J. Appl. Math. & Informatics Vol. **40***(2022), No. 3 - 4, pp. 603 - 618*
https://doi.org/10.14317/jami.2022.603

HIGHER ORDER GALERKIN FINITE ELEMENT METHOD FOR THE GENERALIZED DIFFUSION PDE WITH DELAY

GEMEDA TOLESSA LUBO* AND GEMECHIS FILE DURESSA

Abstract. In this paper, a numerical solution of the generalized diffusion equation with a delay has been obtained by a numerical technique based on the Galerkin finite element method by applying the cubic B-spline basis functions. The time discretization process is carried out using the forward Euler method. The numerical scheme is required to preserve the delay-independent asymptotic stability with an additional restriction on time and spatial step sizes. Both the theoretical and computational rates of convergence of the numerical method have been examined and found to be in agreement. As it can be observed from the numerical results given in tables and graphs, the proposed method approximates the exact solution very well. The accuracy of the numerical scheme is confirmed by computing L_2 and L_∞ error norms.

AMS Mathematics Subject Classification : 65M30, 65M60.
Key words and phrases : Generalized diffusion equation, finite element, cubic B-spline, forward Euler.

1. Introduction

In this work, we consider the generalized diffusion equation with delay

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = r_1 \frac{\partial^2 u(x,t)}{\partial x^2} + r_2 \frac{\partial^2 u(x,t-\tau)}{\partial x^2}, t > 0, 0 < x < L, \\ u(x,t) = \varphi\ (x,t),\ -\tau \le t \le 0, 0 \le x \le L, \\ u(0,t) = u(L,t) = 0, t \ge -\tau, \end{cases} \tag{1}$$

where $r_1 > 0, r_2 \ge 0$ and $\tau > 0$ is a delay constant.
Generalized diffusion equations with delay have attracted significant interest in the last several decades due to their frequent occurrence in real life problems[9, 10, 11, 12, 13]. As they describe diverse physical phenomena like heat transfer, diffusion, mechanics of elastic and plastic materials, fluid mechanics, electrostatics and -dynamics, and many more. The generalized diffusion equation with

Received June 5, 2021. Revised October 8, 2021. Accepted October 11, 2021. *Corresponding author.
© 2022 KSCAM.

the delay has inherent complex nature because of which analytical solutions are hardly obtainable. Therefore, one has to rely mostly on numerical treatments. Jackiewicz and Zubik-Kowal [8] used spectral collocation and waveform relaxation methods to investigate nonlinear partial differential equations with delay. Chen and Wang [5] used the variational iteration method to study a neutral functional differential equation with delays. Partial differential equations(PDEs) with delay were studied by many authors (see for instance [6, 7, 14, 19]). The generalized diffusion equations with a delay are considered in [18, 16, 1, 22]. Up to now, most of the numerical methods available to approximate the diffusion term are based on finite difference methods. Finite difference methods are easy to understand and implement. However, they suffer from a series of drawbacks, such as inflexibility with respect to geometry, the difficulty of generalizing to higher orders of approximation, and the inability to perform true adaptive local refinement. The finite element method (FEM) is an efficient numerical method for solving problems of engineering and mathematical modeling, most specifically PDEs. It is well known that the FEM can be easily designed for high order of accuracy in space. To the best knowledge of the authors, the idea of the cubic B-spline FEM has not been implemented for solving the generalized diffusion equation with a delay so far. Nevertheless, the idea of B-spline FEM is applied to solve Burgers' equations without delay [2, 3, 17]. The proposed method has the order of four in the spatial direction. It gives a better convergence result than the standard second-order central finite-difference. Our best concern is to formulate a numerical scheme of a higher order of accuracy by using cubic B-spline shape functions.

Notations: Denote $\|.\|$ and $\|.\|_r$ as the norm $L_2 = L_2(\Omega)$ and the sobolev space $H^r = H^r(\Omega) = W_2^r(\Omega)$ respectively, so the real valued-function v,

$$\|v\|_r = \quad \|v\|_{L_2} := \left(\int_\Omega v(x)^2 dx \right)^{\frac{1}{2}},$$

and for a positive integer r,

$$\|v\|_r = \quad \|v\|_{H^r} := \left(\sum_{i\leq r} \quad \left\| \frac{\partial^i v(x)}{\partial x^i} \right\|^2 \right)^{\frac{1}{2}}.$$

Let $v(x), w(x)(x \in \Omega)$ be real valued functions.

$$(v, w) := \int_\Omega v(x)w(x)dx, \ (\nabla v, \nabla w) := \int_\Omega \frac{\partial v(x)}{\partial x} \frac{\partial w(x)}{\partial x} dx,$$

and C denote a positive not necessarily the same at different occurrences ,which may depend on r_1, r_2 and t of (1), but independent of h and Δt (the step sizes in t - direction). We denote $u(x,t)$ by u or $u(t)$.

2. Description of the Method

Let $\Delta t = \tau/(m+1)$ be a given step size with $m \geq 1$, the grid points $t_n = n\Delta t (n = 0, 1, \dots)$ and be the approximation of $u(t)$ at $t = t_n = n\Delta t$. For positive integer $\mathbb{N}$, let $\{x_k\}_{k=0}^{N+1}$ be a uniform partition on $\Omega = [0, \pi]$ in the x direction, such that $x_k = kh$, where $h = \pi/(N+1)$ is the step size.
Define the space

$$S_3 = \{\zeta : \zeta \in C^2([0, \pi]), \zeta|_{[x_{k-1}, x_k]} \in P_3, 1 \leq k \leq N+1\},$$

where P_3 is the space of all polynomials of degree ≤ 3. Extending the partition $\{x_k\}_{k=0}^{N+1}$ using $x_k = kh$, $k = -3, -2, -1, N+2, N+3, N+4$. As a basis for S_3, we choose the B -splines, $\{Q_j\}_{j=-1}^{N+2}(x)$, where

