. The potential energies U_b, and U_a are two examples of such functions.

Section 5 presents a brief review of these concepts. One can view the subdifferential, which can be multi-valued, as a generalization of the derivative. The definition of the subdifferential is given in (5.2). For example, let $\phi(x) = |x|$, $x \in \mathbb{R}$. Then $\partial\phi(x) = -1$, for $x < 0$, and $\partial\phi(x) = 1$, for $x > 0$, since ϕ is differentiable for such x. To determine $\partial\phi(0)$, notice that $\phi(0) = 0$. Then the definition of $\partial\phi(0)$ can be interpreted geometrically as the set of all the slopes of the lines that pass through the origin and lie below the graph of ϕ. Thus $\partial\phi(0) = [-1, 1]$. We conclude Section 5 by deriving various expressions for the subdifferentials ∂U_b and ∂U_a. In particular, we show that $\partial U_b = \mathcal{A}$.

Section 6 uses the Extended Hamilton's Principle, which accommodates non-conservative forces, to derive the equations of motion. It contains our main result: the abstract equation of motion for cracked beams and arches is

$$\ddot{y} + \partial U_b(y) + \partial U_a(y) + c_d \dot{y} = p, \tag{1.7}$$

where $\dot{y}, \ddot{y}$ denote the time derivatives.

For beams, it is assumed that the influence of the axial force can be neglected. This case is considered in Section 7. The abstract equation for cracked beams is

$$\ddot{y} + \mathcal{A}y + c_d \dot{y} = p. \tag{1.8}$$

If the beam contains no cracks, then (1.8) becomes $\ddot{y} + y'''' + c_d \dot{y} = p$, which is consistent with the classical Euler-Bernoulli theory.

Our main result in Section 8 is that the "classical" equation for a cracked shallow arch is

$$\ddot{y} + y'''' - \frac{1}{\pi}\left(\beta + \frac{1}{2}\int_0^L |y'(x,t)|^2\,dx\right)\left(y'' + \sum_{i=1}^{m}\theta_i y''(x_i,t)\delta(x-x_i)\right) + c_d\dot{y}$$
$$= p,$$

where $\delta = \delta(x)$ is the delta function.

Motion in viscous media results in the additional term $\mu\mathcal{A}\dot{y}$, $\mu > 0$ in the governing equations. Such a case is referred to as the strong damping motion. If the viscous effects are neglected ($\mu = 0$), we have the weak damping case.

This paper is an important a part of our research program on cracked beams and shallow arches. The equations of motion derived here are further investigated using Lions method. We will derive the existence, uniqueness, and convergence of finite-dimensional approximation for these time-dependent problems. The Modified Shifrin's method will be investigated for its efficiency, as compared with the traditional Transition Matrices method for the eigenvalues and the eigenfunctions. Furthermore, a computational study will be conducted, that may include experiments with real beams and arches.

2. Hilbert spaces

The goal of this section is to introduce Hilbert spaces H, V, H_0^1 suitable for working with cracked elements. See [15] for further details. Suppose that an arch has m cracks at the joint points $0 < x_1 < \cdots < x_m < \pi$. This partition of the interval $[0,\pi]$ is associated with $m+1$ subintervals $l_i = (x_{i-1}, x_i)$, $i = 1,\ldots,m+1$.

Let H be the Hilbert space

$$H = \bigoplus_{i=1}^{m+1} L^2(l_i). \tag{2.1}$$

Let the inner product and the norm in $L^2(l_i)$, $i = 1,\cdots,m+1$, be denoted by $(\cdot,\cdot)_i$ and $|\cdot|_i$ correspondingly. The inner product and the norm in H are defined by

$$(u,v)_H = \sum_{i=1}^{m+1}(u,v)_i, \quad |u|_H^2 = \sum_{i=1}^{m+1}|u|_i^2. \tag{2.2}$$

Consider the Sobolev space $H^2(a,b)$ on a bounded interval $(a,b) \subset \mathbb{R}$, and let $u \in H^2(a,b)$. Then u, u' are continuous functions on $[a,b]$, up to a set of measure zero, and $u'' \in L^2(a,b)$. Therefore, for such u, we will always assume that $u, u' \in C[a,b]$.

Define the linear space

$$V = \left\{ u \in \bigoplus_{i=1}^{m+1} H^2(l_i) \,:\, u(0) = u(\pi) = 0,\ J[u](x_i) = 0,\ i = 1, \cdots, m \right\}. \tag{2.3}$$

We interpret $u \in V$ as a continuous function on $[0, \pi]$, such that $u(0) = u(\pi) = 0$, with $u' \in L^2(0, \pi)$, i.e. $u \in H_0^1(0, \pi)$. Furthermore, $u|_{l_i} \in H^2(l_i)$, and $u'|_{l_i} \in C[x_{i-1}, x_i]$ for $i = 1, 2, \cdots, m+1$.

Define the inner product on V by

$$((u, v))_V = \sum_{i=1}^{m+1} (u'', v'')_i + \sum_{i=1}^{m} J[u'](x_i) J[v'](x_i), \quad \text{for any } u, v \in V, \tag{2.4}$$

where $(u'', v'')_i = \int_{l_i} u''(x) v''(x)\, dx$.

It is clear that $((\cdot, \cdot))_V$ is a symmetric, bilinear form on V. To see that $((u, u))_V = 0$ implies $u = 0$, notice that any function u with $((u, u))_V = 0$ is piecewise linear and continuous on $[0, \pi]$. Furthermore, $J[u'](x_i) = 0$ for any $i = 1, 2, \cdots, m$. Therefore u' is continuous on $[0, \pi]$. In fact, it is a constant there, since $u'' = 0$ a.e. on $[0, \pi]$. Thus u is a linear function on $[0, \pi]$ satisfying the zero boundary conditions at the ends of the interval. Therefore $u = 0$ on $[0, \pi]$, and $((\cdot, \cdot))_V$ is a well-defined inner product on V. The corresponding norm in V is

$$\|u\|_V^2 = \sum_{i=1}^{m+1} |u''|_i^2 + \sum_{i=1}^{m} |J[u'](x_i)|^2, \quad \text{for any } u \in V, \tag{2.5}$$

where $|\cdot|_i$ is the norm in $L^2(l_i)$. It can be shown using the next lemma, that V is a Hilbert space, see [15].

Let $u \in V$. We define the derivatives of u component-wise in the spaces $H^2(l_i)$, that is $u'(x) = (u|_{l_i})'(x), u''(x) = (u|_{l_i})''(x)$, and so on, for $x \in l_i$, $i = 1, \cdots, m+1$. For definiteness, we will assume that the derivative u' is continuous from the right on $[0, \pi]$.

Lemma 2.1. *Let $c \geq 0$ denote various constants independent of $u \in V$. Then*

(i) *The second derivative u'' is bounded in H, and*

$$|u''|_H \leq \|u\|_V. \tag{2.6}$$

(ii) *The derivative u' is bounded on $[0, \pi]$,*

$$\sup\{|u'(x)|,\ x \in [0, \pi]\} \leq c\left(|u'|_H + |u''|_H\right). \tag{2.7}$$

Moreover,

$$|u'|_H \leq c\|u\|_V \quad \textit{and} \quad \sup\{|u'(x)|,\ x \in [0, \pi]\} \leq \|u\|_V. \tag{2.8}$$

(iii) *Function u is Lipschitz continuous, with the Lipschitz constant $c\|u\|_V$. Also, u is bounded on $[0,\pi]$,*

$$\|u\|_\infty = \max\{|u(x)|,\ x \in [0,\pi]\} \le c\|u\|_V \tag{2.9}$$

and

$$|u|_H \le c\|u\|_V. \tag{2.10}$$

Proof. Part (i) follows from the definition (2.5). For part (ii), let $u \in V$. Then its derivative u' is continuous on any interval $[x_{i-1}, x_i]$, $i = 1, \cdots, m+1$. By the mean value theorem for integrals, there exists $c_i \in [x_{i-1}, x_i]$, such that

$$u'(c_i) = \frac{1}{|l_i|}\int_{l_i} u'(s)\,ds.$$

Thus $|u'(c_i)| \le c|u'|_H$. Also, for any $x \in [x_{i-1}, x_i]$,

$$|u'(x) - u'(c_i)| \le \int_{l_i} |u''(s)|\,ds \le c|u''|_H.$$

Therefore, $|u'(x)| \le c\,(|u'|_H + |u''|_H)$ for any $x \in [0,\pi]$, giving (2.7). This inequality implies $|J[u'](x)| \le 2c\,(|u'|_H + |u''|_H)$.

