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THE OPERATORS π_G OF BEST APPROXIMATIONS AND CONTINUOUS METRIC PROJECTIONS†

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Abstract. In this paper, we shall consider some properties of the metric projection as a set valued mapping. For a set G in a metric space E, the mapping $\pi_G; x \to \pi_G(x)$ of E into 2^G is called set valued metric projection of E onto G. We investigated the properties related to the projection $P_{S(\cdot)}(\cdot)$ and $\pi_{S(\cdot)}(\cdot)$ as one-sided best simultaneous approximations.

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1. Introduction

Let E be a normed linear space and G be a linear subspace of E. We define a functional ϑ_G on E by

$$\vartheta_G(x) = \inf_{g \in G} ||x - g||.$$

Every $g_0 \in G$ with this property is called an element of best approximation of x. We shall denote by $P_G(x)$ the set of all elements of best approximation of x by elements of the set G. That is,

$$P_G(x) = \{g_0 \in G \,|\, ||x - g_0|| = \vartheta_G(x)\}.$$

Then one defines a set-valued mapping from the domain $D(\pi_G) \subset E$ to G by the condition

$$\pi_G(x) \in P_G(x).$$

The space E is said to be proximinal if $D(\pi_G) = E$. And the linear subspace G is said to be semi-Chebyshev if for any $x \in D(\pi_G)$, $\pi_G(x)$ is a single-valued mapping. The mapping π_G and ϑ_G are non-linear on $E \backslash G$, but the restriction of π_G to G is linear and $\vartheta_G|_G = 0$.

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In the particular case when G is a Chebyshev set, that is, $P_G(x)$ is the singleton for each $x \in E$, π_G is called the metric projection(or best approximation operator, or Chebyshev map) of E onto G. Then we have π_G is continuous at the origin and ϑ_G is continuous on E, [8].

Some properties of the mapping π_G onto linear subspace of normed linear spaces are the following theorem.

Theorem 1.1. *Let E be a normed linear space and G be a linear subspace of E. Then the mapping π_G is idempotent and continuous at each point $g \in G$.*

Proof. For every $g \in G$ we have, $\pi_G(g) = g$ whence, for an arbitrary $x \in D(\pi_G)$, we have

$$\pi_G^2(x) = \pi_G[\pi_G(x)] = \pi_G(x),$$

that is, the mapping π_G is idempotent. So, for any $x, y \in D(\pi_G)$ we have

$$||x - \pi_G(x)|| \leq ||x - \pi_G(y)|| \leq ||x - y|| + ||y - \pi_G(y)||$$

thus

$$||x - \pi_G(x)|| - ||y - \pi_G(y)|| \leq ||x - y||.$$

Changing in these relations x by y, we obtain the inequalities

$$||y - \pi_G(y)|| - ||x - \pi_G(x)|| \leq ||x - y||$$

for any $x, y \in D(\pi_G)$. That is,

$$|\ ||x - \pi_G(x)|| - ||y - \pi_G(y)||\ | \leq ||x - y||$$

for any $x, y \in D(\pi_G)$. Obviously, $0 \in G$, by taking $y = 0$, we have

$$||x - \pi_G(x)|| \leq ||x||.$$

Finally, for any $x \in D(\pi_G)$,

$$||\pi_G(x)|| \leq ||x - \pi_G(x)|| + ||x|| \leq 2||x||.$$

Consequentially, π_G is continuous at the origin. Let $x \in D(\pi_G)$ and $g \in G$. For any $g' \in G$,

$$||x + g - g'|| \geq ||x - \pi_G(x)|| = ||x + g - (\pi_G(x) + g)||,$$

thus $x + g \in D(\pi_G)$. Since π_G is single-valued on $D(\pi_G)$, we have

$$\pi_G(x + g) = \pi_G(x) + g.$$

Hence, for any sequence $x_n \in D(\pi_G)$ and $x_n \to g_0 \in G$ imply that $\pi_G(x_n) \to \pi_G(g_0) = g_0$, that is, π_G is continuous at each point $g \in G$. □

Proposition 1.2. *Let E be a normed linear space and let G be a finite dimensional Chebyshev subspace of E, hence $D(\pi_G) = E$ and π_G is single-valued on E. Then the mapping π_G is continuous on E.*

Corollary 1.3. *Suppose that G is a Chebyshev subspace of a normed linear space E, and π_G is continuous at each point of $\pi_G^{-1}(0)$. Then the metric projection π_G is continuous.*

Proof. Let $x_0 \in \pi_G^{-1}(0)$. Suppose that there exists a sequence $x_n \to x_0$ with $\pi_G(x_n) \nrightarrow \pi_G(x_0)$. Then $x_n - \pi_G(x_0) \to x_0 - \pi_G(x_0) \in \pi_G^{-1}(0)$. But $\pi_G(x_n - \pi_G(x_0)) = \pi_G(x_n) - \pi_G(x_0) \nrightarrow 0$. It is a contradiction, so the metric projection π_G is continuous. □

2. Semi-continuity and continuity of set-valued metric projections

In the foregoing we have considered the metric projections onto Chebyshev subspaces. In this section, we shall consider the metric projection in its full generally as a set-valued mapping.

