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B-SPLINE TIGHT FRAMELETS FOR SOLVING INTEGRAL ALGEBRAIC EQUATIONS WITH WEAKLY SINGULAR KERNELS

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Abstract. In this paper, we carried out a new numerical approach for solving integral algebraic equations with weakly singular kernels. The novel method is based on the construction of B-spline tight framelets using the unitary and oblique extension principles. Some numerical examples are given to provide further explanation and validation of our method. The result of this study introduces a new technique for solving weakly singular integral algebraic equation and thus in turn will contribute to providing new insight into approximation solutions for integral algebraic equation (IAE).

1. Introduction

A system consisting of two volterra integral equations of the first and second kind is called IAE, which has been widely discussed and arises in numerous mathematical modeling problems in applied science. Kernel identification problems in heat conduction and viscoelasticity [20], as well as the evolution of a chemical reaction within a small cell [16], are some natural examples. In this paper, we introduce a new numerical method for solving a mixed system of

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first and second kind of volterra integral equations (VIE) with weakly singular kernels. More precisely, we consider the following semi-explicit system of the volterra integral equation:

$$\begin{aligned} y(t) &= g_1(t) + (v_{11}y)(t) + (v_{12}z)(t), \\ 0 &= g_2(t) + (v_{21}y)(t) + (v_{22}z)(t), \end{aligned} \tag{1.1}$$

where the singular Volterra integral operators v_{kl} are given by:

$$(v_{kl}q)(t) = \int_0^t (t-s)^{-\alpha} K_{kl}(t,s)\, q(s) ds, \tag{1.2}$$

where $t \in I = [0,T](k,l = 1,2)$, and $0 < \alpha = \frac{n}{m} < 1\,(n,m \in \mathbb{N}, n < m)$, the functions g_1, g_2 and $K_{kl}(t,s)$ are given smooth functions on I and $D = \{(t,s) : 0 \leq s \leq t \leq T\}$ respectively. Furthermore, we assume that $g_2(0) = 0$, $|K_{22}(t,t)| \geq k_0 > 0$, for all $t \in I$. The type of system in (1.1) that satisfies these conditions is called the weakly singular integral algebraic equation of index-1 (WS-IAE).

The semi-explicit system in (1.1) could be written in a matrix form as:

$$AX(t) = G(t) + \int_0^t (t-s)^{-\alpha} K(t,s)\, X(s) ds, \ t \in I, \tag{1.3}$$

where

$$G(t) = (g_1(t), g_2(t))^t, \quad X(t) = (x_1(t), x_2(t))^t,$$

$$K(t,s) = \begin{pmatrix} k_{11}(t,s) & k_{21}(t,s) \\ k_{21}(t,s) & k_{22}(t,s) \end{pmatrix}$$

and A(t) is a singular matrix with continuous entries. Due to the algebraic form of the matrix A(t), system in (1.2) can be represented in different forms. In particular, for the semi-explicit system of volterra integral equations in (1.1), notice that the matrix A should be in the form: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Many different definitions of the IAE index have been discussed in the literature, and they are often closely related. For example, Brunner in [4] investigates the "tractability index" that the algebraic constraints. Subsequently, Chistyakov et al studied the left index [6], and Gear in [14] introduce the concept of the differentiation index, that based on a reduction process for WS-IAE to yield a regular system of IAE, and that plays a significant role in analysis and the construction of numerical techniques for the integral algebraic equation. In general, solving integral algebraic equations with an index of more than one is complicated. Various numerical methods have been developed for solving integral algebraic equations. On the other hand, the numerical solution for IAE with weakly singular kernels is not widely investigating. Brunner and Bulatov in 1998 examined the existence and uniqueness of WS-IAE [5].

Bulatov et al [7] determined the sufficient conditions for the existence of a unique continuous solution. Hadizadeh et al [18] provide a numerical method for solving a system in (1.1) using the Chebyshev collocation method. They investigated the approximate solution of WS-IAEs where the derivatives of its solutions are unbounded at the lowest endpoint of the interval, so they have used some suitable variable transformations to convert the original system to a new one with more regular solutions.

Based on the unitary and oblique extension concepts, B-spline tight framelets systems are generated. We developed a new numerical technique for solving integral algebraic equations with weakly singular kernels (1.1). The organization of this paper is as follows: Section 2 is dedicated to presenting some background information on frames and notations. Section 3 includes the fundamentals of creating B-spline tight framelet systems using unitary and oblique extension principles. The numerical simulation method using the B-spline tight framelet to solve equation (1.1) is introduced in section 4. Finally, some numerical examples are presented to verify the accuracy of the result.