We have

$$\begin{aligned} \int_a^b u''(x)\,dx &= u'|_a^b + \sum_{a<x_i\le b} \left(u'(x_i^-) - u'(x_i^+)\right) \\ &= u'(b) - u'(a) - \sum_{a<x_i\le b} J[u'](x_i). \end{aligned} \tag{2.11}$$

Therefore,

$$|u'(b) - u'(a)| \le \int_0^\pi |u''(x)|\,dx + \sum_{i=1}^m |J[u'](x_i)| \le c\|u\|_V. \tag{2.12}$$

First, choose $a \in [0,\pi]$ be such that $u'(a) \le 0$, which is always possible, since $u(0) = u(\pi) = 0$. Then, by (2.12), for any $b \in [0,\pi]$ we have $u'(b) \le c\|u\|_V$. This establishes the upper bound for $u'(x)$, $x \in [0,\pi]$. Similarly, choosing a such that $u'(a) \ge 0$, we establish the lower bound for $u'(x)$, $x \in [0,\pi]$. Inequalities in (2.8) follow.

To show part (iii), recall that $u \in V$ is continuous on $[0,\pi]$, and by (2.8) its derivative u' is bounded by $\|u\|_V$. Therefore u is Lipschitz continuous there. Furthermore, since $u(0) = 0$, u is bounded on $[0,\pi]$ as claimed in (2.9). Then inequality (2.10) follows. □

At this time we introduce the Hilbert space $H_0^1 = H_0^1(0,\pi)$, with the inner product and the norm given by

$$(u,v)_1 = (u',v')_H, \quad \|u\|_1^2 = |u'|_H^2, \quad u,v \in H_0^1. \tag{2.13}$$

The norm in $(H_0^1)'$ will be denoted by $\|\cdot\|_{-1}$. It can be shown that the identity embedding $i : V \to H_0^1$ is linear, continuous, with a dense range in H_0^1. Furthermore, it is compact.

This allows us to define the Gelfand triple $V \subset H \subset V'$, with the pairing $\langle \cdot, \cdot \rangle_V$ between V and V' extending the inner product in H. This means that given $f \in H = H' \subset V'$, and $v \in V$, we have $\langle f, v\rangle_V = (f, v)_H$.

Also, we have

$$V \subset H_0^1 \subset H \subset \left(H_0^1\right)' \subset V', \tag{2.14}$$

with dense embeddings. Furthermore, the embeddings $V \subset H_0^1 \subset H$ are compact.

3. Variational setting and operator $\mathcal{A}$

Now we introduce the operator $\mathcal{A} : V \to V'$ that "absorbs" the junction boundary conditions. This operator is central to the variational setting of problems for cracked beams and arches. The existence of its eigenvalues and the eigenfunctions is established as well.

Definition 3.1. Define the operator $\mathcal{A}$ on V by

$$\langle \mathcal{A}u, v\rangle_V = \sum_{i=1}^{m+1} (u'', v'')_i + \sum_{i=1}^{m} \frac{1}{\theta_i} J[u'](x_i) J[v'](x_i), \tag{3.1}$$

for any $u, v \in V$. We will also write $\langle \mathcal{A}u, v\rangle$ for $\langle \mathcal{A}u, v\rangle_V$, if it does not cause a confusion.

See Section 1 for the setup for the junction (crack) points x_i, and the flexibilities θ_i. Recall, that a linear operator $A : V \to V'$ is called coercive, if there exists $c > 0$, such that $\langle Au, u\rangle \ge c\|u\|_V^2$ for any $u \in V$.

Lemma 3.2. *Let $\mathcal{A}$ be defined by* (3.1)*. Then $\mathcal{A}$ is a symmetric, continuous, linear, and coercive operator from V onto V'.*

Proof. Clearly, $\mathcal{A}$ is a symmetric linear operator. Since all $\theta_i > 0$, we conclude that there exists a constant $C > 0$, such that $|\langle \mathcal{A}u, v\rangle| \le C\|u\|_V\|v\|_V$. Therefore $\mathcal{A}$ is defined on all of V, and it is bounded.

Similarly,

$$|\langle \mathcal{A}u, u\rangle| = \sum_{i=1}^{m+1} |u''|^2 + \sum_{i=1}^{m} \frac{1}{\theta_i} |J[u'](x_i)|^2 \ge c\|u\|_V^2.$$

Therefore $\mathcal{A}$ is coercive on V, and its range is V', see [21, Theorem 2.2.1]. $\square$

As was mentioned in Section 1, functions $u = u(x)$ modeling an arch with cracks are expected to satisfy certain boundary conditions. For convenience, we restate them here:

$$u(0) = u(\pi) = 0, \quad u''(0) = u''(\pi) = 0 \tag{3.2}$$

and

$$J[u](x_i) = 0, \quad J[u''](x_i) = 0, \quad J[u'''](x_i) = 0, \quad J[u'](x_i) = \theta_i u''(x_i^+), \tag{3.3}$$

for $i = 1, \cdots, m$.

The next theorem is the main result of this section.

Theorem 3.3. *Let the domain of $\mathcal{A}$ be $D(\mathcal{A}) = \{v \in V : \mathcal{A}v \in H\}$.*

(i) *If $u \in D(\mathcal{A})$, then $u|_{l_i} \in H^4(l_i)$, $\mathcal{A}u = u''''$ a.e. on l_i, $i = 1, \cdots, m+1$, and u satisfies conditions* (3.2)–(3.3).
(ii) *Let $f \in H$. Then equation $\mathcal{A}u = f$ in V' has a unique solution $u \in D(\mathcal{A})$.*

Proof. By Lemma 3.2, the operator $\mathcal{A}$ is coercive, and its range is V'. Since $H = H' \subset V'$, condition $f \in H$ implies that $f \in V'$. Therefore equation $\mathcal{A}u = f$ in V' has a solution $u \in D(\mathcal{A})$, which is unique since $\mathcal{A}$ is coercive.

To investigate the properties of functions in $D(\mathcal{A})$, recall that $l_1 = (x_0, x_1)$. Notice that $C_0^\infty(l_1) \subset V$, where it is assumed that the functions from $C_0^\infty(l_1)$ are extended by zero outside of l_1. Thus $v(x) = v'(x) = 0$, for $x = 0$ and any $x \geq x_1$, $v \in C_0^\infty(l_1)$.

Let $u \in D(\mathcal{A})$, so $\mathcal{A}u = f$ for some $f \in H$. By the definition of V, we have $u|_{l_1} \in H^2(l_1)$. For any $v \in C_0^\infty(l_1)$, by the definition of $\mathcal{A}$, we have

$$\langle \mathcal{A}u, v \rangle = \int_{l_1} u''(x)v''(x)\, dx.$$

Integration by parts gives

$$\int_{l_1} u''(x)v''(x)\, dx = \int_{l_1} u(x)v''''(x)\, dx = (f, v)_H = \int_{l_1} f(x)v(x)\, dx.$$

Therefore, $D^{(4)}u = f$ in the sense of the weak derivatives on l_1. Thus $u|_{l_1} \in H^4(l_1)$, and $u'''' = f$ a.e. on l_1. Repeating this argument for other intervals l_i, we conclude that $u|_{l_i} \in H^4(l_i)$, and $u'''' = f$ a.e. on l_i, $i = 1, \cdots, m+1$.

It remains to show the satisfaction of the conditions (3.2)–(3.3). So, let $\mathcal{A}u = f \in H$. Since we have already established that $u|_{l_i} \in H^4(l_i)$, $i = 1, \cdots, m+1$, we can do the Integration by Parts on every interval l_i, to obtain that for *any* $v \in V$

$$(f, v)_H = \langle \mathcal{A}u, v \rangle = \sum_{i=1}^{m+1} (u'', v'')_i + \sum_{i=1}^{m} \frac{1}{\theta}_i J[u'](x_i) J[v'](x_i)$$

$$= \sum_{i=1}^{m+1} (u'''', v)_i - u'''v|_0^\pi + \sum_{i=1}^{m} u'''v|_{x_i^-}^{x_i^+} + u''v'|_0^\pi - \sum_{i=1}^{m} u''v'|_{x_i^-}^{x_i^+}$$

$$+ \sum_{i=1}^{m} \frac{1}{\theta}_i J[u'](x_i) J[v'](x_i).$$

Since $v \in V$, we have $v(0) = v(\pi) = 0$, and v is continuous on $[0, \pi]$. Therefore the above equality can be rewritten as

$$\sum_{i=1}^{m+1} (u'''' - f, v)_i + \sum_{i=1}^{m} J[u'''](x_i) v(x_i) + u''v'|_0^\pi - \sum_{i=1}^{m} u''v'|_{x_i^-}^{x_i^+} \tag{3.4}$$

$$+ \sum_{i=1}^{m} \frac{1}{\theta}_i J[u'](x_i) J[v'](x_i) = 0.$$

The first sum is zero, since $u'''' = f$ a.e. on l_i, $i = 1, \cdots, m+1$. Next, choose a continuously differentiable $v \in V$, which is not zero only in a small neighborhood of $x = 0$, and $v'(0) \neq 0$. Conclude that $u''(0) = 0$. Similarly, $u''(\pi) = 0$.

