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ON C-PARALLEL LEGENDRE AND MAGNETIC CURVES IN THREE DIMENSIONAL KENMOTSU MANIFOLDS†

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Abstract. We find the characterizations of the curvatures of Legendre curves and magnetic curves in Kenmotsu manifolds with C-parallel and C-proper mean curvature vector fields in the tangent and normal bundles. Finally, an illustrative example is presented.

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1. Introduction

Almost contact metric geometry has also been studied by several authors, proving to be a source of nice examples and geometric behaviors for various different topics. By imposing, contact one-form η is closed, it becomes almost Kenmotsu manifold. By adding the condition such that almost complex stucture J is integrable, we call it as Kenmotsu manifold [10].

Almost contact curves play a important role in differential geometry of almost contact metric 3-manifolds. As a one dimensional submanifold, we may consider almost contact curves, that is., almost Legendre curves are Frenet curves in almost contact metric 3-manifolds which belongs to the almost contact distribution. Several authors have studied almost contact curves in contact geometry such as [8], [9], [11], [13], [15]. Baikoussis and Blair, have studied almost contact curves in contact metric 3-manifold and gave the Frenet 3-frame in contact 3-manifold (see [2]).

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Let M be an almost contact metric manifold and $\gamma(s)$ a Frenet curve in M parametrized by the arc-length parameter s. $\alpha(s)$ is a function defined by $\cos[\alpha(s)] = g(T(s), \xi)$ is called as the contact angle. A slant curve is defined by Cho at el. [3] the curve γ with constant contact angle. Especially, slant curves with fixed contact angle as $\frac{\pi}{2}$ are called Legendre curves [1]. Legendre curves in contact metric manifolds have been intensively studied under several different points of view.

Srivastava [15] investigated the properties of almost contact curves in trans-Sasakian 3-manifolds. Lee [11] find equivalent conditions for a Legendre curve with pseudo-Hermitian harmonic mean curvature vector field and proper pseudo-Hermitian mean curvature vector field in Sasakian manifolds. Also the paper gives us that characterized almost contact curves in a Sasakian manifold having the following properties:

- a pseudo-Hermitian parallel mean curvature vector field
- a pseudo-Hermitian proper mean curvature vector field in the normal bundle.

Recently, Inoguchi and Lee have studied almost contact curves in normal almost contact metric 3-manifold and slant curves in normal almost contact metric 3-manifolds (see [8], [9]).

In [5], Güvençe and Özgür author studied C-parallel and C-proper slant curves in $(2n+1)$-dimensional trans-Sasakian manifolds. Moreover in [13] Özgür consider C-parallel and C-proper Legendre curves in $(2n+1)$-dimensional non-Sasakian contact metric manifolds.

On the other hand, Cabrerizo et al. have introduced a notion of magnetic fields on three dimensional Sasakian manifolds as follows:

The magnetic trajectories are curves γ in M^n that satisfy the Lorentz equation

$$\nabla_{\gamma'}\gamma' = \phi\,(\gamma')\,.$$

Majhi and Biswas apply this concept of magnetic curves to the Kenmotz manifold as follows: A curve γ is said to be magnetic curve in a 3-dimensional f-Kenmotsu manifold if $\nabla_{\dot\gamma}\dot\gamma = \phi\dot\gamma$, where ∇ is the Levi-Civita connection (for more details, see [4], [7]).

Motivated by the above studies in the present paper we consider Legendre curves and magnetic curves in Kenmotsu manifolds with C-parallel and C-proper mean curvature vector fields in the tangent and normal bundles.

Consider a regular curve γ in almost contact metric manifold (M, ϕ, ξ, η, g) containing Kenmotsu manifold, the notion of C-parallel (resp., C-proper) can be defined as follows [5]: The C-parallel mean curvature vector field H is defined by $\nabla_{\dot\gamma} H = \lambda\xi$, where λ denotes a non-vanishing differentiable function on M and ∇ the induced Levi-Civita connection. Respectively, the C-proper mean curvature vector field vector field H is $\Delta H = \lambda\xi$, where Δ the operator of Laplacian of M.