$$Q_j(x) = \begin{cases} h^{-3} f_1(x - x_{j-2}), & x \in [x_{j-2}, x_{j-1}], \\ f_2(\frac{x - x_{j-1}}{h}), & x \in [x_{j-1}, x_j], \\ f_2(\frac{x_{j+1} - x}{h}), & x \in [x_j, x_{j+1}], \\ h^{-3} f_1(x_{j+2} - x), & x \in [x_{j+1}, x_{j+2}], \\ 0, & x \notin [x_{j-2}, x_{j+2}], \end{cases} \tag{2}$$

and $f_1(x) = x^3, f_2(x) = 1 + 3x + 3x^2 - 3x^3$. Denote a basis for the space

$$S_h^T = \{\zeta \in S_3; \zeta(0) = \zeta(\pi) = 0\}. \tag{3}$$

The cubic B -spline base for S_h^T can be redefined as follows:

$$\begin{cases} \overline{Q}_0 = Q_0(x) - 4Q_{-1}(x), & \overline{Q}_1(x) = Q_1(x) - Q_{-1}(x), \\ \overline{Q}_j(x) = Q_j(x), & j = 2, 3, \dots, N-1, \\ \overline{Q}_N(x) = Q_N(x) - Q_{N+2}(x), & \overline{Q}_{N+1}(x) = Q_{N+1}(x) - 4Q_{N+2}(x). \end{cases} \tag{4}$$

As the cubic B -splines $Q_j(x)$, the cubic B-splines $\overline{Q}_j(x)$ have the support of at least 4 subintervals.
The weak formulation of (1) for $U \in S_h^T$ is

$$\left(\frac{\partial U(x,t)}{\partial t}, \zeta\right) + r_1(\nabla U(x,t), \nabla\zeta) + r_2(\nabla U(x, t-\tau), \nabla\zeta) = 0, \tag{5}$$

with

$$a(U, \zeta) = \left(\frac{\partial U(x,t)}{\partial t}, \zeta\right) + r_1(\nabla U(x,t), \nabla\zeta) + r_2(\nabla U(x, t-\tau), \nabla\zeta) \; and \; l(\zeta) = 0,$$

$$a(U, \zeta) = l(\zeta), \forall \zeta \in S_h^T,$$

where $a(U, \zeta)$ is symmetric bilinear and $l(\zeta)$ is linear functional.
Application of forward Euler Galerkin method leads to a numerical scheme of the following type

$$\left(\frac{U^n - U^{n-1}}{\Delta t}, \zeta\right) + r_1(\nabla U^{n-1}, \nabla\zeta) + r_2(\nabla U^{n-m-1}, \nabla\zeta) = 0, for \; \forall \zeta \in S_h^T, \tag{6}$$

where $U^n(.) = \varphi(., t_n)$ for $-m \leq n \leq 0$.
Let

$$U^n(x) := \sum_{j=0}^{N+1} \overline{Q}_j(x)\alpha_j^n. \tag{7}$$

Substituting (7) into (6) and choosing $\zeta = \overline{Q}_i(x), i = 0, \ldots, N+1$, we obtain

$$\frac{1}{\Delta t}\sum_{j=0}^{N+1}(\alpha_j^n - \alpha_j^{n-1})(\overline{Q}_i(x), \overline{Q}_j(x)) = -r_1 \sum_{j=0}^{N+1} \alpha_j^{n-1}(\nabla\overline{Q}_i(x), \nabla\overline{Q}_j(x))$$

$$-r_2 \sum_{j=0}^{N+1} \alpha_j^{n-m-1}(\nabla\overline{Q}_i(x), \nabla\overline{Q}_j(x)), \tag{8}$$

which can be rewritten

$$\frac{1}{\Delta t}\sum_{j=0}^{N+1}(\alpha_j^n - \alpha_j^{n-1}) \int_0^{\pi} \overline{Q}_i(x)\overline{Q}_j(x)dx = -r_1 \sum_{j=0}^{N+1} \alpha_j^{n-1} \int_0^{\pi} \overline{Q}_i'(x)\overline{Q}_j'(x)dx$$

$$-r_2 \sum_{j=0}^{N+1} \alpha_j^{n-m-1} \int_0^{\pi} \overline{Q}_i'(x)\overline{Q}_j'(x)dx. \tag{9}$$

Defining the following matrices:

$$E = (e_{i,j})_{i,j=0}^{N+1} = \int_0^{\pi} \overline{Q}_i'(x)\overline{Q}_j'(x)dx, \tag{10}$$

$$D = (d_{i,j})_{i,j=0}^{N+1} = \int_0^{\pi} \overline{Q}_i(x)\overline{Q}_j(x)dx. \tag{11}$$

The entries of the matrices E and D is found from (10) and (11) respectively.

$$E = \frac{3}{10h}\begin{pmatrix} 80 & 43 & -20 & -1 & 0 & \ldots & \ldots & \ldots & 0 \\ 43 & 104 & -14 & -24 & -1 & \ddots & & & \vdots \\ -20 & -14 & 80 & -15 & -24 & -1 & \ddots & & \vdots \\ -1 & -24 & -15 & 80 & -15 & -24 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & -24 & -15 & 80 & -15 & -24 & -1 \\ \vdots & & \ddots & -1 & -24 & -15 & 80 & -14 & -20 \\ \vdots & & & \ddots & -1 & -24 & -14 & 104 & 43 \\ 0 & \ldots & \ldots & \ldots & 0 & -1 & -20 & 43 & 80 \end{pmatrix} \tag{12}$$

and the mass matrix

$$D = \frac{h}{140}\begin{pmatrix} 496 & 773 & 116 & 1 & 0 & \dots & \dots & \dots & 0 \\ 773 & 2296 & 1190 & 120 & 1 & \ddots & & & \vdots \\ 116 & 1190 & 2416 & 1191 & 120 & 1 & \ddots & & \vdots \\ 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 \\ \vdots & & \ddots & 1 & 120 & 1191 & 2416 & 1190 & 116 \\ \vdots & & & \ddots & 1 & 120 & 1190 & 2296 & 773 \\ 0 & \dots & \dots & \dots & 0 & 1 & 116 & 773 & 496 \end{pmatrix} \quad (13)$$