Choose a continuously differentiable $v \in V$, such that $v'(x_i) = 0$, and $v(x_i) = 0$ for all $i = 2, \ldots, m$, but $v(x_1) \neq 0$, $v'(x_1) = 0$. Conclude that $J[u'''](x_1) = 0$. Repeat this procedure for other points x_i, one at a time. Thus $J[u'''](x_i) = 0$, $i = 1, \cdots, m$. We are left with

$$\sum_{i=1}^{m} \left[\frac{1}{\theta}_i J[u'](x_i) J[v'](x_i) - u''v'|_{x_i^-}^{x_i^+} \right] = 0. \tag{3.5}$$

Choose a continuously differentiable $v \in V$, which is not zero only in a small neighborhood of x_1, and such that $v'(x_1) \neq 0$. This implies $J[u''](x_1) v'(x_1) = 0$. Therefore $J[u''](x_1) = 0$. Repeat for other points x_i. Thus $u''(x_i^+) = u''(x_i^-)$ for $i = 1, \cdots, m$. Now we can rewrite (3.5) as

$$\sum_{i=1}^{m} \left[\frac{1}{\theta}_i J[u'](x_i) - u''(x_i^+) \right] J[v'](x_i) = 0.$$

Choose a continuous, piecewise linear $v \in V$, such that $v(x_1) = 1$, v is linear on $[0, x_1]$, and on $[x_1, \pi]$. Note that $J[v'](x_i) = 0$ for $i = 2, \cdots, m$, and $J[v'](x_1) \neq 0$. Conclude that $J[u'](x_1) = \theta_1 u''(x_1^+)$. Repeat for other points x_i, $i = 2, \cdots, m$. Thus u satisfies all the conditions (3.2)–(3.3). $\square$

Remark 3.4. The fact that $u'''' = f$ a.e. on $(0,\pi)$ in Theorem 3.3 does not imply that $u \in H^4(0,\pi)$. This is similar to the fact that the strong derivative p' of a step function p on $(0,\pi)$ is zero a.e. on $(0,\pi)$. However, $p \notin H^1(0,\pi)$.

Finally in this section we discuss the eigenvalues and the eigenfunctions of the operator $\mathcal{A}$. It was shown in Lemma 3.2 that $\mathcal{A}$ is a continuous, linear, symmetric, and coercive operator from V onto V'. Following [21, Section 2.2.1], $\mathcal{A}$ can also be considered as an unbounded operator in H. Using Lemma 2.1, the embedding $V \subset H$ is compact. Therefore the standard spectral theory for Sturm-Liouville boundary value problems is applicable. The eigenfunctions belong to H. Therefore, by Theorem 3.3, they are in the domain $D(\mathcal{A}) \subset V$, thus continuous on $[0,\pi]$, and satisfy conditions (3.2)–(3.3).

We summarize these results in the following lemma.

Lemma 3.5. *Let $\mathcal{A}$ be the operator defined in* (3.1)*. Then*

(i) *there exists an increasing sequence of its real positive eigenvalues $\lambda_1^4, \lambda_2^4, \cdots$ with $\lim_{k\to\infty} \lambda_k^4 = \infty$;*

(ii) *the corresponding eigenfunctions $\varphi_k \in D(\mathcal{A}) \subset V$, $k \geq 1$, and they satisfy the junction conditions* (3.2)–(3.3)*;*

(iii) *the eigenfunctions φ_k satisfy $\mathcal{A}\varphi_k = \lambda_k^4 \varphi_k$ in H, $k \geq 1$. That is, $\varphi_k''''(x) = \lambda_k^4 \varphi_k(x)$ a.e. on every interval l_i, $i = 1, \cdots, m+1$;*

(iv) *the set $\{\varphi_k\}_{k=1}^{\infty}$ is a complete orthonormal basis in H.*

Algorithms for a computational determination of the eigenvalues and the eigenfunctions of $\mathcal{A}$ are discussed in [15].

Remark 3.6. If the arch is uniform, i.e. it has no cracks, then the results presented in this section are simplified. Specifically, the spaces V, H, and the operator $\mathcal{A}$ take the following forms

$$V = H^2(0,\pi) \cap H_0^1(0,\pi), \quad H = L^2(0,\pi), \quad \langle \mathcal{A}u, v\rangle_V = (u'', v'')_H, \tag{3.6}$$

for any $u, v \in V$. See [14] for an investigation of this case.

An efficient method for a computational determination of the eigenvalues and the eigenfunctions of $\mathcal{A}$ (Modified Shifrin's method) is discussed in [15].

4. Physical parameters and their non-dimensional equivalents

This is the only section in the paper that uses physical variables. All the other sections use their non-dimensional equivalents. Since both the physical variables and their equivalents use the same notation, such an arrangement helps to avoid confusing them. The physical variables are contained in Table 1 together with their units. Recall that the newton $N = kg \cdot m \cdot s^{-2}$ is the unit of force.

L	Length of bar	m	
T	Final time	s	
y	Bar deflection	m	$y = y(x,t)$
p	Transverse load/unit of length	$N \cdot m^{-1}$	$p = p(x,t)$
c_d	Damping coefficient	$kg \cdot m^{-1}s^{-1}$	
κ	Bar curvature	m^{-1}	
ρ	Volume mass density	$kg \cdot m^{-3}$	
E	Young's modulus	$N \cdot m^{-2}$	
A	Cross-sectional area	m^2	
I	Area moment of inertia	m^4	
r	Radius of gyration	m	$r = \sqrt{I/A}$
ω_0	t-scale	s^{-1}	
ω	Natural frequency	s^{-1}	
k	Spring stiffness	$N \cdot m/rad$	
θ	Flexibility	m	$\theta = EI/k$
T_k	Kinetic energy	$J = N \cdot m$	
U_a	Potential energy for axial force	J	
U_b	Potential energy due to bending	J	
M	Bending moment magnitude	$N \cdot m$	
S_0	Initial tensile axial force	N	
S_1	Tensile force due to deflection	N	
N	Total axial force	N	$N = S_0 + S_1$
$\mathcal{L}$	Lagrangian	J	$\mathcal{L} = T_k - U_b - U_a$
$\mathcal{W}_{ext}$	External work	J	
$\mathcal{W}_d$	Dissipative work	J	
$\mathcal{W}_{nc}$	Non-conservative work	J	$\mathcal{W}_{nc} = \mathcal{W}_d + \mathcal{W}_{ext}$
e_I	Energy normalization factor	J	
β	Axial force renormalization	1	
$\hat{\mu}$	Crack depth ratio	1	
μ	Normalized dynamic viscosity	1	

TABLE 1. Nomenclature

To use the Extended Hamilton's Principle we need to derive the Lagrangian $\mathcal{L} = T_k - U = T_k - U_b - U_a$, where U_b, U_a and T_k are the potential and the kinetic energies. Also, we need the non-conservative external work $\mathcal{W}_{nc}$, which is the sum of the external work $\mathcal{W}_{ext}$ due to the load p, and the dissipative work $\mathcal{W}_d$ due to the damping force $\mathcal{F}_d$.

Let $y(x,t)$, $0 \le x \le L$, be the position of the arch at the time $t \ge 0$. As usual, the dots denote the time derivatives, and the primes denote the spatial ones. For the kinetic energy we have

$$T_k(y) = \frac{\rho A}{2} \int_0^L |\dot{y}(x,t)|^2 \, dx. \tag{4.1}$$

The external work $\mathcal{W}_{ext}$ by the non-conservative load $p(x,t)$ is

$$\mathcal{W}_{ext}(y) = \int_0^L p(x,t) y(x,t) \, dx. \tag{4.2}$$

The dissipative force $\mathcal{F}_d(x,t) = -c_d \dot{y}(x,t)$, $c_d \geq 0$ is a uniformly distributed viscous damping force acting only in the transverse direction. So, the dissipative work $\mathcal{W}_d$ done by the force is given by

$$\mathcal{W}_d(y) = \int_0^L \mathcal{F}_d(x,t) y(x,t) \, dx = -c_d \int_0^L \dot{y}(x,t) y(x,t) \, dx. \tag{4.3}$$

The potential energy is

$$U(y) = U_b(y) + U_a(y), \tag{4.4}$$

where U_b is the potential energy due to the bending, and U_a is the potential energy due to the axial force. The particular form of these terms depends on the presence of cracks and other factors.