For clarity, $\nabla^{\perp}$ and $\Delta^{\perp}$ stands for the normal connection and Laplacian in the normal bundle, respectively. The condition $\nabla_{\dot{\gamma}} H = \lambda\xi$ does not imply $\nabla^{\perp} H = \lambda\xi$, neither $\Delta_{\dot{\gamma}} H = \lambda\xi$ to $\Delta^{\perp}{}_{\dot{\gamma}} H = \lambda H$. We should consider $\nabla^{\perp}{}_{\dot{\gamma}} H = \lambda\xi$ and $\Delta^{\perp}{}_{\dot{\gamma}} H = \lambda\xi$ corresponding to $\nabla_{\dot{\gamma}} H = \lambda\xi$ and $\Delta_{\dot{\gamma}} H = \lambda\xi$ respectively.

Thus, in the normal bundle, we can naturally define the following notions: H is said to be C-parallel (resp., C-proper) in the normal bundle if $\nabla^{\perp} H = \lambda\xi$ (resp., $\Delta^{\perp} H = \lambda\xi$) where $\Delta^{\perp}$ denotes the operator of covariant differentiation in the normal bundle of M (see [11]).

Introducing the technical apparatus which is required for our framework, and such manifolds have been extensively studied under several points of view in [1], [10], [12], [16], and references cited therein. An exhaustive list of the main results would be a task far beyond the aim of this paper.

The paper is organized as follows: After preliminaries in Section 3, in three dimensional Kenmotsu manifolds, we consider Legendre curves with C-parallel and C-proper mean curvature vector fields, respectively. In the final section, we consider Magnetic curves with C-parallel and C-proper mean curvature vector fields, respectively.

2. Preliminaries

Let γ be a curve in a 3-dimensional Riemannian manifold M which is parameterized by arc length, and let $\nabla_{\dot{\gamma}}$ denote the covariant differentiation along γ with respect to the Levi-Civita connection on M. The parameterized curve γ is called as a Frenet curve if one of the following three cases hold:

- γ is of osculating order 1, i.e,$\nabla_T T = 0$ (geodesic), $T = \dot{\gamma}$. Here, $\cdot$ denotes differentiation with respect to the arc parameter.
- γ is of osculating order 2, i.e., there exist two orthonormal vector fields $T(= \dot{\gamma}), N$ and a non-negative functions κ (curvature) along γ such that $\nabla_T T = \kappa T$, $\nabla_T N = -\kappa T$.
- γ is of osculating order 3, i.e., there exist three orthonormal vectors $T(= \dot{\gamma})$, N, B and two non-negative functions κ(curvature) and τ(torsion) along γ such that

$$\nabla_T T = \kappa N,$$

$$\nabla_T N = -\kappa T + \tau B,$$

$$\nabla_T B = -\tau N.$$

With respect to the Levi-Civita connection, a Frenet curve of osculating order 3 for which k is a positive constant and $\tau = 0$ is called a circle in M; a Frenet curve of osculating order 3 is called a helix in M if κ and τ both are positive constants and the curve is called a generalized helix if $\frac{\kappa}{\tau}$ is a constant.

Let γ be a unit speed Frenet curve of osculating order 3 in M. By a simple calculations, it can be easily seen that

$$\nabla_T\nabla_T T = -\kappa^2 T + \kappa' N + \kappa\tau B \tag{1}$$

$$\nabla_T\nabla_T\nabla_T T = -3\kappa\kappa' T - (\kappa^3 + \kappa\tau^2 - \kappa'')N + (2\kappa'\tau + \kappa\tau')B = \lambda\xi, \tag{2}$$

$$\nabla^\perp_T\nabla^\perp_T T = \kappa' N + \kappa\tau B \tag{3}$$

$$\nabla^\perp_T\nabla^\perp_T\nabla^\perp_T T = -(\kappa\tau^2 - \kappa'')N + (2\kappa'\tau + \kappa\tau')B \tag{4}$$

(For more details see [5]).
Then we have

$$\nabla_T H = -\kappa^2 T + \kappa' N + \kappa\tau B,$$

$$\begin{aligned}\Delta H \quad &= \nabla_T\nabla_T\nabla_T T \\ &= 3\kappa\kappa' T + (\kappa^3 + \kappa\tau^2 - \kappa'')N - (2\kappa'\tau + \kappa\tau')B\end{aligned} \tag{5}$$

$$\begin{aligned}\Delta^\perp H \quad &= \nabla_T\nabla_T\nabla_T T \\ &= (\kappa\tau^2 - \kappa'')N - (2\kappa'\tau + \kappa\tau')B\end{aligned} \tag{6}$$

Let γ be a non geodesic Frenet curve in a contact metric manifold M. In view of [5], we have the following relations:

- γ is a curve with C-parallel mean curvature vector field H if and only if
$$-\kappa^2 T + \kappa' N + \kappa\tau B = \lambda\xi \tag{7}$$
- γ is a curve with C-proper mean curvature vector field H if and only if
$$3\kappa\kappa' T + (\kappa^3 + \kappa\tau^2 - \kappa'')N - (2\kappa'\tau + \kappa\tau')B = \lambda\xi \tag{8}$$
- γ is a curve with C-parallel mean curvature vector field H in the normal bundle if and only if
$$\kappa' N + \kappa\tau B = \lambda\xi \quad \text{or} \tag{9}$$
- γ is a curve with C-proper mean curvature vector field H in the normal bundle if and only if
$$(\kappa\tau^2 - \kappa'')N - (2\kappa'\tau + \kappa\tau')B = \lambda\xi, \tag{10}$$
where λ is a non-zero differentiable function along the curve γ.

An almost contact manifold is defined by $(2n+1)$-dimensional differentiable manifold M satisfying its structural group $GL_{2n+1}\mathbb{R}$ of linear frame bundle is reducible to $\mathrm{U}(n) \times \{1\}$ (see [6]). This gives us a following crucial identities:

$$\phi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1 \tag{11}$$

$$\phi\xi = 0, \eta \circ \phi = 0, \eta(X) = g(X, \xi) \tag{12}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), g(X, \phi Y) = -g(\phi X, Y), \tag{13}$$

for any vector fields $X, Y \in \chi(M)$ where I is the identity of the tangent bundle TM, ϕ is a tensor field of $(1,1)$-type, η is a 1-form, ξ is a vector field and g is a metric tensor field (see [1]).

An almost contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be a trans-sasakian manifold if $(M^{2n+1} \times \mathbb{R}; J; G)$ belongs to the class W_4 of the Hermitian manifolds, where J is the almost complex structures on $M^{2n+1} \times \mathbb{R}$ defined by, $J(Z, f\frac{d}{dt}) = (\phi Z - f\xi, \eta(Z)\frac{d}{dt})$; where the pair $J(Z, f\frac{d}{dt})$ denotes a tangent vector to $(M^{2n+1} \times \mathbb{R}$. Z and $f\frac{d}{dt}$ being tangent to $M^{2n+1}(\phi, \xi, \eta, g)$ are said to be normal if the structure J is integrable. The necessary and sufficient condition for (ϕ, ξ, η, g) to be normal is

$$[\phi, \phi] + 2\xi \otimes \eta = 0,$$

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ defined by

$$[\phi, \phi](X,Y) = [\phi X, \phi Y] + \phi^2[X,Y] - \phi[\phi X, Y] - \phi[X, \phi Y],$$

for any vector field Z on M^{2n+1} and smooth function f on $M^{2n+1} \times \mathbb{R}$. This may be expressed by the condition

$$(\nabla_X \phi)(Y) = \alpha(g(X,Y)\xi - \eta(Y)) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X),$$

for smooth function α and β on M^{2n+1}.

We say that (M, ϕ, ξ, η, g) is an Kenmotsu manifold if the covariant differentiation of ϕ satisfies [10]:

$$(\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{14}$$

For an Kenmotsu manifold from (2.2) it follows that

$$\nabla_X \xi = X - \eta(X)\xi \tag{15}$$

In a three dimensional Riemannian manifold, we always have

$$\begin{aligned} R(X,Y)Z =& g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ & - \frac{r}{2}g(Y,Z)X - g(X,Z)Y, \end{aligned}$$

In a three dimensional Kenmotsu manifold, we have [14]

$$R(X,Y)Z = (\frac{r}{2}+2)\big(g(Y,Z)X - g(X,Z)Y\big) - (\frac{r}{2}+3)\Big\{\eta(X)(g(Y,Z)\xi - g(\xi,Z)Y) + \eta(Y)(g(\xi,Z)X - g(X,Z)\xi)\Big\}$$

$$S(X,Y) = (\frac{r}{2}+2)g(Y,Z)X - (\frac{r}{2}+3)\eta(X)\eta(Y),$$

where r is a scalar curvature of M.
From (2.5), we obtain

$$R(X,Y)\xi = -[\eta(Y)X - \eta(X)Y],$$

and (2.11) yields

$$S(X,\xi) = -\eta(X).$$

3. Legendre Curves with C-parallel and C-proper mean curvature vector fields in the tangent and normal bundles

Definition. A unit speed curve γ on a smooth manifold is called Legendre curve if it satisfies $\eta(T) = 0$.