In matrix notation, the numerical scheme (6) can be expressed as

$$\begin{cases} D\alpha^n = (D - r_1 \Delta t E)\alpha^{n-1} - r_2 \Delta t E \alpha^{n-m-1}, for\ n > 0, \\ \alpha^n = \gamma^n, for - m \leq n \leq 0, \end{cases} \quad (14)$$

with γ^n being the corresponding vector of the components of the given initial approximation of $\varphi(t_n)$, $\alpha^n := (\alpha_0, \dots, \alpha_{N+1})^T$, and hence obviously has a unique solution for positive integer n. The algebraic system (14) is solved by using matrix inverse method.

3. Stability Analysis

Definition 3.1. If the solution U^n corresponding to any sufficiently differentiable function $\varphi_h(0,t) = \varphi_h(\pi,t)$ satisfies

$$\lim_{n\to\infty} U^n = 0, x \in [0,\pi], \quad (15)$$

then the zero of solution (6) is asymptotically stable.

Lemma 3.1. [19]*Let $\gamma_m(z) = \alpha(z)z^m - \beta(z)$ be a polynomial, where $\alpha(z)$and $\beta(z)$ are polynomials of constant degree. Then, the polynomial $\gamma_m(z)$ is a Schur polynomial for $m \geq 1$ if and only if the following condition hold:*

(i) $\alpha(z) = 0 \Rightarrow |z| < 1$,
(ii) $|\beta(z)| \leq |\alpha(z)|, \forall z \in C, |z| = 1,$ *and*
(iii) $\gamma_m(z) \neq 0, \forall z \in C, |z| = 1.$

Let $K := [x_i, x_{i-1}]$ be an element of the finite element, and $\tilde{K} := [-1,1]$ be the reference element in η -plane. Then,

$$\int_K Q_i Q_j dx = \frac{h}{2}\int_{\tilde{K}} \tilde{Q}_i \tilde{Q}_j d\eta, \int_K \nabla Q_i \nabla Q_j dx = \frac{2}{h}\int_{\tilde{K}} \nabla\tilde{Q}_i \nabla\tilde{Q}_j d\eta,$$

By (8),

$$\alpha^n = \frac{2}{h}\tilde{D}^{-1}\left(\frac{h}{2}\tilde{D} - \frac{2r_1\Delta t}{h}\tilde{E}\right)\alpha^{n-1} - \frac{4r_2\Delta t}{h^2}D^{-1}\tilde{E}\alpha^{n-m-1}$$

$$\alpha^n = \left(I - \frac{4r_1\Delta t}{h^2}\tilde{D}^{-1}\tilde{E}\right)\alpha^{n-1} - \frac{4r_2\Delta t}{h^2}\tilde{D}^{-1}\tilde{E}\alpha^{n-m-1}. \tag{16}$$

Let $\alpha^n = \lambda^n C_1$, where C_1 is a constant vector and λ^n is an eigenvalue. Then the characteristic of (16) is

$$\lambda^m - \left(1 - \frac{4r_1\Delta t}{h^2}\lambda_{\tilde{D}^{-1}\tilde{E}}\right)\lambda^{m-1} + \frac{4r_2\Delta t}{h^2}\lambda_{\tilde{D}^{-1}\tilde{E}} = 0. \tag{17}$$

Theorem 3.2. *Suppose that* $0 \leq r_2 < r_1$. *Then the solution of proposed numerical scheme is delay independently asymptotically stable if and only if* $\frac{\Delta t}{h^2} \leq \frac{1}{4r_1\lambda_{\tilde{D}^{-1}\tilde{E}}}$.

Proof. Denote $\alpha(\lambda) = \lambda - (1 - \frac{4r_1\Delta t}{h^2}\lambda_{\tilde{D}^{-1}\tilde{E}})$ and $\beta(\lambda) = -\frac{4r_2\Delta t}{h^2}\lambda_{\tilde{D}^{-1}\tilde{E}}$.
(a) If $\alpha(\lambda) = 0$, then $|\lambda| = \left|(1 - \frac{4r_1\Delta t}{h^2}\lambda_{\tilde{D}^{-1}\tilde{E}})\right| < 1$ if and only if $\frac{\Delta t}{h^2} \leq \frac{1}{2r_1\lambda_{\tilde{D}^{-1}\tilde{E}}}$.
(b) For $\forall\lambda \in \mathbb{C}$, $|\lambda| = 1$, we can assume $\lambda = \cos\theta + i\sin\theta$, then $\frac{\Delta t}{h^2} \leq \frac{1}{4r_1\lambda_{\tilde{D}^{-1}\tilde{E}}}$.

$$\begin{aligned}|\alpha(\lambda)|^2 &= \left|\lambda - 1 + \frac{4r_1\Delta t}{h^2}\lambda_{\tilde{D}^{-1}\tilde{E}}\right)\right|^2 \\ &= 2(1-\cos\theta)\left(1 - \frac{4r_1\Delta t}{h^2}\lambda_{\tilde{D}^{-1}\tilde{E}}\right) + \left(\frac{4r_1\Delta t}{h^2}\lambda_{\tilde{D}^{-1}\tilde{E}}\right)^2 \\ &\geq \left(\frac{4r_1\Delta t}{h^2}\lambda_{\tilde{D}^{-1}\tilde{E}}\right)^2 > \left(\frac{4r_2\Delta t}{h^2}\lambda_{\tilde{D}^{-1}\tilde{E}}\right)^2 = |\beta(\lambda)|^2.\end{aligned}$$