To simplify the notations the t-dependency of y is suppressed, if it does not cause a confusion.

Beam. Suppose that we have a uniform beam with no cracks, as shown in Figure 2(a). The magnitude $M(x)$ of the bending moment vector $\vec{M}(x)$ of the beam at any point $x \in (0, L)$ is given by $M(x) = EI\kappa(x)$, where $\kappa(x)$ is the curvature of the beam at x. Assuming $|y'| \ll 1$, we get $\kappa(x) = y''(x)$, and $M(x) = EIy''(x)$. Accordingly, the bending potential energy of the beam is given by

$$U_b(y) = \frac{EI}{2} \int_0^L (y'')^2 \, dx. \tag{4.5}$$

Now suppose that the beam has cracks as in Section 1, and in Figure 2(b). The cracks are at $0 < x_1 < \cdots < x_m < L$. The standard approach to modeling a crack is to represent it as a massless rotational spring with the spring constant k, and the flexibility θ.

The spring constant k relates the torque $(N \cdot m)$ to the angle of rotation. In our case this relationship takes the form $EIy''(x) = kJ[y'](x)$, or $J[y'](x) = \theta y''(x)$, where

$$\theta = \frac{EI}{k}. \tag{4.6}$$

Thus the unit of the flexibility θ is m.

If the beam has a rectangular cross-section, as shown in Figure 1, then the area moment of inertia I of the rectangle can be computed explicitly, and (4.6) can be simplified further. If the crack is double-sided, then by [19, Eq. (2.8)-(2.10)], the expression for the flexibility θ becomes

$$\theta = 6\pi H\hat{\mu}^2(0.535 - 0.929\hat{\mu} + 3.500\hat{\mu}^2 - 3.181\hat{\mu}^3 + 5.793\hat{\mu}^4), \tag{4.7}$$

where H is the half-height of the beam cross-section, and $\hat{\mu} = a/H$.

If the crack is single-sided, then by [19, Eq. (2.8)-(2.10)]

$$\theta = 6\pi H\hat{\mu}^2(0.6384 - 1.035\hat{\mu} + 3.7201\hat{\mu}^2 - 5.1773\hat{\mu}^3 + 7.553\hat{\mu}^4 - 7.332\hat{\mu}^5), \tag{4.8}$$

where H is the entire height of the beam cross-section, and $\hat{\mu} = a/H$.

The potential energy of the rotational spring with the spring constant k is

$$U_{crack} = \frac{1}{2}k\alpha^2,$$

where α is the angle of twist from its equilibrium position in radians. Since $J[y'](x) \approx \alpha$ for small jumps in the slope of the beam, we conclude that the total bending potential energy of the cracked beam with m cracks is

$$U_b(y) = \frac{EI}{2}\int_0^L (y'')^2\,dx + \frac{EI}{2}\sum_{i=1}^m \frac{1}{\theta_i}|J[y'](x_i)|^2. \tag{4.9}$$

Shallow arch. First, consider the uniform case. Following [22], the axial force N in the uniform arch shown in Figure 2(a) is represented as the sum of two components $N = S_0 + S_1$, where S_0 is the initial axial tensile force, and S_1 is the axial tensile force due to deflection. That is, the force is positive if it is tensile, and negative if it is compressing. The value of S_0 is assumed to be given, and the unknown force S_1 can be found through the deflection $y = y(x)$ as follows.

Let ΔL be the elongation of the arch due to the deflection. By the definition of the Young's modulus E

$$S_1 = EA\frac{\Delta L}{L}. \tag{4.10}$$

The elongation is

$$\begin{aligned}\Delta L &= \int_0^L \sqrt{1 + |y'(x)|^2}\,dx - L \qquad (4.11)\\ &\approx \int_0^L \left(1 + \frac{1}{2}|y'(x)|^2\right)dx - L = \frac{1}{2}\int_0^L |y'(x)|^2\,dx.\end{aligned}$$

Then

$$S_1 = \frac{EA}{2L}\int_0^L |y'(x)|^2\,dx. \tag{4.12}$$

The potential energy U_a due to the axial force N is

$$U_a(y) = \frac{EA}{2L}(\Delta L^*)^2, \tag{4.13}$$

where $\Delta L^* = \frac{NL}{EA}$ is the elongation of the arch caused by the total axial force $N = S_0 + S_1$. Therefore

$$U_a(y) = \frac{L}{2EA}(S_0 + S_1)^2 = \frac{L}{2EA}\left(S_0 + \frac{EA}{2L}\int_0^L |y'(x)|^2\, dx\right)^2. \tag{4.14}$$

Now suppose that the arch has cracks as in Section 1, and in Figure 2(b). To derive its axial potential energy $U_a(y)$, note that there exists a sequence of smooth functions u_n, such that $u_n \to y$ in $H_0^1(0, L)$, as $n \to \infty$. For such functions, the potential energy $U_a(u_n)$ is given by the expression in (4.14), i.e.

$$U_a(u_n) = \frac{L}{2EA}\left(S_0 + \frac{EA}{2L}\int_0^L |u_n'(x)|^2\, dx\right)^2, \quad n = 1, 2, \cdots \tag{4.15}$$

Because of the continuity of this functional on $H_0^1(0, L)$, we conclude that we can pass to the limit in (4.15), as $n \to \infty$, to get

$$U_a(y) = \frac{L}{2EA}\left(S_0 + \frac{EA}{2L}\int_0^L |y'(x)|^2\, dx\right)^2, \tag{4.16}$$

that is, the same expression as (4.14), even for an arch with cracks.

Non-dimensional variables. Now we find the non-dimensional equivalents for the above physical quantities. Define the t-scale ω_0, and the radius of gyration r by

$$\omega_0 = \left(\frac{\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}}, \quad r = \sqrt{\frac{I}{A}}. \tag{4.17}$$

Then make the change of variables

$$x \leftarrow \frac{\pi x}{L}, \quad y \leftarrow \frac{y}{r}, \quad p \leftarrow \frac{p}{EIr}\left(\frac{L}{\pi}\right)^4, \quad t \leftarrow \omega_0 t, \quad c_d \leftarrow \frac{c_d}{\rho A \omega_0}. \tag{4.18}$$

Temporarily distinguish the notation for the original physical variables by assigning them the 0 subscript, and their non-dimensional equivalents by assigning them the n subscript. For example, (4.18) transformation for t says that $t_n = \omega_0 t_0$. Then

$$y_0(x_0, t_0) = r y_n(x_n, t_n) = r y_n\left(\frac{\pi x_0}{L}, \omega_0 t_0\right). \tag{4.19}$$

Therefore,

$$y_0'(x_0, t_0) = r\frac{\pi}{L} y_n'(x_n, t_n), \quad y_0''(x_0, t_0) = r\left(\frac{\pi}{L}\right)^2 y_n''(x_n, t_n) \tag{4.20}$$

and

$$\dot{y}_0(x_0,t_0) = r\omega_0\dot{y}_n(x_n,t_n), \quad \ddot{y}_0(x_0,t_0) = r\omega_0^2\ddot{y}_n(x_n,t_n). \tag{4.21}$$

Let the energy normalization factor e_I be defined by

$$e_I = \frac{L}{\pi}\rho A r^2\omega_0^2 = \frac{L}{\pi}\rho I\omega_0^2 = \left(\frac{\pi}{L}\right)^3 r^2 EI. \tag{4.22}$$

Then the kinetic energy $T_{k,o}$ from (4.1) is transformed according to

$$\begin{aligned} T_{k,0}(y_0) &= \frac{\rho A}{2}\int_0^L |\dot{y}_0(x_0,t_0)|^2\,dx_0 = \frac{L}{\pi}\rho A r^2\omega_0^2\frac{1}{2}\int_0^\pi |\dot{y}_n(x_n,t_n)|^2\,dx_n \\ &= e_I\frac{1}{2}|\dot{y}_n|_H^2 = e_I T_{k,n}(y_n). \end{aligned} \tag{4.23}$$

Consistent with the definition of the flexibility θ in (4.6), we use the transformation $\theta \leftarrow \pi\theta/L$. Therefore, for the bending potential energy (4.9) we have

$$\begin{aligned} U_{b,o}(y_o) &= \frac{EI}{2}\int_0^L |y_o''(x_o)|^2\,dx_o + \frac{EI}{2}\sum_{i=1}^m \frac{1}{\theta_{i,o}}|J[y_o'](x_{i,o})|^2 \\ &= \left(\frac{\pi}{L}\right)^3 r^2 EI\left[\frac{1}{2}\int_0^\pi |y_n''(x_n)|^2\,dx_n + \frac{1}{2}\sum_{i=1}^m \frac{1}{\theta_{i,n}}|J[y_n'](x_{i,n})|^2\right] \\ &= e_I U_{b,n}(y_n). \end{aligned} \tag{4.24}$$

The potential energy U_a due to the axial force is given by (4.16). After the substitution (4.18) we get

$$\begin{aligned} U_{a,0}(y_0) &= \frac{L}{2EA}\left(S_0 + \frac{EA}{2L}r^2\frac{\pi}{L}\int_0^\pi |y_n'(x_n)|^2\,dx_n\right)^2 \\ &= \frac{L}{2EA}\frac{(EA)^2\pi^2r^2r^2}{L^4}\left(\frac{L^2S_0}{EA\pi r^2} + \frac{1}{2}\int_0^\pi |y_n'(x_n)|^2\,dx_n\right)^2 \\ &= \frac{e_I}{2\pi}\left(\beta + \frac{1}{2}\int_0^\pi |y_n'(x_n)|^2\,dx_n\right)^2, \end{aligned}$$

where the non-dimensional $\beta \in \mathbb{R}$ is a renormalization of the axial force S_0.