2. Preliminaries

Duffin and Schaeffer introduced frames in 1952 to help in the analysis of a particular form of non-harmonic Fourier series. Frames for $L^2(\mathbb{R})$ were constructed by Daubechies and others. Integral equations have been solved using wavelets. Beylkin was the first to apply the wavelet approach to solve integral equations, in 1991. Many numerical techniques for solving different forms of integral equations, such fredholm and volterra integral equations are discussed.

Now, we will recall some concepts that essential for this article.

If a compactly supported function $\phi(x) \in L^2(\mathbb{R})$ satisfies the equation:

$$\phi(x) = 2\sum h_0[k]\phi(2x-k), \quad \text{where } \; k \in \mathbb{Z}, \tag{2.1}$$

then it is considered refinable for some finite supported sequence $h_0[k] \in l^2(\mathbb{Z})$. The sequence h_0 is called the refinable mask or the low pass filter for ϕ.

A compactly supported wavelet ψ is called having an order of vanishing moments m if

$$\int x^k \psi(x)dx = 0, \text{ for all } 0 \le k \le m-1.$$

Definition 2.1. A sequence of elements $\{f_k\}_{k=1}^{\infty}$ in $L^2(\mathbb{R})$ is a frame for $L^2(\mathbb{R})$ if there exist constants $A, B > 0$ such that

$$A\left\|f\right\|^2 \le \sum_{k=1}^{\infty} \left|\langle f, f_k\rangle\right|^2 \le B\left\|f\right\|^2.$$

The numbers A, B are the frame bounds. A frame is tight if $A = B$.

For a function$f \in L^2(\mathbb{R})$, the Fourier transform of f is defined by

$$\hat{f} = \int_{\mathbb{R}} f(x) e^{-i\omega x} dx,\ \omega \in \mathbb{R}.$$

For a given sequence $\{h_k, k \in \mathbb{Z}\}$, the Fourier series is given by

$$\hat{h}(\omega) = \sum_{k \in \mathbb{Z}} h(k) e^{-i\omega k}.$$

3. Main theorems: B-spline tight framelets

In many fields, such as geometric modeling and applied mathematics, quasi-affine tight framelets systems generated by the unitary extension principle and oblique extension principle play a key role. For example, it has been used to obtain approximate solution for integral equations such as VIE of first and second kind [1], and other areas see for example [17]. In particular, we used the B-spline function of various orders to obtain framelets from a compactly supported refinable function.

The B-spline function of order n, is defined by using convolution as in [16].

$$B_n(x) = B_{n-1}(x) * B_1(x) = \int_{-1/2}^{1/2} B_{n-1}(x-t) dt\ n \geq 2,\ x \in \mathbb{R}, \tag{3.1}$$

where $B_1(x) = \chi_{[\frac{-1}{2}, \frac{1}{2}]}(x)$ is the characteristic function of the interval $[\frac{-1}{2}, \frac{1}{2}]$.

The Fourier transform for the B-spline of order n is given by

$$\hat{B}_n(\omega) = e^{\frac{-ij\omega}{2}} \left(\frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}} \right)^n,$$

where $j = 1$ when n is odd and zero otherwise, and its refinable mask is defined as

$$\hat{h_0}(\omega) = \exp\left(\frac{-ij\omega}{2}\right) \left(\cos(\frac{\omega}{2})\right)^n.$$

It is clear that $B_n(x)$ is a polynomial of degree n-1, more details for the B-spline can be found in references [2], [3], and [19] and some application in [21]. According to Ron and Dyn, the periodic exponential B-splines, possess an essential property of translation invariance and satisfy a generalized Hermite—Genocchi formula [13]. Recently, the development of approximate approaches using the B-spline function has a lot of interest. The graphs of the B-splines for $n = 2, 3, 4$ are presented in Figure 1.

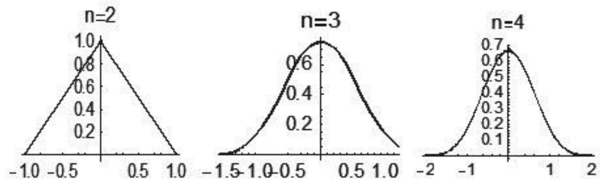

FIGURE 1. The B-spline function of different orders.

3.1. **B-spline tight framelet by unitary extension principle (B-UEP).** Dong and Shen's introduce a theorem, which utilizes the unitary extension principle (UEP) to generate a tight frame for $L^2(\mathbb{R})$ using a multiresolution analysis provided by a refinable function.