Else, if $\frac{\Delta t}{h^2} > \frac{1}{r_1\lambda_{\tilde{D}^{-1}\tilde{E}}}$, then $|\alpha(\lambda)|^2 < \left(\frac{4\Delta t}{h^2}\lambda_{\tilde{D}^{-1}\tilde{E}}\right)^2(r_1)^2$. Choose $r_2^2 = \frac{r_1^2 + \frac{|\alpha(\lambda)|^2}{(\frac{4r_1\Delta t}{h^2}\lambda_{\tilde{D}^{-1}\tilde{E}})^2}}{2}$, so $\frac{|\alpha(\lambda)|^2}{(\frac{4r_2\Delta t}{h^2}\lambda_{\tilde{D}^{-1}\tilde{E}})} < r_2 < r_1$ and $|\alpha(\lambda)| < r_2\frac{4r_2\Delta t}{h^2}\lambda_{\tilde{D}^{-1}\tilde{E}} = |\beta(\lambda)|$. Therefore, $\frac{\Delta t}{h^2} \leq \frac{1}{4r_1\lambda_{\tilde{D}^{-1}\tilde{E}}}$ is a sufficient and necessary condition.
(c) By (b), it is straightforward. The proof is completed. □

4. Convergence Analysis

Introduce the elliptic or Ritz projection R_h on to S_h^T as the orthogonal projection with respect to $(\nabla v, \nabla w)$, such, so that

$$(\nabla R_h v, \nabla\zeta) = (\nabla v, \nabla\zeta), \forall\zeta \in S_h^T, for\ v \in H_0^1(\Omega). \tag{18}$$

Lemma 4.1. [15] *Assume that for any* $v \in H^s(\Omega) \cap H_0^1(\Omega)$,

$$\inf_{\zeta \in \overline{S}_h} \{\|v-\zeta\| + \|\nabla(v-\zeta)\|\} \le Ch^s,$$

for $1 \le s \le r$ *holds. Then, with* R_h *defined by (18) we have* $\|R_h v - \zeta\| + h\|\nabla(R_h v - v)\| \le Ch^s \|v\|_s$, *for any* $v \in H^s(\Omega) \cap H_0^1(\Omega)$, $1 \le s \le r$.

Let $u(t) := u(.,t)$ and $u : [0, +\infty) \to H_0^1(\Omega)$. Define $D_h : H_0^1(\Omega) \to S_h^T$ by

$$r_1(\nabla D_h u(t) - \nabla u(t), \nabla\zeta) + r_2(\nabla D_h u(t-\tau) -\nabla u(t-\tau), \nabla\zeta) = 0, for\ t > 0\ \zeta \in S_h^T \tag{19}$$

$$D_h u(t) = R_h u(t) = R_h \varphi(t), for\ -\tau \le t \le 0. \tag{20}$$

The number r is referred to as the order of accuracy of the family S_h^T. For the piecewise linear functions in a plane domain $r = 2$. In the case $r > 2$, S_h^T often consists of piecewise polynomials of degree at most $r-1$. For instance, $r = 4$ in the case of piecewise cubic polynomial subspaces ([20], see p. 4). Let u_h and u be the solutions of (1). Then error can be quantified by bounding the norm of the error $u_h(t) - u(t)$ in terms of the mesh spacing h of the finite element mesh. This can be generalized in the following remarks.

Remark 4.1. Using polynomial with degrees $p \ge 1$ as basis we expect an error bound of

$$\|u_h(t) - u(t)\| \le Ch^{p+1},$$

where C is a problem-dependent constant independent of h and the constant $p+1$ indicates the order of convergence of the FEM, as the mesh spacing h decreases.

Theorem 4.2. *Let* u *and* U^n *be the exact and approximation solution of (1) respectively.Assume that* $\|u(t) - R_h u(t)\|_i \le Ch^4 \|u(t)\|$, $\|u_t(t) - R_h u_t(t)\| \le Ch^4 \|u_t(t)\|$, $-\tau \le t \le 0$ *and* $\|\varphi_h(t) - \varphi(t)\| \le Ch^4$, *then*

$$\|U^n - u(t_n)\| \le C(h^4 + \Delta t), for\ any\ n = 1, 2, ...$$

Proof. Denote

$$U^n - u(t_n) = (U^n - D_h u(t_n)) + (D_h u(t_n) - u(t_n)) = \mu^n + \rho^n,$$

and $\rho^n(t) = \rho(t_n)$ is bounded as in [15],

$$\left(\frac{\mu^n - \mu^{n-1}}{\Delta t}, \zeta\right) + r_1(\nabla\mu^{n-1}, \nabla\zeta) + r_2(\mu^{n-m-1}, \zeta) = -(W^n, \zeta), \forall\zeta \in S_h^T,$$

where

$$W^n = \frac{D_h u(t_n) - D_h u(t_{n-1})}{\Delta t} - u_t(t_n) = (D_h - I)\tilde{\partial}u(t_n) + (\tilde{\partial}u(t_n) - u_t(t_n))$$

$$=: W_1^n + W_2^n, \tilde{\partial}u(t_n) := \frac{u_t(t_n) + u_t(t_{n-1})}{\Delta t}.$$

Setting $\zeta = \mu^{n-1}$,gives

$$\left(\frac{\mu^n - \mu^{n-1}}{\Delta t}, \mu^{n-1}\right) + r_1 \left\|\mu^{n-1}\right\|_1^2 + r_2(\mu^{n-m-1}, \mu^{n-1}) = -(W^n, \mu^{n-1}).$$