Similarly, we conclude that $\mathcal{W}_{ext}$, and $\mathcal{W}_d$ are transformed by (4.18) into their non-dimensional equivalents in the same way:

$$T_k \leftarrow \frac{T_k}{e_I}, \quad U_a \leftarrow \frac{U_a}{e_I}, \quad U_b \leftarrow \frac{U_b}{e_I}, \quad \mathcal{W}_{ext} \leftarrow \frac{\mathcal{W}_{ext}}{e_I}, \quad \mathcal{W}_d \leftarrow \frac{\mathcal{W}_d}{e_I}. \tag{4.25}$$

Therefore, the expressions for the non-dimensional quantities are

$$T_k(y) = \frac{1}{2}|\dot{y}|_H^2, \quad \mathcal{W}_{ext}(y) = (p,y)_H, \quad \mathcal{W}_d(y) = -c_d(\dot{y},y)_H, \tag{4.26}$$

$$U_b(y) = \frac{1}{2}\left(|y''|_H^2 + \sum_{i=1}^{m} \frac{1}{\theta_i} |J[y'](x_i)|^2 \right), \tag{4.27}$$

$$U_a(y) = \frac{1}{2\pi}\left(\beta + \frac{1}{2}|y'|_H^2 \right)^2, \tag{4.28}$$

where $\beta \in \mathbb{R}$.

Natural beam frequencies.

Equation of harmonic transverse oscillations $v = v(x)$ of a uniform beam defined on interval $(0, L)$ is

$$EIv''''(x) = \omega^2 \rho A v(x), \quad 0 < x < L. \tag{4.29}$$

For a cracked beam, equation (4.29) is satisfied on every subinterval (x_{i-1}, x_i), $i = 1, \cdots, m+1$.

Using the transformations to the non-dimensional variables (4.17)–(4.21), we get

$$EIr\left(\frac{\pi}{L}\right)^4 v_n''''(x_n, t_n) = \omega^2 \rho A r v_n(x_n, t_n),$$

where $v_n(x_n, t_n)$ is $v(x, t)$ in the new (non-dimensional) variables. Dropping the subscript n, we obtain the equation for v in the non-dimensional quotients

$$v'''' = \omega^2 \left(\frac{L}{\pi}\right)^4 \frac{\rho A}{EI} v.$$

Comparing this equation with the definition of the eigenvalues and the eigenfunctions $\varphi_k'''' = \lambda_k^4 \varphi_k$, we conclude that the natural beam frequencies are given by

$$\omega_k = \lambda_k^2 \left(\frac{\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}}, \quad k \geq 1. \tag{4.30}$$

5. Convex functions and subdifferentials

Subdifferentials provide the proper mathematical framework for the abstract formulation of equations of motion. Following [3, Section 1.2], let X be a Hilbert space. A function $\phi : X \to (-\infty, +\infty]$ is called *proper and convex* on X, if ϕ is not identically $+\infty$, and

$$\phi((1-\lambda)x + \lambda y) \leq (1-\lambda)\phi(x) + \lambda\phi(y), \tag{5.1}$$

for any $x, y \in X$, and $\lambda \in [0, 1]$. The function ϕ is called *lower-semicontinuous* on X, if every level set $\{x \in X : \phi(x) \leq c\}$, $c > -\infty$, is closed in X.

Given a proper, convex, lower-semicontinuous function ϕ on X, the *subdifferential* $\partial\phi : X \to X'$ is defined by

$$\partial\phi(x) = \{x^* \in X' \ : \ \phi(y) \geq \phi(x) + \langle x^*, y - x\rangle\}, \tag{5.2}$$

for any $y \in X$. Thus $\partial\phi \subset X \times X'$.

In general, $\partial\phi$ does not have to be defined everywhere on X. Furthermore, the mapping $x \to \partial\phi(x) \subset X'$ can be multi-valued. Geometrically, if $\partial\phi(x)$ is single-valued at $x \in X$, then $y \to \phi(x) + \langle \partial\phi(x), y - x\rangle$ is the tangent plane to the graph of ϕ at x.

Let $D(\phi) = \{x \in X : \phi(x) < \infty\}$, and $D(\partial\phi) = \{x \in X : \partial\phi(x) \neq \emptyset\}$. Note that if $x \in D(\partial\phi)$, then $x \in D(\phi)$. Indeed, ϕ is a proper function. Therefore $D(\phi) \neq \emptyset$. Choose $y \in D(\phi)$, and $x^* \in \partial\phi(x)$. Then $\phi(x) \leq \phi(y) - \langle x^*, y - x\rangle < \infty$, as claimed.

Let $f : X \to \mathbb{R}$. The *directional derivative* $f'(x; y)$ of f at $x \in X$ in the direction $y \in X$ is defined by

$$f'(x; y) = \lim_{\alpha \to 0+} \frac{f(x + \alpha y) - f(x)}{\alpha}. \tag{5.3}$$

A function f is said to be Gâteaux differentiable at $x \in X$, if there exists $(\nabla f)(x) \in X'$, such that

$$f'(x; y) = \langle \nabla f(x), y\rangle, \tag{5.4}$$

for any $y \in X$. The linear functional $(\nabla f)(x)$ is called the Gâteaux derivative of f at $x \in X$.

If the convergence in (5.3) is uniform in y on bounded subsets, then f is said to be *Fréchet differentiable.*

The Gâteaux differentiability of f is a more restrictive condition, than f is having a subdifferential. The following lemma is proved in [3, Section 1.2]:

Lemma 5.1. *If ϕ is convex and Gâteaux differentiable at $x \in X$, then $\partial\phi(x) = \nabla\phi(x)$ and $\partial\phi(x)$ is a singleton.*

Recall that an operator $A : X \to X'$ is called symmetric, if $\langle Au, v\rangle = \langle Av, u\rangle$, for any $u, v \in D(A)$.

Theorem 5.2. *Let X be a Hilbert space, and $A : X \to X'$ be a linear, continuous, and symmetric operator such that $D(A) = X$, and $\langle Au, u\rangle \geq 0$ for any $u \in X$. Then function $\phi : X \to \mathbb{R}$ defined by*

$$\phi(u) = \frac{1}{2}\langle Au, u\rangle, \quad u \in X, \tag{5.5}$$

is convex, proper and lower-semicontinuous on X. Moreover, it is Fréchet differentiable on X with $\nabla\phi(u) = \partial\phi(u) = Au$ for any $u \in X$, and $D(\phi) = D(\partial\phi) = X$.