Definition 3.1. ([11]) For the set of functions $\Psi := \{\psi_1, \psi_2, ..., \psi_n\}$. A framelets system for $L^2(\mathbb{R})$ generated by Ψ is defined as

$$X(\Psi) = \{\psi_{l,j,k}, 1 \leq l \leq n, j, k \in \mathbb{Z}\}, \tag{3.2}$$

where

$$\psi_{l,j,k} = D^j T_k \psi_l = 2^{\frac{j}{2}} \psi_l(2^j. - k),\ l = 1, ..., n,$$

$$\psi_l = 2 \sum_{k \in \mathbb{Z}} h_l[k] \phi(2x - k).$$

The sequence $\{h_l[k]\}_{l=1}^n$ is called the high pass filter of the system.

Framelets may be used to approximate a function, and to have a proper estimate for a function, we need approximation schemes, one of which is the quasi-interpolation scheme, which is defined as follows:

$$\mathcal{P}_n : f \to \sum_{k \in \mathbb{Z}} \langle \phi, \phi_{n,k} \rangle \, \phi_{n,k}. \tag{3.3}$$

Now, we consider the following two lemmas before introduce the unitary extension principle theorem [12].

Lemma 3.2. *Let $\phi(x) \in L^2(\mathbb{R})$ be a refinable function with a refinable mask h_0, and the sequence $\{h_l\}_{l=1}^r$ satisfies the condition* (3.3)*. Then*

$$\mathcal{P}_n f = \mathcal{P}_{n-1} f + \sum_{l=1}^{r} \sum_{k \in \mathbb{Z}} \langle f, \psi_{l,n-1,k} \rangle \, \psi_{l,n-1,k}.$$

Lemma 3.3. *Let $\phi(x) \in L^2(\mathbb{R})$ be a refinable function and the operator $\mathcal{P}_n$ be defined in* (3.3)*. Then, for all function $g \in L^2(\mathbb{R})$*

$$\lim_{n\to\infty} \mathcal{P}_n g = g \;\; \textit{and} \;\; \lim_{n\to-\infty} \mathcal{P}_n g = 0.$$

Theorem 3.4. *Let $\phi \in L^2(\mathbb{R})$ be a compactly supported refinable function with its refinable mask h_0, and let $\{h_l\}_{l=1}^r$ be a set of finitely supported sequence. Then the system $X(\Psi)$ defined in* (3.2) *forms a tight frame for $L^2(\mathbb{R})$ provided that*

$$\sum_{l=0}^{r} \left|\hat{h}_l(\omega)\right|^2 = 1 \;\; \textit{and} \;\; \sum_{l=0}^{r} \left|\hat{h}_l(\omega)\,\overline{\hat{h}_l(\omega+\pi)}\right|^2 = 0 \tag{3.4}$$

hold for $\omega \in [-\pi, \pi]$.

Proof. Using Lemma 3.2, we have

$$\mathcal{P}_n f = \mathcal{P}_{n-1} f + \sum_{l=1}^{r} \sum_{k\in\mathbb{Z}} \langle f, \psi_{l,n-1,k}\rangle\, \psi_{l,n-1,k}.$$

By induction and consider $m = n - 1$, we obtain

$$\mathcal{P}_n f = \mathcal{P}_m f + \sum_{l=1}^{r} \sum_{j=m}^{n-1} \sum_{k\in\mathbb{Z}} \langle f, \psi_{l,j,k}\rangle\, \psi_{l,j,k}.$$

Let $m \to -\infty$ and applying Lemma 3.3, we have

$$\mathcal{P}_n f = \sum_{l=1}^{r} \sum_{j<n} \sum_{k\in\mathbb{Z}} \langle f, \psi_{l,j,k}\rangle\, \psi_{l,j,k}.$$

Then, taking $n \to \infty$ on both sides and using lemma 3.3

$$f = \sum_{l=1}^{r} \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \psi_{l,n,k}\rangle\, \psi_{l,n,k}.$$

Therefore, $X(\Psi)$ is a tight framelets for $L^2(\mathbb{R})$. □

For an arbitrary $f \in L^2(\mathbb{R})$, by using the truncated quasi-interpolation operators on framelets, it can be represented as:

$$\mathcal{P}_n f = \sum_{l=1}^{r} \sum_{j<n} \sum_{k\in\mathbb{Z}} \langle f, \psi_{l,j,k}\rangle\, \psi_{l,j,k}.$$

Numerical solutions to integral algebraic equations with weakly singular kernels could be found by using this representation.

Example 3.5. Let $\phi_0(.)$ be the B-spline function of a second order, with low pass filter $\hat{h}_0 = \frac{1}{4}\left(1+e^{-i\omega}\right)^2$. Then the Fourier transform of the high pass filter $h_1 = \frac{-1}{4}\left(1-e^{-i\omega}\right)^2$ and $h_2 = \frac{-\sqrt{2}}{4}\left(1-e^{-2i\omega}\right)$. The corresponding system $X(\Psi) = \{\psi_1, \psi_2\}$ is a framelets, where

$$2\psi_1 = \frac{-1}{2}\left(|x-2| - 2|2x-3| + 6|x-1| - 2|2x-1| + |x|\right),$$
$$\psi_2 = \frac{1}{2}\left(|x-2| - 2|2x-3| + 2|2x-1| - |x|\right).$$

Curves for B2-tight framelet functions produced by UEP ψ_1 and ψ_2 are illustrated in Figure 2.