By applying Schwartz inequality,

$$\left(\frac{\mu^n - \mu^{n-1}}{\Delta t}, \mu^{n-1}\right) + \left\|\mu^{n-1}\right\|_1^2 \leq C(\left\|\mu^{n-m-1}\right\|_1^2 + \|W^n\| \left\|\mu^{n-1}\right\|).$$

So

$$\|\mu^n\|^2 + \Delta t \left\|\mu^{n-1}\right\|_1^2 \leq C(\left\|\mu^{n-1}\right\|^2 + \Delta t \left\|\mu^{n-m-1}\right\|_1^2 + (\Delta t)^2 \|W^n\|^2).$$

Without loss of generality, we can assume $n \in ((k-1)m, km], k \in N$. Then

$$\Delta t \left\|\mu^{n-1}\right\|_1^2 \leq C(\left\|\mu^{n-1}\right\|^2 + \Delta t \left\|\mu^{n-m-1}\right\|_1^2 + (\Delta t)^2 \|W^n\|^2)$$

$$\leq C(\left\|\mu^{n-1}\right\|^2 + \left\|\mu^{n-m-1}\right\|^2 + \Delta t \left\|\mu^{n-2m}\right\|_1^2 + (\Delta t)^2(\|W^n\|^2 + \left\|W^{n-m-1}\right\|^2))$$

$$\leq \cdots \leq C\left(\sum_{i=0}^{k-1} \left\|\mu^{n-im-1}\right\|^2 + \Delta t \left\|\mu^{n-km-1}\right\|_1^2 + (\Delta t)^2 \sum_{i=0}^{k-1} \left\|W^{n-im-1}\right\|^2\right).$$

Therefore

$$\|\mu^n\|^2 \leq C\left(\sum_{i=0}^{k-1} \left\|\mu^{n-im-1}\right\|^2 + \Delta t \left\|\mu^{n-km-1}\right\|_1^2 + (\Delta t)^2 \sum_{i=0}^{k-1} \left\|W^{n-im-1}\right\|^2\right)$$

By the assumption of the theorem and using the discrete Gronwall inequality(see[4]),

$$\|\mu^n\|^2 \leq C\left(\left\|\mu^0\right\|^2 + \Delta t \left\|\mu^{n-km-1}\right\|_1^2 + (\Delta t)^2 \sum_{i=0}^{k-1} \left\|W^{n-im-1}\right\|^2\right). \tag{21}$$

We write

$$W_1^n = (D_h - I)\tilde{\partial} u(t_n) = (\Delta t)^{-1} \int_{t_{n-1}}^{t_n} d/dt(D_h - I)u(t)dt,$$

So

$$(\Delta t)^2 \sum_{i=0}^{k-1} \left\|W^{n-im-1}\right\|^2 = \sum_{i=0}^{k-1} \left(\int_{t_{n-im-1}}^{t_{n-im}} d/dt(D_h - I)u(t)dt\right)^2$$

$$\leq \sum_{i=0}^{k-1} \left(\int_{t_{n-im-1}}^{t_{n-im}} Ch^4 \|u_t(t)\| \, dt\right)^2 \leq Ch^{2(4)}. \tag{22}$$

Moreover

$$\left\|\Delta t W_2^i\right\| = u(t_i) - u(t_{i-1}) - \Delta t u_t(t_i) = -\int_{t_{i-1}}^{t_i} (t_i - t_{i-1}) u_{tt}(t) dt,$$

So that

$$(\Delta t)^2 \sum_{i=0}^{k-1} \left\| W_2^{n-im-1} \right\|^2 = \sum_{i=0}^{k-1} \left\| \int_{t_{n-im-1}}^{t_{n-im}} (t - t_{n-im-1}) u_{tt}(t) dt \right\|^2$$

$$\leq (\Delta t)^2 \sum_{i=0}^{k-1} \left\| \int_{t_{n-im-1}}^{t_{n-im}} u_{tt}(t) dt \right\|^2 \leq C(\Delta t)^2, \tag{23}$$

from(22) and (23)

$$\| U^n - u(t_n) \| \leq C(h^4 + \Delta t).$$

This complete the proof. □

5. Numerical Experiments

We carry out numerical experiments to illustrate our theoretical results. The L_∞ and L_2 error norms are computed by:

$$L_\infty = \max_{0 \leq n \leq N+1} |t_n) - (U^n)|, L_2 = \sqrt{h \sum_{i=0}^{N+1} |u(t_n) - (U^n)|^2}$$

Order of convergence is denoted by

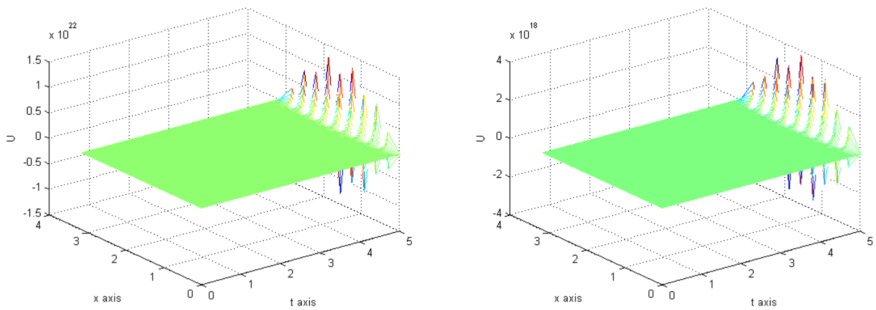
$$Order(r) = \frac{log(Error(N)/Error(2N))}{log(2 - \frac{1}{N+1})}$$

5.1. Stability Test.

Example 5.1. We conduct numerical experiments to illustrate our numerical stability analysis of the method when applied to