Let $\gamma : I \subset \mathbb{R} \to M$ be a non-geodesic Legendre curve of osculating order 3 in a Kenmotsu manifold. By using the definition of a Legendre curve, we obtain

$$\eta(T) = 0. \tag{16}$$

$$\eta(N) = -\frac{1}{\kappa}. \tag{17}$$

Theorem 3.1. There does not exist a non-geodesic Legendre curve $\gamma : I \subset \mathbb{R} \to M$ of osculating order 3, which has C-parallel mean curvature vector field in a three dimensional Kenmotsu manifold M.

Proof. Let γ be a curve with C-parallel mean curvature vector field. Then (7) it follows that

$$-\kappa^2 T + \kappa' N + \kappa\tau B = \lambda\xi.$$

Taking inner product of the foregoing equation with T, we have

$$\kappa = 0.$$

Therefore γ is a geodesic. This completes the proof. □

Theorem 3.2. Let $\gamma : I \subset \mathbb{R} \to M$ be a non-geodesic Legendre curve of osculating order 3 in a three dimensional Kenmotsu manifold. Then γ has C-parallel mean curvature vector field in the normal bundle if and only if

$$\kappa \neq \text{constant},$$

$$\tau^2 = \frac{\kappa'^2}{\kappa^2}(\kappa^2 - 1),$$

$$\xi = -\frac{1}{\kappa}N - \frac{\tau}{\kappa'}B,$$

and

$$\lambda = -\kappa\kappa'.$$

Proof. If

$$\kappa \neq \text{constant},$$

then from (9), we get

$$\kappa' N + \kappa\tau B = \lambda\xi. \tag{18}$$

Then taking the inner product of (18) with N, we have

$$\kappa' = \lambda\eta(N). \tag{19}$$

Using (17) in (19) and keeping in mind that $\kappa \neq 0$, as γ is non-geodesic. We get

$$\kappa' = \lambda(-\frac{1}{\kappa}),$$

and hence

$$\kappa'\kappa = -\lambda. \tag{20}$$

Again taking inner product of (18) with B, we obtain

$$\kappa\tau = \lambda\eta(B),$$

It follows that

$$\eta(B) = \frac{\kappa\tau}{-\kappa\kappa'} = -\frac{\tau}{\kappa'}. \tag{21}$$

Since $\xi \in span\{N, B\}$, then

$$\xi = \eta(N)N + \eta(B)B.$$

Using (17) and (21), we get

$$\xi = (-\frac{1}{\kappa})N + (-\frac{\tau}{\kappa'})B.$$

Also taking inner product of (18) with ξ, we have

$$\kappa'\eta(N) + \kappa\tau\eta(B) = \lambda.$$

Using (17) and (21) in the foregoing equation, we obtain

$$\kappa'(-\frac{1}{\kappa}) + \kappa\tau(-\frac{\tau}{\kappa'}) = -\kappa\kappa'.$$

Therefore

$$\tau^2 = \frac{\kappa'^2}{\kappa^2}(\kappa^2 - 1).$$

If is κ constant, then from (20) we have

$$\lambda = 0,$$

which is a contradiction, as λ is non-zero differential function.
The converse statement is trivial. This completes the proof of the theorem. □

Theorem 3.3. Let $\gamma : I \subset \mathbb{R} \to M$ be a non-geodesic Legendre curve of osculating order 3 in a three dimensional Kenmotsu manifold. Then γ is a curve with C-proper mean curvature vector field if and only if

$$\kappa = constant,$$

$$\lambda = -\kappa^2(\kappa^2 + \tau^2),$$

$$\xi = -\frac{1}{\kappa}N + \frac{2\kappa'\tau + \kappa\tau'}{\kappa^2(\kappa^2 + \tau^2)},$$

and

$$\kappa^2(\kappa^2 + \tau^2)^2 + (2\kappa'\tau + \kappa\tau')^2 = \kappa^4(\kappa^2 + \tau^2)^2.$$