Proof. To see that φ is convex, let $0 \leq \lambda \leq 1$, and $u, v \in X$. Then

$$\begin{aligned}\phi(\lambda u + (1-\lambda)v) &= \frac{1}{2}\left[\lambda^2\langle Au, u\rangle + 2\lambda(1-\lambda)\langle Au, v\rangle + (1-\lambda)^2\langle Av, v\rangle\right] \\ &= \frac{1}{2}\left[\lambda\langle Au, u\rangle + (1-\lambda)\langle Av, v\rangle - \lambda(1-\lambda)\langle A(u-v), (u-v)\rangle\right] \\ &\leq \lambda\phi(u) + (1-\lambda)\phi(v).\end{aligned}$$

Since A is continuous, function ϕ is continuous on X. In particular, it is lower-semicontinuous on X. From the continuity of A, we have $\|Av\|_{X'} \leq C\|v\|_X$, for some $C > 0$. Since

$$\phi(v) - \phi(u) - \langle Au, v-u\rangle = \frac{1}{2}\langle A(v-u), v-u\rangle,$$

we conclude that

$$|\phi(v) - \phi(u) - \langle Au, v-u\rangle| \leq \frac{C}{2}\|v-u\|_X^2. \tag{5.6}$$

Therefore $Au = (\nabla\phi)(u)$ is the Gâteaux (even Fréchet) derivative of ϕ at $u \in X$, and $\partial\phi = A$ is its single-valued subdifferential, with $D(\partial\phi) = D(\varphi) = X$. □

Theorem 5.3. *Let V be the Hilbert space defined in* (2.3), *and $\mathcal{A}$ be the linear operator defined in* (3.1). *Let*

$$\varphi(u) = \frac{1}{2}\langle\mathcal{A}u, u\rangle, \quad u \in V. \tag{5.7}$$

(i) *Then $D(\partial\varphi) = V$, and $\partial\varphi(u) = \mathcal{A}u$, for any $u \in V$.*
(ii) *If $u \in D(\mathcal{A}) \subset V$, then $\partial\varphi(u) = u''''$ a.e. on every interval l_i, $i = 1, \cdots, m+1$.*
(iii) *If $u \in H_0^1(0,\pi) \cap H^4(0,\pi) \subset D(\mathcal{A})$, then $\partial\varphi(u) = u''''$ a.e. on $[0,\pi]$.*

Proof. By Theorem 5.2 with $X = V$, function $\varphi : V \to \mathbb{R}$ is proper, convex, and lower-semicontinuos on V, and (i) follows. For a general $u \in V$, the expression for $\partial\varphi(u) \in V'$ is complicated. However, if we assume that u is somewhat more regular, then we can get a simpler expression for it.

Suppose that condition (ii) is satisfied. Then, by Theorem 3.3, we have $\mathcal{A}u = u''''$ a.e. on every interval l_i, $i = 1, \cdots, m+1$. Thus, we can say that $\partial\varphi(u) = u''''$ a.e. on every such interval.

Suppose further, that $u \in H_0^1(0,\pi) \cap H^4(0,\pi) \subset D(\mathcal{A})$. Then $u'''' \in L^2(0,\pi)$, and u' is smooth. Thus $J[u'](x_i) = 0$ for any $i = 1, \cdots, m$, and we have

$$\langle\partial\varphi(u), v\rangle = \langle\mathcal{A}u, v\rangle = \int_0^\pi u''(x)v''(x)\,dx = \int_0^\pi u''''(x)v(x)\,dx, \tag{5.8}$$

for any $v \in V$. Therefore, in this case $\partial\varphi(u) = u''''$ a.e. on $[0,\pi]$. □

Theorem 5.4. *Let $H_0^1 = H_0^1(0,\pi)$ be the Hilbert space defined in (2.13) and $\langle\cdot,\cdot\rangle_1$ be the duality pairing between H_0^1 and $(H_0^1)'$. Let $\mathcal{B}$ be the linear operator defined by*

$$\langle \mathcal{B}u, v\rangle_1 = (u', v')_H, \quad u, v \in H_0^1 \tag{5.9}$$

and

$$\psi(u) = \frac{1}{2}\langle \mathcal{B}u, u\rangle, \quad u \in H_0^1. \tag{5.10}$$

(i) *Then $D(\partial\psi) = H_0^1$ and $\partial\psi(u) = \mathcal{B}u \in (H_0^1)'$, for any $u \in H_0^1$.*

(ii) *If $u \in V \subset H_0^1$, then*

$$\partial\psi(u) = \mathcal{B}u = -\sum_{i=1}^{m} J[u'](x_i)\delta(x - x_i) - u'', \tag{5.11}$$

where $\delta(x-a)$, $a \in [0,\pi]$ is the element of $(H_0^1)'$, defined by $\langle\delta(x - a), v\rangle_1 = v(a)$, for any $v \in H_0^1$.

(iii) *If $u \in D(\mathcal{A}) \subset V$, then*

$$\partial\psi(u) = \mathcal{B}u = -\sum_{i=1}^{m} \theta_i u''(x_i)\delta(x - x_i) - u'', \tag{5.12}$$

which is still an element of $(H_0^1)'$.

Proof. By the definition, the operator $\mathcal{B} : H_0^1 \to (H_0^1)'$ is continuous, symmetric and coercive on H_0^1. In particular, $\mathcal{B}$ is positive, and its range is $(H_0^1)'$. Theorem 5.2 is applicable with $X = H_0^1$, and $A = \mathcal{B}$. We conclude that the function $\psi : H_0^1 \to \mathbb{R}$ is proper, convex, and lower-semicontinuos on H_0^1. Furthermore, $D(\partial\psi) = H_0^1$, and $\partial\psi(u) = \mathcal{B}u \in (H_0^1)'$, for any $u \in H_0^1$, as claimed in (i).

As in Theorem 5.3, a simpler expression for the subdifferential $\partial\psi(u)$ can be obtained assuming an additional regularity of $u \in H_0^1$. Suppose that $u \in V \subset H_0^1$. Then we have

$$\begin{aligned}\langle \mathcal{B}u, v\rangle_1 = (u', v')_H &= \int_0^\pi u'(x)v'(x)\,dx \\ &= -\sum_{i=1}^{m} J[u'(x_i)]v(x_i) - \sum_{i=1}^{m+1}(u'', v)_i,\end{aligned} \tag{5.13}$$

for any $v \in H_0^1$. Therefore, in this case, we have (5.11).

Suppose further, that $u \in D(\mathcal{A}) \subset V$. Then, by Theorem 3.3, u satisfies conditions (3.2)–(3.3). In particular, $J[u'](x_i) = \theta_i u''(x_i)$. Thus, with this additional assumption on u, we have (5.12). □

6. Extended Hamilton's principle

To derive the governing equations for beams and arches, we use the Extended Hamilton's Principle, which accommodates non-conservative forces, see [17].

The Principle states that the motion $y = y(x,t)$ of the system gives a stationary value to the *action* of the system, that is, to the integral

$$I(y) = \int_{t_1}^{t_2} (\mathcal{L} + \mathcal{W}_{nc})\, dt, \tag{6.1}$$

where the Lagrangian $\mathcal{L} = T_k - U$ is the difference between the kinetic energy T_k and the potential energy $U = U_b + U_a$. The term $\mathcal{W}_{nc} = \mathcal{W}_d + \mathcal{W}_{ext}$ for the non-conservative external work $\mathcal{W}_{nc}$ is the sum of the external work $\mathcal{W}_{ext}$, due to the load p, and the dissipative work $\mathcal{W}_d$, due to a damping force $\mathcal{F}_d$. Thus (6.1) becomes

$$I(y) = \int_{t_1}^{t_2} \left(T_k(y) - U_b(y) - U_a(y) + \mathcal{W}_d(y,\dot{y}) + \mathcal{W}_{ext}(y)\right)\, dt, \tag{6.2}$$

where the components of the action are given in (4.26)–(4.28).

The traditional way of expressing the fact that the motion of the system gives a stationary value to the functional I defined in (6.2), is to say that its *variation* $\delta I = 0$. Let us examine this statement in more detail within the framework of the Hilbert spaces.

Recall that the Hilbert spaces V, H_0^1, and H were defined in Section 2. Define

$$W = \{y \ : \ y \in L^2(0,T;V),\ \dot{y} \in L^2(0,T;H),\ \ddot{y} \in L^2(0,T;V')\},$$

where the derivatives are taken in the sense of distributions, [21, Chapter 2]. This is a Hilbert space with the standard inner product and the norm.

Given $\eta \in W$, the directional derivative $I'(y;\eta)$ is defined by

$$I'(y;\eta) = \lim_{\alpha \to 0+} \frac{I(y+\alpha\eta) - I(y)}{\alpha}, \tag{6.3}$$

which is consistent with (5.3).

The Gâteaux derivative $\nabla I(y)$ (if exists) is an element of the dual W' such that

$$I'(y;\eta) = \langle \nabla I(y), \eta \rangle_W, \tag{6.4}$$

for any $\eta \in W$, see Section 5, and [3, Section 1.2].

Functional I has a stationary value at y means that $\nabla I(y) = 0$, which gives the meaning to the infinitesimal variation equation $\delta I = 0$.

To derive the governing equation, one has to transform the stationary value equation $\nabla I(y) = 0$ into a more explicit form by computing the directional derivatives of I along the specially chosen $\eta \in W$. Doing so, we arrive at the following main result.

Theorem 6.1. *Equation*

$$\ddot{y} + \partial U_b(y) + \partial U_a(y) + c_d \dot{y} = p, \tag{6.5}$$

is the abstract governing equation for beams and arches in V', a.e. for $t \in [0, T]$. Note: $\partial U_b : V \to V'$, and $\partial U_a : H_0^1 \to (H_0^1)'$.