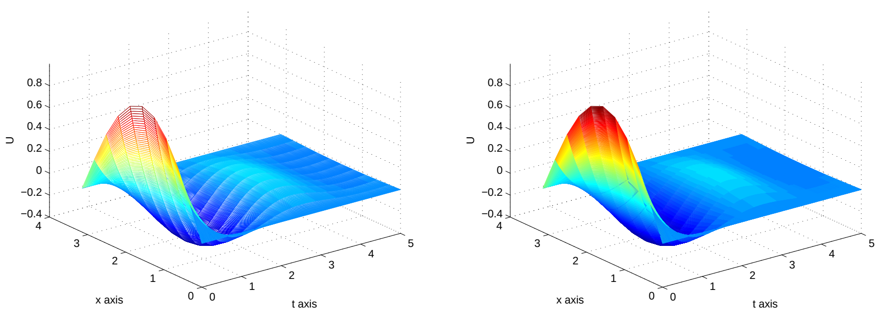
$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = r_1 \frac{\partial^2 u(x,t)}{\partial x^2} + r_2 \frac{\partial^2 u(x,t-\tau)}{\partial x^2}, t > 0, 0 < x < \pi, \\ u(x,t) = \varphi\ (x,t),\ -\tau \leq t \leq 0, 0 \leq x \leq \pi, \\ u(0,t) = u(\pi,t) = 0, t \geq -\tau. \end{cases} \tag{24}$$

First, we take the initial function as $\varphi(x,t) = sin(x), \tau = 1, r_1 = 1.5, r_2 = 1$ such that the trivial solution of (1) is asymptotically stable.'



(A) (B)

FIGURE 1. Solution of (24) with parameter values a) $N = 10, m = 40$ b) $N = 10, m = 50$



(A) (B)

FIGURE 2. Solution of (24) with parameter values a) $N = 10, m = 100$ b) $N = 10, m = 200$

Numerical results are obtained and plotted at time $T = 5$ using different($\Delta t = \tau/(m+1)$, $h = \pi/(N+1)$). From figures 1(A) and 1(B), it is easily seen that the numerical solution is unstable for the step sizes $\Delta t = \frac{1}{40}$ and $\Delta t = \frac{1}{50}$. The numerical solution is asymptotically stable for $\Delta t = \frac{1}{100}$ and $\Delta t = \frac{1}{500}$ in figures 2(A) and 2(B), respectively. These agree with the theoretical results described in Theorem 3.2.

.

5.2. Convergence Test.

Example 5.2. [21] We use the following equation to show the convergence results:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = r_1 \frac{\partial^2 u(x,t)}{\partial x^2} + r_2 \frac{\partial^2 u(x,t-\tau)}{\partial x^2} + h(x,t), t > 0, 0 < x < \pi, \\ u(x,t) = \varphi\ (x,t),\ -\tau \leq t \leq 0, 0 \leq x \leq \pi, \\ u(0,t) = u(\pi,t) = 0, t \geq -\tau, \end{cases} \tag{25}$$

where the initial function and the exact are respectively $\varphi(x,t) = \sin(x)$ and $u(x,t) = exp(-t)\sin(x)$. The added $h(x,t)$ can be specified by substituting the exact solution in (25).
Here, we take the parameters $r_1 = 1, r_2 = 0.5, \tau = 0.5$, and compute the problem on $[0,\pi] \times [0,2]$ for different space and temporal step sizes($\Delta x = \pi/(N+1), \Delta t = \tau/(m+1)$).

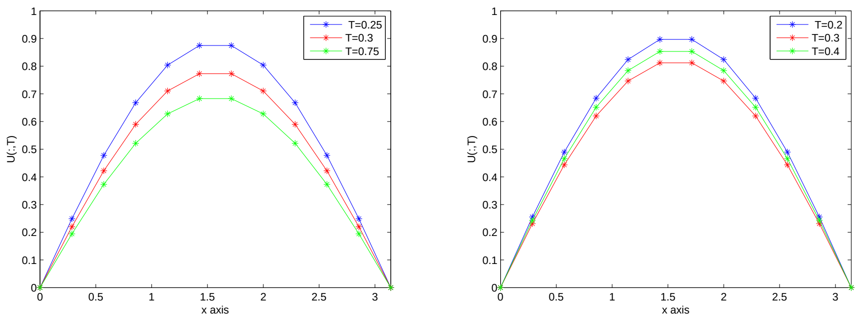
TABLE 1. Errors and convergence orders ($\Delta t \approx \Delta x^4$)

	Central difference method($\theta = 1/2$)[21]				Present method			
N	L_2	Order	L_∞	Order	L_2	Order	L_∞	Order
10	1.80E-03	-	1.44E-03	-	1.12E-03	-	8.86E-04	-
20	3.22E-04	2.48	2.57E-04	2.48	7.02E-05	4.29	5.58E-05	4.27
40	8.59E-05	1.90	6.86E-05	1.90	4.39E-06	4.14	3.50E-06	4.14
80	2.14E-05	2.00	1.71E-05	2.00	2.74E-07	4.07	2.19E-07	4.07

TABLE 2. Errors and convergence orders ($\Delta t \approx 0.5\Delta x^4$)

	Central difference method($\theta = 1$)[21]				Present method			
N	L_2	Order	L_∞	Order	L_2	Order	L_∞	Order
10	1.34E-02	-	1.07E-02	-	2.23E-03	-	1.76E-03	-
20	3.25E-03	2.04	2.59E-03	2.04	1.40E-04	4.28	1.12E-04	4.26
40	8.10E-04	2.00	6.46E-04	2.00	8.77E-06	4.14	6.99E-06	4.14
80	2.02E-04	2.00	1.61E-04	2.00	5.48E-07	4.07	4.37E-07	4.07