For a set G in a metric space E, the mapping $P_G : x \to P_G(x)$ of E into 2^G(the collection of all subsets of G) is called the set-valued metric projection of E onto G. Let X, Y be metric spaces, a set valued mapping $F : X \to 2^Y$ and $x_0 \in X$. Then F is said to be :

(1) upper semicontinuous(u.s.c.) at x_0 if for any open set V in Y such that $F(x_0) \subset V$, then there exists a neighborhood U of x_0 such that $F(x) \subset V$ for each $x \in U$.

(2) lower semicontinuous(l.s.c.) at x_0 if for any open set V in Y such that $F(x_0) \cap V \neq \emptyset$, then there exists a neighborhood U of x_0 such that $F(x) \cap V \neq \emptyset$ for each $x \in U$.

(3) upper Hausdorff semicontinuous(u.H.s.c.) at x_0 if for any $\varepsilon > 0$, there exists a neighborhood U of x_0 such that $F(x) \subset B_\varepsilon(F(x_0))$ for each $x \in U$.

(4) lower Hausdorff semicontinuous(l.H.s.c.) at x_0 if for any $\varepsilon > 0$, there exists a neighborhood U of x_0 such that $F(x_0) \subset B_\varepsilon(F(x))$ for each $x \in U$.

(5) Hausdorff continuous at x_0 if F is both lower and upper Hausdorff semicontinuous at x_0.

(6) A set G is an "*approximatively compact*" set in a metric space E if for every $x \in E$ and $\{g_n\} \subset G$ with $\lim_{n\to\infty} d(x, g_n) = d(x, G)$ there exists a subsequence $\{g_{n_k}\}$ converging to an element of G.

For lower semi-continuity the easier implications are reversed, if $U : E \to 2^G$ is l.H.s.c. at x_0, then U is l.s.c.at x_0 and conversely, if U is l.s.c.at x_0 and $U(x_0)$ is compact, then U is l.H.s.c. at x_0, [9]. However it is not known any example of a l.s.c. metric projection P_G which is not l.H.s.c. For metric projections onto linear subspaces we have the following stronger result.

Theorem 2.1. [10] *For a linear subspace G of a normed linear space E and for $x_0 \in E$, the following statements are equivalent:*

(1) P_G is u.s.c.at x_0

(2) P_G is u.H.s.c.at x_0 and $P_G(x_0)$ is compact.

We have the following extension of Proposition 1.2.

Corollary 2.2. *Suppose that a set G is an approximatively compact in a metric space E. Then P_G is u.s.c.*

Proof. Since the set G is an approximatively compact in a metric space E, G is proximinal and $P_G(x)$ is closed and bounded. Let N be an arbitrary closed subset of G. We shall show that the set

$$B = \{x \in E \,:\, P_G(x) \cap N \neq \emptyset\}$$

is closed, which will complete the proof. Let $\{x_n\}$ be a sequence in B with converging to an element $x \in E$. There exists a sequence $\{g_n\} \subset G$ such that

$$\lim_{n\to\infty} d(x, g_n) = d(x, G).$$

Consequently, G being *approximatively compact*, there exists a subsequence $\{g_{n_k}\}$ converging to an element $g_0 \in G$ such that $d(x, g_0) = d(x, G)$, that is $g_0 \in P_G(x) \cap N$, whence $x \in B$. □

Proposition 2.3. *For every finite-dimensional linear subspace G of a normed linear space E, P_G is u.s.c., and hence also u.H.s.c.*

If G is a proximinal set, but not approximatively compact, then the conclusions of corollary 2.2. and proposition 2.3 may be no longer valid, as shown by the following example : Let $E = \ell^2$ and let G be the sequence

$$g_1 = 0, g_n = \{1, \frac{1}{n}, \underbrace{0, \ldots, 0}_{n-1}, 1, 0, \ldots\}\ (n = 2, 3, \cdots).$$

3. Continuity of the metric projection $\pi_{S(\cdot)}(\cdot)$

In this chapter, we observe the properties related to the metric projection in the space $C(X)$ with L_1-norm, $C_1(X)$. We will consider exact conditions on a finite-dimensional subspace S of $C_1(X)$ which imply that there exists a one-sided best simultaneous L_1-approximation for each compact subset $F \subset C_1(X)$ from $S(F)$ and we find the characterizations of the one-sided best simultaneous L_1-approximation. Moreover, we have a necessary and sufficient conditions on a subspace S of $C_1(X)$ in order that for each compact set F, the metric projection $P_{S(F)}(F)$ is semi-continuous and continuous.