Proof. Let $\eta(x,t) = \zeta(t)v(x)$, where $\zeta : [t_1, t_2] \to \mathbb{R}$ is a smooth function satisfying $\zeta(t_1) = \zeta(t_2) = 0$, and $v \in V$.

To obtain $\nabla I(y)$, we compute the corresponding directional derivatives for every term in (6.2) for the same $\eta = \zeta v$. First, notice that

$$T_k(y + \alpha\eta) - T_k(y) = \alpha(\dot{y}, v)_H \dot{\zeta} + \frac{\alpha^2}{2} |\dot{\zeta}|^2 \, |v|_H^2. \tag{6.6}$$

By the definition of the directional derivative,

$$\int_{t_1}^{t_2} T_k'(y; \eta)\, dt = \int_{t_1}^{t_2} (\dot{y}, v)_H \dot{\zeta}\, dt.$$

Integrating by parts in t, and using $\zeta(t_1) = \zeta(t_2) = 0$, we get

$$\int_{t_1}^{t_2} T_k'(y; \eta)\, dt = \int_{t_1}^{t_2} (\dot{y}(t), v)_H \dot{\zeta}(t)\, dt = -\int_{t_1}^{t_2} \langle \ddot{y}(t), v \rangle_V \zeta(t)\, dt. \tag{6.7}$$

Similarly, we have

$$\int_{t_1}^{t_2} \mathcal{W}_{ext}'(y; \eta)\, dt = \int_{t_1}^{t_2} (p(t), v)_H \zeta(t)\, dt, \tag{6.8}$$

where the dependency of $p = p(x,t)$ on x is suppressed.

The work $\mathcal{W}_d$ done by the non-conservative force $\mathcal{F}_d$ is not covered by the classical Hamilton's Principle. To accommodate such a case, the Principle is extended by introducing the Rayleigh's dissipation functional, for which the action is *defined* in a non-variational manner, see [17] for details. In our case it implies that

$$\int_{t_1}^{t_2} \mathcal{W}_d'(y; \eta)\, dt = -c_d \int_{t_1}^{t_2} (\dot{y}(t), v)_H \zeta(t)\, dt. \tag{6.9}$$

The expressions for the potential energies U_b and U_a depend on whether the system is a beam or an arch, and on the presence of cracks. They are given in (4.27)–(4.28). In every case, $U_b(u)$ is a convex, lower-semicontinuous function on V, and $U_a(u)$ is a convex, lower-semicontinuous function on H_0^1. Therefore they admit subdifferentials ∂U_b and ∂U_a, on V and H_0^1 correspondingly, see Section 5.

If U_b and U_a are Gateaux differentiable, then the subdifferentials are equal to their Gateaux derivatives. Thus, we have for $\eta = \zeta v$

$$\int_{t_1}^{t_2} U_b'(y;\eta)\,dt = \int_{t_1}^{t_2} \langle \partial U_b(y(t)), v\rangle_V \zeta(t)\,dt, \tag{6.10}$$

$$\int_{t_1}^{t_2} U_a'(y;\eta)\,dt = \int_{t_1}^{t_2} \langle \partial U_a(y(t)), v\rangle_1 \zeta(t)\,dt. \tag{6.11}$$

The stationary value equation $\nabla I(y) = 0$, for $I(y)$ given by (6.2), implies

$$\begin{aligned} 0 &= (\nabla I(y), \eta)_W = I'(y;\eta) \\ &= \int_{t_1}^{t_2} \left[(-c_d\dot{y} + p(t), v)_H - \langle \ddot{y} + \partial U_b(y(t)), v\rangle_V - \langle \partial U_a(y(t)), v\rangle_1\right] \zeta(t)\,dt. \end{aligned} \tag{6.12}$$

Since ζ is an arbitrary smooth function on $[t_1, t_2]$, we get

$$(-c_d\dot{y} + p(t), v)_H - \langle \ddot{y} + \partial U_b(y(t)), v\rangle_V - \langle \partial U_a(y(t)), v\rangle_1 = 0,$$

for any $v \in V$, and almost any $t \in [0, T]$. This can further be written as

$$\langle -\ddot{y} - \partial U_b(y(t)) - \partial U_a(y(t)) - c_d\dot{y} + p(t), v\rangle_V = 0. \tag{6.13}$$

Thus (6.13) implies equation (6.5). □

7. Beam equation

In this section we use Theorem 6.1 to derive the governing equations for uniform and cracked beams. The strong damping case is also considered.

Uniform beam. Suppose that we have a uniform beam with no cracks, as shown in Figure 2(a). The classical Euler-Bernoulli beam theory [16] assumes that the beam motion is due mainly to its bending, while the influence of the axial force is negligible. Thus we let $U_a = 0$.

By (4.27) the bending potential energy of the uniform beam is given by

$$U_b(y) = \frac{1}{2}|y''|_H^2. \tag{7.1}$$

Define operator $\mathcal{A} : V \to V'$ by

$$\langle \mathcal{A}u, v\rangle = (u'', v'')_H, \tag{7.2}$$

for any $u, v \in V$. The operator $\mathcal{A}$ is linear, symmetric, and coercive on V. Then

$$U_b(u) = \frac{1}{2}\langle \mathcal{A}u, u\rangle, \tag{7.3}$$

for any $u \in V$. By Theorem 5.2, U_b is a convex, lower-semicontinuous function on V. Furthermore, $\partial U_b(u) = \mathcal{A}u$. From (6.5), the abstract Uniform Beam equation in V' is

$$\ddot{y} + \mathcal{A}y + c_d\dot{y} = p. \tag{7.4}$$

Assume that $u \in D(\mathcal{A})$. By Theorem 5.3, $\mathcal{A}u = u''''$ for such u. Therefore, with this assumption, equation (7.4) can be written as

$$\ddot{y} + y'''' + c_d\dot{y} = p, \tag{7.5}$$

which is the classical Euler-Bernoulli form of the Beam equation for beams without cracks. To obtain this equation in physical variables, reverse the substitutions in Section 4.

Beam with cracks. Now suppose that the beam has cracks as in Section 1, and in Figure 2(b). Continuing with the classical Euler-Bernoulli beam theory we disregard the axial force, and let $U_a = 0$.

By (4.27), the bending potential energy of the cracked beam with m cracks is

$$U_b(y) = \frac{1}{2}\left(|y''|_H^2 + \sum_{i=1}^{m} \frac{1}{\theta}_i |J[y'](x_i)|^2\right). \tag{7.6}$$

Arguing as in Theorem 5.3, we get that U_b is a convex, lower-semicontinuous function on V.

Define operator $\mathcal{A} : V \to V'$ as in equation (3.1), that is,

$$\langle \mathcal{A}u, v\rangle = \sum_{i=1}^{m+1} (u'', v'')_i + \sum_{i=1}^{m} \frac{1}{\theta}_i J[u'](x_i)J[v'](x_i), \tag{7.7}$$

for any $u, v \in V$. By Lemma 3.2, the operator $\mathcal{A}$ is linear, symmetric, and coercive on V. Then

$$U_b(u) = \frac{1}{2}\langle \mathcal{A}u, u\rangle, \tag{7.8}$$

for any $u \in V$.

By Theorem 5.2, $\partial U_b(u) = \mathcal{A}u$. Therefore the abstract equation for the beam with cracks is

$$\ddot{y} + \mathcal{A}y + c_d\dot{y} = p, \tag{7.9}$$

which is satisfied in V', a.e. for $t \in [0, T]$.

Furthermore, using Theorem 5.3, if $u \in D(\mathcal{A})$, then u satisfies the boundary conditions of the problem, i.e. (3.2) and (3.3), as well as $\mathcal{A}u = u''''$ a.e. on every interval l_i, $i = 1, \cdots, m+1$. Then equation (7.9) can be written as

$$\ddot{y} + y'''' + c_d\dot{y} = p. \tag{7.10}$$

We can call it the classical Beam equation for beams with cracks. While this equation looks the same as the Beam equation for beams with no cracks (7.5), its abstract formulation (7.9) uses the operator $\mathcal{A}$ defined by (7.7), rather than by (7.2). In particular, $y(\cdot, t) \in V$, a.e. $t \in [0, T]$.

Strong damping. Viscous effects on the beam and arch motion are discussed in [2, 11]. Considerations based on the Voigt model for viscoelasticity result in the additional term $\mu \mathcal{A} \dot{y}$ in the governing equations. Here $\mu > 0$ is a non-dimensional normalized dynamic viscosity coefficient.