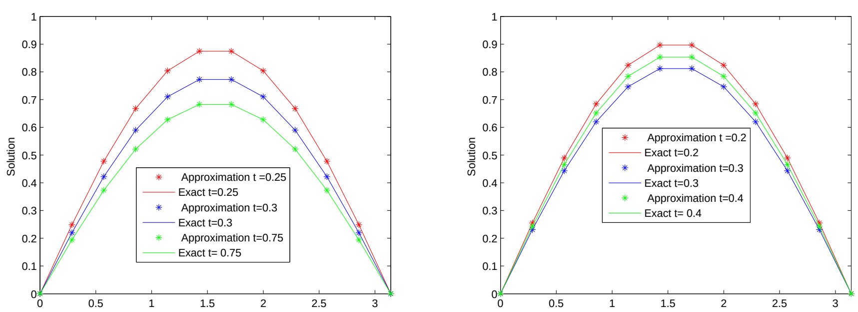
Numerical errors and the corresponding orders are listed in Table 1 and 2. As it can be seen from the tables, there is a noticeable decrease in both error norms when mesh sizes decrease. These results confirm the convergence of the proposed numerical scheme. The calculated error norms are compared with the results obtained in [21]. Additionally, they also imply that the numerical method gives a better order of convergence than the standard second-order central finite-difference. In Figure 3, the graph of approximation solutions for example 5.2 at different times is given. In Figures 4 - 6, the exact and numerical solutions of the problem are drawn on the same coordinate axis. In figures 7 and 8, we present the graph of analytical solution and numerical solution. By comparing the two solutions, we can see that the solution obtained by the present method is similar to the one obtained by the analytical method.



(A) (B)

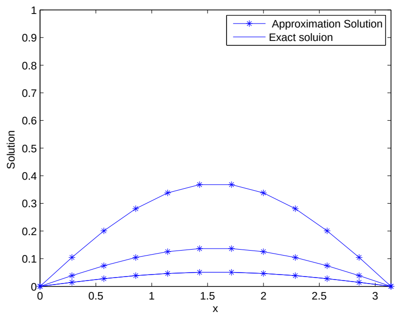
FIGURE 3. The numerical solutions of example 5.2 at different times ($N = 10$ and $m = 100$)

.

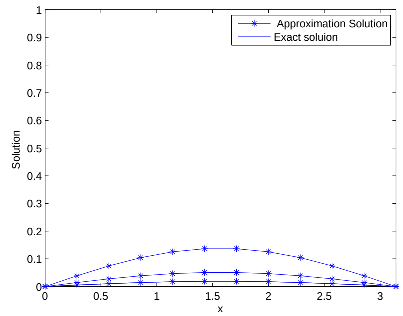


(A) (B)

FIGURE 4. Comparisons between approximate and exact solutions of example 5.2 ($N = 10$ and $m = 100$)

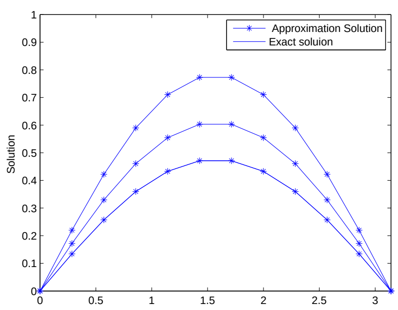


(A)

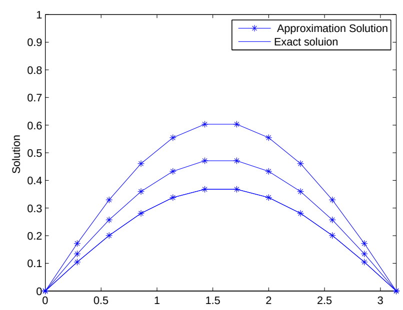


(B)

FIGURE 5. Exact and numerical solutions for example 5.2
(a) $T = 2, T = 3, T = 4,\ N = 10, m = 100$, and $\tau = 1$.
(b) $T = 1, T = 2, T = 3, N = 10, m = 100$, and $\tau = 1$.

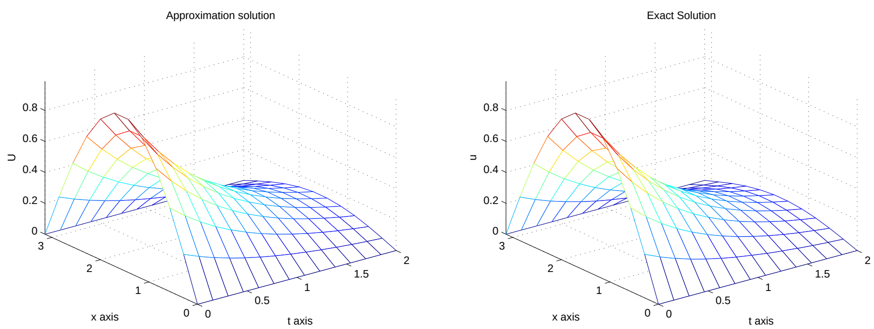


(A)



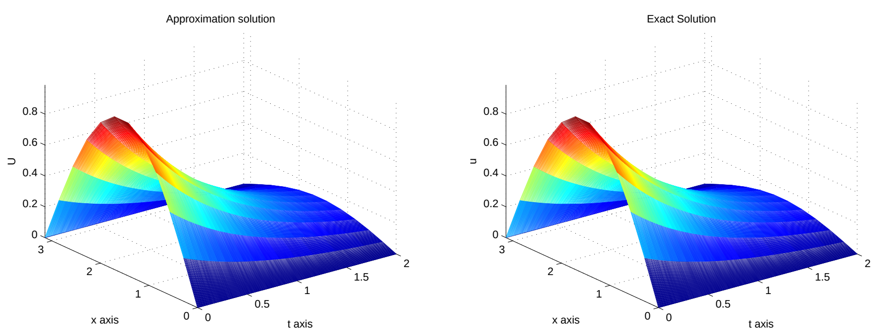
(B)

FIGURE 6. Exact and numerical solutions for example 5.2
(a) $T = 0.25, T = 0.5, T = 0.75, N = 10, m = 100$, and $\tau = 0.5$.
(b) $T = 0.5, T = 0.75, T = 0.5, N = 10, m = 100$, and $\tau = 0.5$.