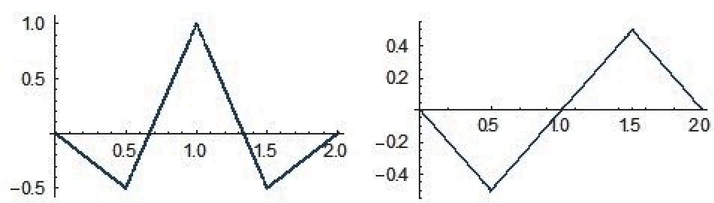

FIGURE 2. The framelets functions B2-UEP.

Example 3.6. For the compactly supported refinable function $\phi(x) = B_4(x)$, define

$$3\hat{h}_1 = \frac{1}{4}\left(1-e^{-i\omega}\right)^4, \quad \hat{h}_2 = \frac{-1}{4}\left(1-e^{-i\omega}\right)^3\left(1+e^{-i\omega}\right),$$
$$\hat{h}_3 = \frac{-\sqrt{6}}{16}\left(1-e^{-i\omega}\right)^2\left(1+e^{-i\omega}\right)^2,$$
$$\hat{h}_4 = \frac{-1}{4}\left(1-e^{-i\omega}\right)\left(1+e^{-i\omega}\right)^3,$$

then, the system $X(\Psi) = \{\psi_1, \psi_2, \psi_3, \psi_4\}$ is a framelets for $L^2(\mathbb{R})$, where

$$2\psi_1 = \frac{1}{3}\Big(|x-4|^3 - |2x-1|^3 + 28|x-3|^3 - 7|2x-5|^3 + 70|x-2|^3$$
$$-7|2x-3|^3 - 28|x-1|^3 - |2x-1|^3 + |x|^3\Big),$$
$$\psi_2 = \frac{-1}{3}\Big(|x-4|^3 - 6|x-3.5|^3 + 14|x-3|^3 - 14|x-2.5|^3 + 14|x-1.5|^3$$
$$-14|x-1|^3 + 6|x-0.5|^3 - |x|^3\Big),$$

$$2\psi_3 = \frac{-1}{2\sqrt{6}}\Big(|x-4|^3 - 4|x-3.5|^3 + 4|x-3|^3 + 4|x-2.5|^3 - 10|x-2|^3$$
$$+4|x-1.5|^3 + 4|x-1|^3 - 4|x-0.5|^3 + |x|^3\Big),$$
$$\psi_4 = \frac{1}{3}\Big(|x-4|^3 - 2|x-3.5|^3 - 2|x-3|^3 + 6|x-2.5|^3 - 6|x-1.5|^3$$
$$+2|x-1|^3 + 2|x-0.5|^3 - |x|^3\Big).$$

The framelets functions ψ_1, ψ_2, ψ_3, and ψ_4 generated based on using a fourth-order B-spline function (B_4) are represented in Figure 3.

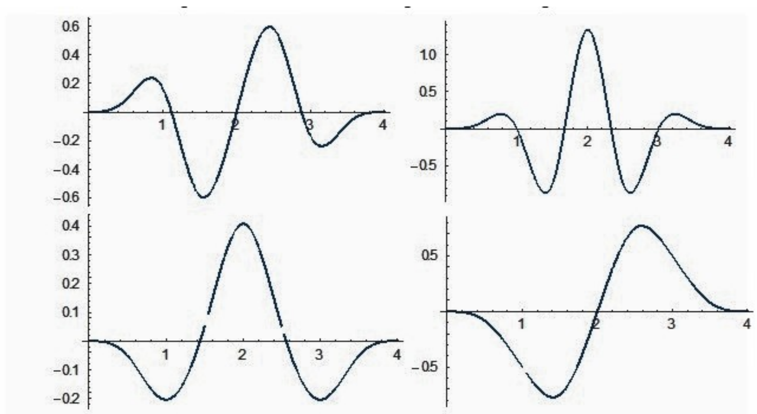

FIGURE 3. The tight framelets functions B4-UEP.

3.2. **B-spline tight framelet by oblique extension principle (B-OEP).** With a high approximation order truncated wavelet system and high order vanishing moments in the generators, the oblique extension concept could be utilized to construct a tight framelete. The unitary extension theory has been generalized several times since 1997. The oblique extension principle of [9] and [10] is the first generalization of the unitary extension principle. The aim of generalizing the unitary extension principle is to obtain a spline tight wavelet system with higher approximation power.