If such a term is present, we refer to the model as having the *strong damping*. Otherwise, if $\mu = 0$, the model is for the *weak damping*. In particular, equations (7.9) and (7.10) describe the weak beam damping motion case. The corresponding non-dimensional abstract and classical equations in the presence of the strong damping $\mu > 0$ are

$$\ddot{y} + \mathcal{A} y + \mu \mathcal{A} \dot{y} + c_d \dot{y} = p \tag{7.11}$$

and

$$\ddot{y} + y'''' + \mu \dot{y}'''' + c_d \dot{y} = p. \tag{7.12}$$

8. Arch equation

In this section Theorem 6.1 is used to derive the governing equations for uniform and cracked arches.

Uniform shallow arch. The potential energy for the arch contains the term U_a, due to the axial force. By (4.28),

$$U_a(y) = \frac{1}{2\pi} \left(\beta + \frac{1}{2} |y'|_H^2 \right)^2 . \tag{8.1}$$

To compute the subdifferential of U_a, note that for $u \in V$,

$$\partial U_a(u) = \frac{1}{\pi} \left(\beta + \frac{1}{2} |y'|_H^2 \right) \partial \psi(u), \tag{8.2}$$

where ψ is

$$\psi(u) = \frac{1}{2} |u'|_H^2 = \frac{1}{2} \langle \mathcal{B} u, u \rangle, \quad u \in V, \tag{8.3}$$

see Theorem 5.4. Since there are no cracks in the arch, we have $\partial \psi(u) = -u''$, per (5.11).

Using the subdifferential $\partial U_b(u) = \mathcal{A} u$, for $\mathcal{A}$ defined by (7.2), the abstract form of the uniform shallow arch equation is

$$\ddot{y} + \mathcal{A} y - \frac{1}{\pi} \left(\beta + \frac{1}{2} |y'|_H^2 \right) y'' + c_d \dot{y} = p, \tag{8.4}$$

in V', a.e. on $[0,T]$.

Its classical form is

$$\ddot{y} + y'''' - \frac{1}{\pi}\left(\beta + \frac{1}{2}\int_0^\pi |y'(x,t)|^2\,dx\right)y'' + c_d\dot{y} = p, \tag{8.5}$$

cf. [22, eq. (6)].

Arch with cracks. Now suppose that the arch has cracks as in Section 1, and in Figure 2(b). Its axial potential energy $U_a(y)$ has the same expression as (8.1), even if the arch has cracks.

However, the subdifferential $\partial U_a(u)$ is different from the one in the smooth case. It has been computed in Theorem 5.4 as

$$\partial\psi(u) = \mathcal{B}u = -\sum_{i=1}^m J[u'](x_i)\delta(x-x_i) - u'', \tag{8.6}$$

for $u \in V$. The bending potential U_b is given by (7.6). Then its subdifferential $\partial U_b(u) = \mathcal{A}u$, and we get from (6.5) that

$$\ddot{y} + \mathcal{A}y - \frac{1}{\pi}\left(\beta + \frac{1}{2}|y'|_H^2\right)\left(\sum_{i=1}^m J[y'](x_i)\delta(x-x_i) - y''\right) + c_d\dot{y} = p, \tag{8.7}$$

is the abstract equation for a shallow arch with cracks in V', a.e. $t \in [0,T]$.

Then, assuming that the function y is smooth as discussed in Theorem 5.4, we can use (5.12) for the subdifferential $\partial U_a(u)$, and $\partial U_b(u) = \mathcal{A}u = u''''$. This results in

$$\ddot{y} + y'''' - \frac{1}{\pi}\left(\beta + \frac{1}{2}|y'|_H^2\right)\left(\sum_{i=1}^m \theta_i y''(x_i,t)\delta(x-x_i) - y''\right) + c_d\dot{y} = p, \tag{8.8}$$

which can be called the "classical" form of the shallow arch equation with cracks. These equations are also referred to as describing the weak arch damping motion.

Strong damping. Equations (8.7) and (8.8) describe the weak arch damping $\mu = 0$ motion case. The corresponding non-dimensional abstract and "classical" equations in the presence of the strong damping $\mu > 0$ are

$$\ddot{y}+\mathcal{A}y+\mu\mathcal{A}\dot{y}-\frac{1}{\pi}\left(\beta + \frac{1}{2}|y'|_H^2\right)\left(\sum_{i=1}^m J[y'](x_i)\delta(x-x_i) - y''\right)+c_d\dot{y} = p \tag{8.9}$$

and

$$\ddot{y} + y'''' + \mu\dot{y}'''' - \frac{1}{\pi}\left(\beta + \frac{1}{2}|y'|_H^2\right)\left(\sum_{i=1}^{m}\theta_i y''(x_i,t)\delta(x-x_i) - y''\right) + c_d\dot{y} = p, \tag{8.10}$$

see Section 7.

Acknowledgments: This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2020R1I1A1A01065032). And this paper was supported by Education and Research promotion program of the KOREATECH in 2021.

References

[1] J.M. Ball, *Initial-boundary value problems for an extensible beam*, J. Math. Anal. Appl., **42**(1) (1973), 61-90.
[2] J.M. Ball, *Stability theory for an extensible beam*, J. Diff. Equ., **14**(3) (1973), 399-418.
[3] V. Barbu, *Nonlinear differential equations of monotone type in Banach spaces*, Springer-Verlag, New York, 2010.
[4] S. Caddemi and I. Calio, *Exact closed-form solution for the vibration modes of the EulerBernoulli beam with multiple open cracks*, J. Sound Vibration, **327**(3) (2009), 473-489.
[5] S. Caddemi and A. Morassi, *Multi-cracked Euler-Bernoulli beams: Mathematical modeling and exact solutions*, Inter. J. Solids Structures, **50**(6) (2013), 944-956.
[6] F. Cannizzaro and A. Greco and S. Caddemi and I. Calio, *Closed form solutions of a multi-cracked circular arch under static loads*, Inter. J. Solids Structures, **121** (2017), 944-956.
[7] M.N. Cerri and G.C. Ruta, *Detection of localised damage in plane circular arches by frequency data*, Journal of Sound and Vibration, **270**(1) (2004), 39-59.
[8] T.G. Chondros and A.D. Dimarogonas and J. Yao, *A continuous cracked beam vibration theory*, J. Sound Vibration,**215**(1) (1998), 17-34.
[9] S. Christides and A.D.S. Barr, *One-dimensional theory of cracked Bernoulli-Euler beams*, International Journal of Mechanical Sciences, **26**(11) (1984), 639-648.
[10] A.D. Dimarogonas, *Vibration of cracked structures: a state of the art review*, Eng. Fracture Mechanics, **55**(5) (1996), 831-857.
[11] E. Emmrich and M. Thalhammer, *A class of integro-differential equations incorporating nonlinear and nonlocal damping with applications in nonlinear elastodynamics: Existence via time discretization*, Nonlinearity, **24**(9) (2011), 2523-2546.
[12] S. Gutman and J. Ha, *Shallow arches with weak and strong damping*, J. Kor. Math. Soc., **54** (2017), 945-966.
[13] S. Gutman and J. Ha, *Uniform attractor of shallow arch motion under moving points load*, J. Math. Anal. Appl., **464**(1) (2018), 557-579.

[14] S. Gutman, J. Ha and S. Lee, *Parameter identification for weakly damped shallow arches*, J. Math. Anal. Appl., **403**(1) (2013), 297-313.

[15] J. Ha and S. Gutman and S. Shon, *Variational setting for cracked beams and shallow arches*, to appear in Arch. Appl. Mech., (2022).

[16] S.M. Han, H. Benaroya and T. Wei, *Dynamics of transversely vibrating beams using four engineering theories*, J. Sound Vibration, **225(5)** (1999), 935-988.

[17] J. Kim, G. F. Dargush and Y. Ju, *Extended framework of Hamilton's principle for continuum dynamics*, Inter. J. Solids Structures, **50**(20) (2013), 3418-3429.

[18] H.P. Lin, S.C. Chang and J.D. Wu, *Beam vibrations with an arbitrary number of cracks*, J. Sound Vibration, **258**(5) (2002), 987-999.

[19] W.M. Ostachowicz and M. Krawczuk, *Analysis of the effect of cracks on the natural frequencies of a cantilever beam*, J. Sound Vibration **150**(2) (1991), 191-201.

[20] E.I. Shifrin and R. Ruotolo, *Natural frequencies of a beam with an arbitrary number of cracks*, J. Sound Vibration, **222**(3) (1999), 409-423.

[21] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Springer, New York, 1997.

[22] S. Woinowsky-Krieger, *The effect of axial force on the vibration of hinged bars*, J. Appl. Mech., **17** (1950), 35-36.