Proof. Let γ be a curve with C-proper mean curvature vector field. Taking inner product of (8) with T, we have

$$\kappa\kappa' = 0.$$

Since γ is not geodesic (that is, $\kappa \neq 0$), we get

$$\kappa' = 0, \tag{22}$$

which implies that κ is constant.
Again taking inner product of (8) with N, we have

$$\kappa^3 + \kappa\tau^2 - \kappa'' = \lambda\eta(N). \tag{23}$$

Then using (17) and (22) in (23), we get

$$\lambda = -\kappa^2(\kappa^2 + \tau^2). \tag{24}$$

Also taking inner product of (8) with B and using (24), yields

$$\eta(B) = \frac{2\kappa'\tau + \kappa\tau'}{\kappa^2(\kappa^2 + \tau^2)}. \tag{25}$$

Since $\xi \in span\{N, B\}$, we obtain

$$\xi = -\frac{1}{\kappa}N + \frac{2\kappa'\tau + \kappa\tau'}{\kappa^2(\kappa^2 + \tau^2)}B.$$

Since ξ is a unit vector, we have

$$\eta(N)^2 + \eta(B)^2 = 1. \tag{26}$$

Using (17) and (25) in (26), we get

$$\kappa^2(\kappa^2 + \tau^2)^2 + (2\kappa'\tau + \kappa\tau')^2 = \kappa^4(\kappa^2 + \tau^2)^2. \tag{27}$$

The converse statement is trivial. □

Theorem 3.4. Let $\gamma : I \subset R \to M$ be a non-geodesic Legendre curve of osculating order 3 in a three dimensional Kenmotsu manifold. Then γ is a curve with C-proper mean curvature vector field in the normal bundle if and only if

$$\lambda = \kappa(\kappa'' - \kappa\tau^2),$$

$$\xi = (-\frac{1}{\kappa})N + \frac{2\kappa'\tau + \kappa\tau'}{\lambda}B,$$

and

$$\eta(N)^2 + \eta(B)^2 = 1.$$

Proof. Let γ is a curve with C-proper mean curvature vector field in the normal bundle. Therefore from (10), we have

$$(\kappa\tau^2 - \kappa'')N - (2\kappa'\tau + \kappa\tau')B = \lambda\xi,$$

Taking inner product of the foregoing equation with N, we have

$$\kappa\tau^2 - \kappa'' = \lambda\eta(N). \tag{28}$$

Using (17) in (28) yields

$$\lambda = \kappa(\kappa'' - \kappa\tau^2).$$

Again taking inner product of (10) with B, we get

$$\eta(B) = -\frac{2\kappa'\tau + \kappa\tau'}{\lambda}.$$

Since $\xi \in span\{N, B\}$, we obtain

$$\xi = -\frac{1}{\kappa}N + \frac{2\kappa'\tau + \kappa\tau'}{\lambda}B.$$

Since ξ is a unit vector. Then

$$\eta(N)^2 + \eta(B)^2 = 1.$$

□

The converse statement is trivial. This completes the proof of the theorem.

4. Magnetic curves with C-parallel and C-proper mean curvature vector fields in the tangent and normal bundles

Definition. A unit speed curve γ on a smooth manifold is called magnetic curve with respect to Levi-Civita connection if it satisfies $\nabla_T T = \phi T$.

Let $\gamma : I \subset \mathbb{R} \to M$ be a non-geodesic magnetic curve of osculating order 3 in a three dimensional Kenmotsu contact metric manifold. From the definition of a magnetic curve, we obtain

$$\eta(T) = \sqrt{1-\kappa^2}. \tag{29}$$

$$\eta(N) = 0. \tag{30}$$

$$\eta(B) = -\frac{\kappa\sqrt{1-\kappa^2}}{\tau}. \tag{31}$$

If $\kappa = 1$, then the curve is Legendre curve. We already discussed Legendre curves with C-parallel mean curvature vector field and C-proper mean curvature vector field in the previous Section. Therefore for the next section, we consider $\kappa \neq \pm 1$.