Theorem 3.7. *Suppose that the refinable function* φ *with the mask* $\hat{h_0}$, *and there exists a* 2π*-periodic function* Θ *that is nonnegative, essentially bounded, continuous at the origin with* $\Theta(0) = 1$. *Also, if* $\xi \in [-\pi, \pi]$ *and* $\xi+\pi \in [-\pi, \pi]$, *where* Θ *satisfying the following equalities*

$$
\begin{aligned}
2\left|\hat{h}_0(\xi)\right|^2\Theta(2\xi)+\sum_{l=1}^{n}\left|\hat{h}_l(\xi)\right|^2&=\Theta(\xi),\\
\hat{h_0}(\xi)\overline{\hat{h}_0(\xi+\pi)}\Theta(2\xi)+\sum_{l=1}^{n}\hat{h}_l(\xi)\,\overline{\hat{h}_l(\xi+\pi)}&=0.
\end{aligned}
\tag{3.5}
$$

Then the wavelet system $X(\Psi)$ defined by $\hat{h_0},\cdots,\hat{h_n}$ is a tight wavelet frame.

In order to construct the B-spline tight framelets produced by the OEP, the appropriate approximation $\Theta(\xi)$ should be chosen at the origin to $\frac{1}{|\hat{\varphi}|^2}=O(|.|^{2l})$. For example, if φ is a B-spline of order m, we must choose Θ as a 2π-periodic function that approximates is $\left|\frac{\xi/}{\sin(\xi/)}\right|^{2m}$ at $\xi=0$.

Example 3.8. Take

$$\hat{h}_0(\omega)=\frac{\left(1+e^{-i\omega}\right)^2}{4}$$

and

$$\Theta(\omega)=\frac{4}{3}-\frac{e^{-i\omega}}{6}-\frac{e^{i\omega}}{6},$$

where φ is the linear B-spline function, $B_2(x)$. Define $\Psi=\{\psi_1,\psi_2\}$, where the corresponding high pass filters are:

$$2\hat{h}_1(\omega)=\frac{-1}{4}\left(1-e^{-i\omega}\right)^2$$

and

$$2\hat{h}_2(\omega)=\frac{-\sqrt{6}}{24}\left(1-e^{-i\omega}\right)^2\left(e^{-i\omega}+4e^{-i(2\omega)}+e^{-i(3\omega)}\right).$$

In particular,

$$
\begin{aligned}
2\hat{\psi}_1(\omega)&=\frac{-1}{\omega^2}\left(1-e^{-i\omega/2}\right)^4,\\
\hat{\psi}_2(\omega)&=\frac{-1}{\sqrt{6}\omega^2}\left(1-e^{-i\omega/2}\right)^4\left(e^{-i\omega/2}+4e^{-i\omega}+e^{-i(3\omega)/2}\right).
\end{aligned}
$$

Then, the symmetric framelets are given by

$$2\psi_1=\frac{-1}{2}\left(|x-2|-2\,|2x-3|+6\,|x-1|-2\,|2x-1|+|x|\right)$$

and

$$2\psi_2=\frac{-1}{2\sqrt{6}}\left(|x-3|-9\,|x-2|+8\,|2x-3|-9\,|x-1|+|x|\right).$$

The tight framelets functions ψ_1 and ψ_2 generated by OEP based on the B-spline function B_2, are given in Figure 4.

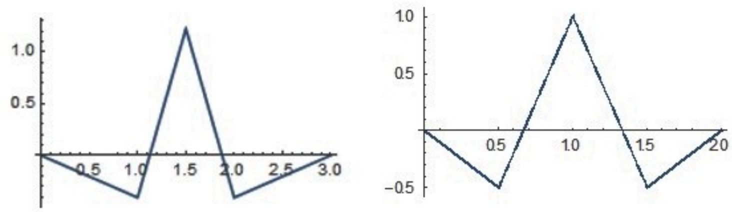

FIGURE 4. The framelets functions B2-OEP.

Example 3.9. Consider the B-spline of order 3, B_4, and take the periodic function Θ defined by:

$$\Theta(\omega) = \frac{2452}{945} - \frac{1657}{840}\cos(\omega) + \frac{44}{105}\cos(2\omega) - \frac{311}{7560}\cos(3\omega).$$