Now we define a norm on the space of all ℓ-tuples of functions in $C(X)$ as follow: for any ℓ elements $f_1, \cdots, f_\ell$ in $C(X)$, let $F = (f_1, \cdots, f_\ell)$ and

$$||F|| = ||(f_1, \cdots, f_\ell)|| = \max_{(w_1,\cdots,w_\ell)\in A} ||\sum_{i=1}^{\ell} w_i f_i||_1$$

where X is a compact subset of $\mathbb{R}^N$ and $A = \{(w_1, \cdots, w_\ell) \,|\, \sum_{i=1}^{\ell} w_i = 1$ and $w_i > 0$ for $i = 1, \cdots, \ell\}$. Let S be a finite subspace of $C(X)$. We define the set, for $i = 1, \cdots, \ell$,

$$S(f_i) = \{s \in S \,|\, s(x) \leq f_i(x) \text{ for } x \in X\}$$

and the set

$$S(F) = \bigcap_{i=1}^{\ell} S(f_i)$$

in normed linear space $(C(X), ||\cdot||_1)$. By the definition of $S(F)$, if S contains a strictly positive function, then $S(F)$ is non-empty for every ℓ-tuple F of functions in $C(X)$. Throughout this chapter we shall restrict to those F for which $S(F)$ is non-empty.

For each ℓ-tuple F of $C(X)$, $S(F)$ is a closed and convex subset of $C_1(X)$. The metric projection $P : F \to P_{S(F)}(F)$ is a set valued map the approximating set depends on some ℓ-tuple F. Now we consider a continuity of the projection $P_{S(\cdot)}(\cdot)$.

Remark 3.1. For each ℓ-tuple $F = (f_1, \cdots, f_\ell)$ of functions in $C_1(X)$, a minimizing sequence in $S(F)$ is bounded, so the sequence has a subsequence which is convergent in $S(F)$. Thus $S(F)$ is approximatively compact relative to F.

Mabizela [11] prove that for each F, $S(F)$ is a m-dimensional subspace of $C_1(X)$, if F_n converges to F_0, then the map $S(\cdot)$ is Hausdorff continuous on F_0. Thus we can show that the following proposition.

Proposition 3.1. [12] *For each ℓ-tuple $F = (f_1, \cdots, f_\ell)$ of functions in $C_1(X)$, the metric projection $P : F \to P_{S(F)}(F)$ is upper semicontinuous.*

Corollary 3.2. *Let $f \in C_1(X)$ be given. Assume that $S(f)$ is an "approximatively compact" subset of S. If the map $f \to S(f)$ is Hausdorff continuous at f then $P : f \to P_{S(f)}(f)$ is upper semicontinuous.*

Theorem 3.3. *Let S be a set in $C_1(X)$. If S is a Chebyshev and "approximatively compact", then $\pi_{S(\cdot)}(\cdot)$ is continuous on $C_1(X)$.*

Proof. By hypothesis, S is a Chebyshev, we can take $P_{S(\cdot)}(\cdot)=\pi_{S(\cdot)}(\cdot)$. So $\pi_{S(\cdot)}(\cdot)$ is upper semicontinuous by corollary 3.2. The proof completes by to show that $\pi_{S(\cdot)}(\cdot)$ is lower semicontinuous. For any closed set V, let

$$\pi^{-1}_{S(\cdot)}(V) = \{F \mid \pi_{S(F)}(F) \cap V \neq \emptyset\}.$$

It suffices to show that $\pi^{-1}_{S(\cdot)}(V)$ is closed. Let $\{F_n\} \subset \pi^{-1}_{S(F)}(V)$ be a sequence such that $\{F_n\}$ converge to F_0 with respect to Hausdorff metric, denote that $H(F_n, F_0) \to 0$ as $n \to \infty$. We can take $v_n \in \pi_{S(F_n)}(F_n) \cap V$ for all $n \in \mathbb{N}$. Since S is approximatively compact, there is a subsequence $\{v_{n_k}\}$ which converge to $v_0 \in S$ with $v_0 \in \pi_{S(F_0)}(F_0) \cap V$, that is, $\pi^{-1}_{S(\cdot)}(V)$ is closed. □

The motivation is the one-sided best approximation of an element, which has been studied by R. Bojanic, R. DeVore(1966), H. Strauss(1982), G. Nürnberger (1985), A. Pinkus and V. Totik(1986). They proved, by looking at Gaussian-type quadrature formula, the existence of $f \in C[a, b]$ with more than the best one-sided L_1-approximation from π_n. They also showed, for f continuous on $[a, b]$ and differentiable on (a, b), the uniqueness of the best one-sided L_1-approximation from π_n. DeVore essentially generalized these results to T- and ET_2-systems [1].

It is also natural to raise the problem of extending to study weakly semi-continuous P_G and those P_G which are Lipschitzian for the Hausdorff metric on $2^G \setminus \emptyset$, or to find a suitable generalization of linearity for set-valued P_G and then give characterizations of those linear subspaces G for which P_G has this property.

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