Theorem 4.1. Let $\gamma : I \subset \mathbb{R} \to M$ be a non-geodesic magnetic curve with curvature $\kappa \neq \pm 1$ of osculating order 3 in a three dimensional Kenmotsu manifold. Then γ is a curve with C-parallel mean curvature vector field if and only if

$$\kappa = \text{constant},$$

$$\lambda = -\frac{\kappa^2}{\sqrt{1-\kappa^2}},$$

$$\xi = \sqrt{1-\kappa^2}T + \frac{\kappa\tau}{\lambda}B,$$

and

$$\eta(T)^2 + \eta(B)^2 = 1.$$

Proof. Taking inner product of (7) with N and using (30), we get

$$\kappa' = 0.$$

This implies

$$\kappa = \text{constant}. \tag{32}$$

Again taking inner product of (7) with T and using (29), we have

$$\lambda = -\frac{\kappa^2}{\sqrt{1-\kappa^2}}. \tag{33}$$

Also taking inner product of (7) with ξ , we obtain

$$\eta(B) = \frac{\kappa\tau}{\lambda}. \tag{34}$$

Since $\xi \in span\{T, B\}$ and using (29) and (34), we get

$$\xi = \sqrt{1-\kappa^2}T + \frac{\kappa\tau}{\lambda}B. \tag{35}$$

Since ξ is a unit vector. Then

$$\eta(T)^2 + \eta(B)^2 = 1. \tag{36}$$

The converse of the theorem is trivial. This completes the proof of the theorem.
□

Theorem 4.2. Let $\gamma : I \subset \mathbb{R} \to M$ be a non-geodesic magnetic curve with curvature $\kappa \neq \pm 1$ of osculating order 3 in a three dimensional Kenmotsu manifold. Then γ is a curve with C-proper mean curvature vector field if and only if

$$\kappa'' = \kappa(\kappa^2 + \tau^2), \tag{37}$$

$$\lambda = \frac{3\kappa\kappa'}{\sqrt{1-\kappa^2}}, \tag{38}$$

$$\xi = \frac{3\kappa\kappa'}{\lambda}T - \frac{2\kappa'\tau + \kappa\tau'}{\lambda}B, \tag{39}$$

and

$$\eta(T)^2 + \eta(B)^2 = 1. \tag{40}$$

Proof. Let γ be a curve with C-proper mean curvature vector field. Taking inner product of (8) with N, we get

$$\kappa'' = \kappa(\kappa^2 + \tau^2).$$

Again taking inner product of (8) with T, we have

$$\eta(T) = \frac{3\kappa\kappa'}{\lambda}. \tag{41}$$

Using (29) in (41) yields

$$\lambda = \frac{3\kappa\kappa'}{\sqrt{1-\kappa^2}}.$$

Also taking inner product of (8) with B, we obtain

$$\eta(B) = -\frac{2\kappa'\tau + \kappa\tau'}{\lambda}. \tag{42}$$

Since $\xi \in span\{T, B\}$ and using (41) and (42), we have

$$\xi = \frac{3\kappa\kappa'}{\lambda}T - \frac{2\kappa'\tau + \kappa\tau'}{\lambda}B.$$

Since ξ is a unit vector. Thus

$$\eta(T)^2 + \eta(B)^2 = 1.$$

The converse of the theorem is trivial. This completes the proof of the theorem.
□

Theorem 4.3. There does not exist a non-geodesic magnetic curve $\gamma : I \subset \mathbb{R} \to M$ of osculating order 3 in three dimensional Kenmotsu manifold with C-parallel mean curvature vector field in the normal bundle.

Proof. Let γ have C-parallel mean curvature vector field in the normal bundle. Taking inner product with T, we get

$$\eta(T) = 0.$$

Therefore the curve is Legendre curve. Again from (29), we have

$$\kappa = \pm 1. \tag{43}$$

Using (43) in (31) yields

$$\eta(B) = 0. \tag{44}$$

Taking inner product of (9) with ξ and using (44), we get

$$\lambda = 0,$$

which is a contradiction, as λ is non-zero differential function. This completes the proof. □

Theorem 4.4. There does not exist a non-geodesic magnetic curve $\gamma : I \subset \mathbb{R} \to M$ of osculating order 3 in three dimensional Kenmotsu manifold with C-proper mean curvature vector field in the normal bundle.

Proof. If possible, let γ is a curve with C-proper mean curvature vector field in the normal bundle. Taking inner product of (10) with N, we get

$$\kappa'' = \kappa\tau^2. \tag{45}$$

Again taking inner product of (10) with B, we have

$$\eta(B) = -\frac{2\kappa'\tau + \kappa\tau'}{\lambda} \tag{46}$$

Also taking inner product of (10) with T, we obtain

$$\eta(T) = 0. \tag{47}$$

Then from (29) and (47) we have

$$\kappa = \pm 1. \tag{48}$$

Using (45),(46) and (48) it follows that

$$\eta(B) = 0. \tag{49}$$

Taking inner product of (10) with ξ and using (49) and (30) we get

$$\lambda = 0,$$

which is a contradiction. This completes the proof of the theorem. □

5. Example

Let us consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$E_1 = e^{-z}\frac{\partial}{\partial x}, \quad E_2 = e^{-z}\frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z},$$

are linearly independent at each point of M. Let g be the metric defined by

$$\begin{aligned} g(E_i, E_j) \quad &= 1 \quad \text{for} \quad i = j, \\ &= 0 \quad \text{for} \quad i \neq j. \end{aligned}$$