Then, it is a quasi-affine tight framelet function are:

$$\begin{aligned} 2\psi_1 =& \nu_1[|x-5|^3 - 35\,|x-4|^3 + 20\,|2x-7|^3 - 350\,|x-3|^3 + 56\,|2x-5|^3 \\ &- 350\,|x-2|^3 + 20\,|2x-3|^3 - 35\,|x-1|^3 + |x|^3], \end{aligned}$$

$$\begin{aligned} 2\psi_2 =& \nu_2[199658352\,|x-6|^3 - 16955218889\,|x-5|^3 \\ &+ 14161261961\,|2x-9|^3 - 361590873308\,|x-4|^3 \\ &+ 86356459199\,|2x-7|^3 - 851590490870\,|x-3|^3 \\ &+ 86356459199\,|2x-5|^3 + 361590873308\,|x-2|^3 \\ &+ 14161261961\,|2x-3|^3 - 16955218889\,|x-1|^3 + 99568352\,|x|^3] \end{aligned}$$

and

$$\begin{aligned} 2\psi_3 =& \nu_3[7775\,|x-7|^3 - 76902\,|x-6|^3 + 405720\,|x-5|^3 - 14140\,|2x-9|^3 \\ &- 1657425\,|x-4|^3 + 358488\,|2x-7|^3 - 1657425\,|x-3|^3 \\ &- 14140\,|2x-5|^3 + 405720\,|x-2|^3 - 76902\,|x-1|^3 + 7775\,|x|^3]. \end{aligned}$$

where $\nu_1 \approx 0.01105$, $\nu_2 \approx -3.8 \times 10^{-11}$, and $\nu_3 \approx 1.54 \times 10^{-6}$.

Figure 5 shows the symmetric wavelet functions ψ_1, ψ_2, and ψ_3 derived from the OEP.

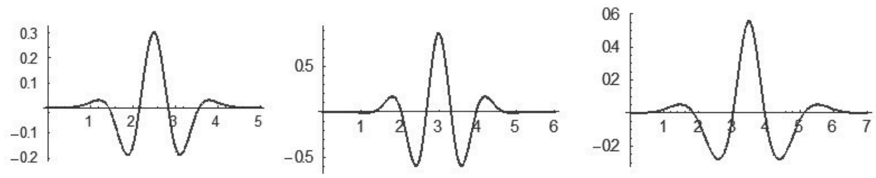

FIGURE 5. The framelets functions B4-OEP: ψ_1, ψ_2,and ψ_3 respectively.

4. Solving WSIAE by B-Spline Tight Framelets

In this section, we will obtain a numerical solution for the unknown functions y(t) and z(t) of equation (1.1) by truncating the quasi-affine framelets. The approximation solutions $\mathcal{P}_n(x_1)$ and $\mathcal{P}_n(x_2)$ for $x_1(t)$ and $x_2(t)$ are as follows:

$$\mathcal{P}_n(x_1) = \sum_{l=1}^{r} \sum_{j,k} \zeta_l^{j,k} \psi_{l,j,k} \tag{4.1}$$

and

$$\mathcal{P}_n(x_2) = \sum_{l=1}^{r} \sum_{j,k} \eta_l^{j,k} \psi_{l,j,k}. \tag{4.2}$$

Notice that j and k will be choosing to get a good accurate representing in substituting equations (4.1) and (4.2) into the system in (1.1) yields:

$$\begin{aligned} 2\mathcal{P}_n(x_1) = g_1(t) &+ \int_0^t (t-s)^{-\alpha} K_{11}(t,s)\, \mathcal{P}_n(x_1) ds \\ &+ \int_0^t (t-s)^{-\alpha} K_{12}(t,s)\, \mathcal{P}_n(x_2) ds, \\ 0 = g_2(t) &+ \int_0^t (t-s)^{-\alpha} K_{21}(t,s)\, \mathcal{P}_n(x_1) ds \\ &+ \int_0^t (t-s)^{-\alpha} K_{22}(t,s)\, \mathcal{P}_n(x_2) ds. \end{aligned} \tag{4.3}$$

Using (4.1) and (4.2), we get

$$\begin{aligned}\sum_{l=1}^{r}\sum_{j,k}\zeta_l^{j,k}\psi_{l,j,k} &= g_1(t)+\sum_{l=1}^{r}\sum_{j,k}\zeta_l^{j,k}\left(\int_0^t (t-s)^{-\alpha}K_{11}(t,s)\psi_{l,j,k}ds\right)\\ &+\sum_{l=1}^{r}\sum_{j,k}\eta_l^{j,k}\left(\int_0^t (t-s)^{-\alpha}K_{12}(t,s)\psi_{l,j,k}ds\right),\\ 0 &= g_2(t)+\sum_{l=1}^{r}\sum_{j,k}\zeta_l^{j,k}\left(\int_0^t (t-s)^{-\alpha}K_{21}(t,s)\psi_{l,j,k}ds\right)\\ &+\sum_{l=1}^{r}\sum_{j,k}\eta_l^{j,k}\left(\int_0^t (t-s)^{-\alpha}K_{22}(t,s)\psi_{l,j,k}ds\right).\end{aligned}\tag{4.4}$$