(A) (B)

FIGURE 7. Approximation solution and analytical solution for example 5.2 ($N = 10$ and $m = 5$)



(A) (B)

FIGURE 8. Approximation solution and analytical solution for example 5.2 ($N = 10$ and $m = 100$)

6. Conclusion

In this paper, a numerical solution of the generalized diffusion equation with delay is obtained using the Galerkin finite element method based on cubic B -spline shape functions. The performance of the method was examined on two test problems, and its accuracy was shown by computing L_2 and L_∞ error norms. The numerical results obtained demonstrate that the proposed scheme is a remarkably successful numerical technique for solving the generalized diffusion equation with delay. Accuracy in spatial direction is improved by using the cubic B-spline basis functions.

References

1. A.M.A. Adam, E.B.M. Bashier, M.H.A. Hashim, and K.C. Patidar, *Fitted Galerkin spectral method to solve delay partial differential equations*, Math. Meth. Appl. Sci. **39** (2016), 3102-3115.
2. E.A. Aksan, *numerical solution of burgers' equation by finite element method constructed on the method of discretization in time*, Appl. Math.Comput. **170** (2005), 895-904.
3. E.A. Aksan, *An application of cubic B-spline Finite element method for the burgers equation*, Therm. Sci. **22** (2018), 195-202.
4. H. Brunner, *C Collocation Methods for Volterra Integral and Related Functional Differential Equation*, Cambridge university press, 2004.
5. X. Chen and L. Wang, *The variational iteration method for solving a neutral functional-differential equation with proportional delays*, Comput. Math. Appl. **59** (2010), 2696-2702.
6. P. Garcia, M.A. Castro, J.A. Martin and A. Sirvent, *Numerical solutions of diffusion mathematical models with delay*, Math. Comput. Model. **50** (2009), 860-868.
7. P. Garcia, M.A. Castro, J.A. Martin and A. Sirvent, *Convergence of two implicit numerical schemes for diffusion mathematical models with delay*, Math. Comput. Model. **52** (2010), 1279-1287.
8. Z. Jackiewicz and B. Zubik-Kowal, *Spectral collocation and waveform relaxation methods for nonlinear delay partial differential equations*, Appl. Numer. Math. **56** (2006), 433-443.
9. J. Kongson and S. Amornsamankul, *A model of the signal transduction process under a delay*, EA J. Appl. Math. **7** (2017), 741-751.
10. R. Kumar, A.K. Sharma, and K. Agnihotri, *Dynamics of an innovation diffusion model with time delay*, EA J. Appl.Math. **7** (2017), 455-481.
11. D. Li and J. Wang, *Unconditionally optimal error analysis of crank -nicolson galerkin fems for a strongly nonlinear parabolic system*, J. Sci. Comput. **72** (2017), 892-915.
12. D. Li, J. Zhang, Z. Zhang, *Unconditionally optimal error estimates of a linearized galerkin method for nonlinear time fractional reaction subdiusion equations*, J. Sci. Comput. **76** (2018), 848-866.
13. D. Li.and J. Zhang, *Effcient implementation to numerically solve the nonlinear time fractional parabolic problems on unbounded spatial domain*, J. Comput. **322** (2016), 415-428.
14. D. Li, J. Zhang, Z. Zhang, *Unconditionally optimal error estimates of a linearized galerkin method for nonlinear time fractional reaction subdiusion equations*, J. Sci. Comput. **76** (2018), 848-866.
15. H. Liang, *Convergence and asymptotic stability of galerkin methods for linear parabolic equations with delays*, Appl. Math. Comput. **264** (2015), 160-178.
16. J.A. Martin, F. Rodriguez and R. Company, *Analytic solution of mixed problems for the generalized diffusion with delay*, Math. Comput. Model. **40** (2004), 361-369.
17. T. Ozis, A. Esen, and S. Kutluay, *Numerical solution of burgers' equation by quadratic B-splinef Finite elements*, Appl. Math. Comput. **165** (2005), 237-249.
18. E. Reyes, F. Rodriguez and J.A. Martin, *Analytic-numerical solutions of diffusion mathematical models with delays*, Comput. Math. Appl. **56** (2008), 743-753.
19. H. Tian, *Asymptotic stability analysis of the linear θ-method for linear parabolic differential equations with delay*, J. Diff. Equ. Appl. **15** (2009), 473-487.
20. V. Thomée, *Galerkin finite element methods for parabolic problems*, Springer Science & Business Media, 2007.
21. F. Wu, D. Li, J. Wen, and J. Duan, *Stability and convergence of compact finite difference method for parabolic problems with delay*, Appl. Math. Comput. **322** (2018), 129-139.
22. Q. Zhang, M. Chen, Y. Xu, and D. Xu, *Compact θ-method for the generalized delay diffusion equation*, Appl. Math. Comput. **316** (2018). 357-369.

Gemeda Tolessa Lubo received M.Sc. from Haramaya University, and currently he is a Ph.D. scholar in Wollega University. His research interests include numerical solution of delay differential equations.

Department of Mathematics, Wollega University, Nekemte, Ethiopia.
e-mail: gemedatolesa@gmail.com

Gemechis File Duressa received his M.Sc. degree from Addis Ababa University, Ethiopia and Ph.D. degree from National Institute of Technology, Warangal, India. He is currently working as a professor of Mathematics at Jimma University. His research interests include Numerical Methods for Singularly Perturbed Differential Equations(both ODE and PDE). So far, he has published more than 115 research articles in reputable journals.

Department of Mathematics, College of Natural Sciences, Jimma University, Jimma, Ethiopia.
e-mail: gammeef@gmail.com