Here i and j runs from 1 to 3. Let η be the 1-form defined by $\eta(Z) = g(Z, E_3)$, for any vector field Z tangent to M. Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi E_1 = -E_2, \quad \phi E_2 = E_1, \quad \phi E_3 = 0.$$

Then we have

$$[E_1, E_2] = 0, \quad [E_1, E_3] = E_1, \quad [E_2, E_3] = E_2,$$

From Koszul's formula, the Riemannian connection we have

$$\nabla_{E_1}E_1 = -E_3, \quad \nabla_{E_1}E_2 = 0, \quad \nabla_{E_1}E_3 = E_1,$$

$$\nabla_{E_2}E_1 = 0, \quad \nabla_{E_2}E_2 = -E_3, \quad \nabla_{E_2}E_3 = E_2,$$

$$\nabla_{E_3}E_1 = 0, \quad \nabla_{E_3}E_2 = 0, \quad \nabla_{E_3}E_3 = 0.$$

Since

$$\nabla_X \xi = X - \eta(X)\xi$$

is satisfied for $\xi = E_3$. Hence the manifold is a Kenmotsu manifold [5].
Consider the curve $\gamma : I \to M = \mathbb{R}^3$ defined by $\gamma(s) = (\sqrt{\frac{2}{3}}s, \sqrt{\frac{1}{3}}s, 1)$.
Thus

$$\begin{aligned} g(\dot{\gamma}, E_3) \quad &= \eta(\dot{\gamma}) \\ &= g(\sqrt{\frac{2}{3}}E_1 + \sqrt{\frac{1}{3}}E_2, E_3) \\ &= 0, \end{aligned} \tag{50}$$

and

$$\begin{aligned} g(\dot{\gamma}, \dot{\gamma}) \quad &= g(\sqrt{\frac{2}{3}}E_1 + \sqrt{\frac{1}{3}}E_2, \sqrt{\frac{2}{3}}E_1 + \sqrt{\frac{1}{3}}E_2) \\ &= 1. \end{aligned} \tag{51}$$

Therefore the curve γ is Legendre curve.
Then by simple calculations

$$\begin{aligned}\nabla_T T = \nabla_{\dot\gamma}\dot\gamma \qquad &= \nabla_{(\sqrt{\frac{2}{3}}E_1+\sqrt{\frac{1}{3}}E_2)}(\sqrt{\frac{2}{3}}E_1+\sqrt{\frac{1}{3}}E_2)\\ &= -E_3. \end{aligned} \tag{52}$$

Therefore the curvature κ is given by

$$\kappa = |\nabla_T T| = 1, \quad N = \frac{1}{\kappa}\nabla_T T = -E_3. \tag{53}$$

Then

$$B = T \times N = -\sqrt{\frac{1}{3}}E_1 + \sqrt{\frac{2}{3}}E_2. \tag{54}$$

Thus by simple calculations the torsion of the curve γ is $\tau = 0$. Hence curve γ satisfies

$$3\kappa\kappa' T + (\kappa^3 + \kappa\tau^2 - \kappa'')N - (2\kappa'\tau + \kappa\tau')B = -E_3 = (-1)\xi, \text{ for } \lambda = -1. \tag{55}$$

Moreover, the following equations hold:

$$\kappa = 1 = \text{constant},$$

$$\lambda = -1 = -\kappa^2(\kappa^2 + \tau^2),$$

$$\xi = -\frac{1}{\kappa}N + \frac{2\kappa'\tau + \kappa\tau'}{\kappa^2(\kappa^2+\tau^2)}B$$

and

$$\kappa^2(\kappa^2+\tau^2)^2 + (2\kappa'\tau + \kappa\tau')^2 = \kappa^4(\kappa^2+\tau^2)^2.$$

Therefore γ is a curve with C-proper mean curvature vector field. Thus the theorem 3.3 is verified.

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