The system in (4.4) may be represented as:

$$\begin{aligned}g_1(t) &= \sum_{l=1}^{r}\sum_{j,k}\zeta_l^{j,k}\left[\psi_{l,j,k}(t)-\int_0^t (t-s)^{-\alpha}K_{11}(t,s)\psi_{l,j,k}ds\right]\\ &-\sum_{l=1}^{r}\sum_{j,k}\eta_l^{j,k}\left(\int_0^t (t-s)^{-\alpha}K_{12}(t,s)\psi_{l,j,k}ds\right),\\ g_2(t) &= -\sum_{l=1}^{r}\sum_{j,k}\zeta_l^{j,k}\left(\int_0^t (t-s)^{-\alpha}K_{21}(t,s)\psi_{l,j,k}ds\right)\\ &-\sum_{l=1}^{r}\sum_{j,k}\eta_l^{j,k}\left(\int_0^t (t-s)^{-\alpha}K_{22}(t,s)\psi_{l,j,k}ds\right).\end{aligned}\tag{4.5}$$

Based on the framelets system $\{\psi_{l,j,k}\}_{l=1}^{r}$ $j,k \in \mathbb{Z}$, where $-n \leq j \leq n$ and $-2^n \leq k \leq 2^n$, then the linear system (4.5) has $4nr(2^{n+1}-1)$ of unknowns $\{\zeta_l^{j,k}\}$ and $\{\eta_l^{j,k}\}$. In interval $[a,b]$, let the collocation points are $\{t_i, i=1,...,2nr(2^n-1)\}$.

Inserting the collocation point t_i in the system (4.5), we obtain

$$\begin{aligned} g_1(t_i) &= \sum_{l=1}^{r}\sum_{j,k}\zeta_l^{j,k}\left[\psi_{l,j,k}(t_i) - \int_0^{t_i}(t_i-s)^{-\alpha}K_{11}(t_i,s)\psi_{l,j,k}ds\right] \\ &\quad - \sum_{l=1}^{r}\sum_{j,k}\eta_l^{j,k}\left(\int_0^{t_i}(t_i-s)^{-\alpha}K_{12}(t_i,s)\psi_{l,j,k}ds\right), \\ g_2(t_i) &= -\sum_{l=1}^{r}\sum_{j,k}\zeta_l^{j,k}\left(\int_0^{t_i}(t_i-s)^{-\alpha}K_{21}(t_i,s)\psi_{l,j,k}ds\right) \\ &\quad - \sum_{l=1}^{r}\sum_{j,k}\eta_l^{j,k}\left(\int_0^{t_i}(t_i-s)^{-\alpha}K_{22}(t_i,s)\psi_{l,j,k}ds\right). \end{aligned} \tag{4.6}$$

The unknown coefficient $\zeta_l^{j,k}$ and $\eta_l^{j,k}$ will then be determined by solving this system, which may be written as a matrix:

$$KG = C,$$

where

$$2G(t) = \left([G_1],[G_2]\right)^T, \quad C = \left([\zeta],[\eta]\right)^T, K(t,s) = \begin{pmatrix}[K_{11}] & [K_{21}] \\ [K_{21}] & [K_{22}]\end{pmatrix}.$$

Then, the column vectors C and G are $2^{n+1}(2n+1)\times 1$, and the block $[K_{ij}]$ for $i,j=1,2$ in matrix K is given by

$$[K_{ij}] = \begin{pmatrix} k_{i,j}(1,-n,-2^n) & \cdots & k_{i,j}(1,n,2^{n-1}) \\ \vdots & \ddots & \vdots \\ k_{i,j}(2nr(2^{n+1}-1),-n,-2^n) & \cdots & k_{i,j}(2nr(2^{n+1}-1),n,2^n-1) \end{pmatrix}, \tag{4.7}$$

where

$$\begin{aligned} k_{1,1} &= \psi_{l,j,k}(t_i) - \int_0^{t_i}(t_i-s)^{-\alpha}K_{11}(t_i,s)\psi_{l,j,k}ds, \\ k_{1,2} &= \int_0^{t_i}(t_i-s)^{-\alpha}K_{12}(t_i,s)\psi_{l,j,k}ds, \\ k_{2,1} &= -\int_0^{t_i}(t_i-s)^{-\alpha}K_{21}(t_i,s)\psi_{l,j,k}ds, \\ k_{2,2} &= -\int_0^{t_i}(t_i-s)^{-\alpha}K_{22}(t_i,s)\psi_{l,j,k}ds. \end{aligned}$$

5. Numerical Experiments

We apply the B-spline tight framelets method constructed in the previous section to solve equations in (1.3). The following numerical examples are discussed to demonstrate the validity and efficiency of our method.

Example 5.1. Consider the weakly singular integral algebraic equations system defined by

$$AX(t) = G(t) + \int_0^t (t-s)^{-\frac{1}{3}} K(t,s)\, X(s) ds,\ t \in [0,1],$$

with

$$2k_{11}(t,s) = t+s+2, \quad k_{12}(t,s) = ts,$$
$$k_{21}(t,s) = (t+s)^2,\ k_{22}(t,s) = 1+st^2.$$

Let $g_1(t)$, and $g_2(t)$ are chosen such that the exact solutions are $x_1(t) = t, x_2(t) = t$. Applying the B-spline tight framelets method generated by the unitary extension principle, then the error at different points t is presented in Table 1.

Table 1. The errors of Example 5.1 using B2-UEP

t	$\lvert x_1 - \mathcal{P}_n(x_1)\rvert$	$\lvert x_2 - \mathcal{P}_n(x_2)\rvert$
0.2	2.7×10^{-5}	1.18×10^{-4}
0.4	5×10^{-5}	1.27×10^{-4}
0.6	3.6×10^{-5}	3×10^{-5}
0.8	5.9×10^{-5}	7.1×10^{-5}

Table 2 shows the absolute error of the computed solution using the B-spline tight framelets method generated by the oblique extension principle.

Table 2. The errors of Example 5.1 using B2-OEP

t	$\lvert x_1 - \mathcal{P}_n(x_1)\rvert$	$\lvert x_2 - \mathcal{P}_n(x_2)\rvert$
0.2	8×10^{-5}	7.8×10^{-5}
0.4	7.2×10^{-5}	1.1×10^{-4}
0.6	6.6×10^{-5}	4.3×10^{-5}
0.8	7.7×10^{-5}	5.7×10^{-5}

Example 5.2. Consider the weakly singular integral algebraic equations system defined by:

$$AX(t) = G(t) + \int_0^t (t-s)^{-\frac{1}{4}} K(t,s)\, X(s) ds,\ t \in [0,1],$$

where

$$2k_{11}(t,s) = e^{(\sqrt{s}+t)}(t^2+s^4+3), \quad k_{12}(t,s) = \cos s^{\frac{1}{4}}(t+s)$$
$$k_{21}(t,s) = e^{(\sqrt{s}+t^2+1)}(t+s), \; k_{22}(t,s) = \sin(s^{\frac{1}{4}}+1)(1+st)$$
$$G(t) = (g_1(t), g_2(t))^T, \quad X(t) = (x_1(t), x_2(t))^T.$$

Let $g_1(t), g_2(t)$ are chosen such that the exact solution is $x_1(t) = \exp(t^{\frac{1}{2}})$, $x_2(t) = \sin(t^{\frac{1}{4}})$.

Applying the B-spline tight framelets method generated by the unitary extension principle, then the error at different points t is presented in Table 3 with the results of the method in [8].

TABLE 3. The errors of Example 5.2 using B2-UEP

t	$\lvert x_1 - \mathcal{P}_n(x_1)\rvert$	$\lvert x_2 - \mathcal{P}_n(x_2)\rvert$
0.1	5.05×10^{-5}	3.3×10^{-3}
0.3	4.7×10^{-3}	1.46×10^{-2}
0.5	1×10^{-3}	5.3×10^{-2}
Result in [8] with N=4	5×10^{-3}	1.94×10^{-2}

For the case of B_2-UEP, Figure 6 shows a comparison and good agreement between the approximate solution and the exact solution.

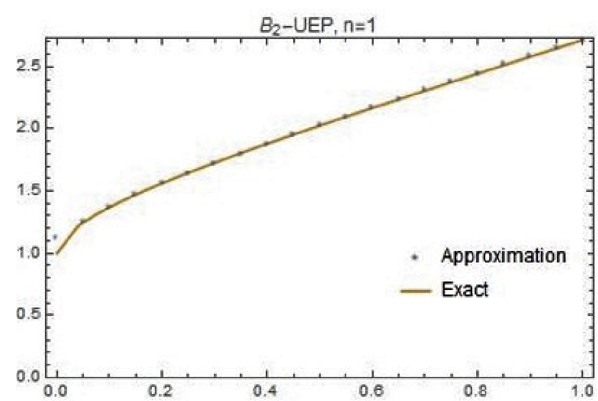

FIGURE 6. Comparison of exact solution to approximation solution for x_1 in Example 6.

6. Conclusion

We introduce a new, efficient numerical scheme for solving integral algebraic equations of index 1 with weakly singular kernels based on B-spline tight framelets generated by using UEP and OEP. Moreover, numerical examples have been carried out to test the efficiency of the proposed method. The results demonstrate that the presented technique is accurate, consistent, and converges to the exact solution. In future work, we will further improve our numerical approach to solve various types of equations, such as fractional differential equations and an integro-differential equations.

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