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THE ZERO-DIVISOR GRAPHS OF $\mathbb{Z}(+)\mathbb{Z}_n$ AND $(\mathbb{Z}(+)\mathbb{Z}_n)[X]\!]$

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Abstract. Let $\mathbb{Z}$ be the ring of integers and let $\mathbb{Z}_n$ be the ring of integers modulo n. Let $\mathbb{Z}(+)\mathbb{Z}_n$ be the idealization of $\mathbb{Z}_n$ in $\mathbb{Z}$ and let $(\mathbb{Z}(+)\mathbb{Z}_n)[X]\!]$ be either $(\mathbb{Z}(+)\mathbb{Z}_n)[X]$ or $(\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!]$. In this article, we study the zero-divisor graphs of $\mathbb{Z}(+)\mathbb{Z}_n$ and $(\mathbb{Z}(+)\mathbb{Z}_n)[X]\!]$. More precisely, we completely characterize the diameter and the girth of the zero-divisor graphs of $\mathbb{Z}(+)\mathbb{Z}_n$ and $(\mathbb{Z}(+)\mathbb{Z}_n)[X]\!]$. We also calculate the chromatic number of the zero-divisor graphs of $\mathbb{Z}(+)\mathbb{Z}_n$ and $(\mathbb{Z}(+)\mathbb{Z}_n)[X]\!]$.

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1. Introduction

1.1. Preliminaries. In order to help the reader's better understanding, this subsection is devoted to review some preliminaries.

Let R be a commutative ring with identity and let M be a unitary R-module. Then the *idealization* of M in R (or *trivial extension* of R by M) is a commutative ring

$$R(+)M := \{(r, m) \mid r \in R \text{ and } m \in M\}$$

under the usual addition and the multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ for all $(r_1, m_1), (r_2, m_2) \in R(+)M$. It is obvious that $(1, 0)$ is the identity of $R(+)M$. For more on the idealization, the readers can refer to [4, 8].

Let G be an (undirected) graph. Recall that G is *connected* if there is a path between any two distinct vertices of G. The graph G is said to be *complete* if any two distinct vertices are adjacent. The complete graph with n vertices is denoted by K_n. The graph G is called a *null graph* (or *edgeless graph*) if G has no edges, and we denote by $\overline{K}_n$ the null graph with n vertices. An *independent*

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set (or *stable set*) in G is a set of pairwise nonadjacent vertices. The graph G is a *bipartite graph* if the vertex set of G is the union of two disjoint independent sets. In this case, the disjoint independent sets are called the *partite sets* of G. The graph G is a *complete bipartite graph* if G is a bipartite graph such that two distinct vertices are adjacent if and only if they belong to different partite sets. If one of the partite sets of a complete bipartite graph G is a singleton set, then we call G a *star graph*. We denote the complete bipartite graph by $K_{m,n}$, where m and n are the cardinal numbers of the partite sets. We also denote the star graph by $K_{1,n}$. For vertices a and b in G, $d(a,b)$ denotes the length of the shortest path from a to b. If there is no such path, then $d(a,b)$ is defined to be ∞; and $d(a,a)$ is defined to be zero. The *diameter* of G, denoted by $\mathrm{diam}(G)$, is the supremum of $\{d(a,b) \mid a \text{ and } b \text{ are vertices of } G\}$. The *girth* of G, denoted by $\mathrm{g}(G)$, is defined as the length of the shortest cycle in G. If G contains no cycles, then $\mathrm{g}(G)$ is defined to be ∞. A subgraph H of G is an *induced subgraph* of G if two vertices of H are adjacent in H if and only if they are adjacent in G. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colors needed to color the vertices of G so that no two adjacent vertices share the same color. A *clique* C in G is a subset of the vertex set of G such that the induced subgraph of G by C is a complete graph. A *maximal clique* in G is a clique that cannot be extended by including one more adjacent vertex. For more on graph theory, the readers can refer to [14].

1.2. The zero-divisor graph of a commutative ring. Let R be a commutative ring with identity and let $\mathrm{Z}(R)$ be the set of nonzero zero-divisors of R. The *zero-divisor graph* of R, denoted by $\Gamma(R)$, is the simple graph with vertex set $\mathrm{Z}(R)$, and for distinct $a, b \in \mathrm{Z}(R)$, a and b are adjacent if and only if $ab = 0$. Clearly, $\Gamma(R)$ is the null graph if and only if R is an integral domain.

In [6], Beck first introduced the concept of the zero-divisor graphs of commutative rings and in [3], Anderson and Naseer continued to study Beck's investigation. In their papers, all elements of R are vertices of the zero-divisor graph and the authors were mainly interested in colorings. In [2], Anderson and Livingston gave the present definition of $\Gamma(R)$ in order to emphasize the study of the interplay between graph-theoretic properties of $\Gamma(R)$ and ring-theoretic properties of R. Later, in [5], Axtell and Stickles studied the zero-divisor graph of idealizations. It was shown that $\Gamma(R)$ is connected with $\mathrm{diam}(\Gamma(R)) \leq 3$ [2, Theorem 2.3]; and $\mathrm{g}(\Gamma(R)) \leq 4$ [11, (1.4)].

For more on the zero-divisor graph of a commutative ring, the readers can refer to a survey article [1].

Let $\mathbb{Z}$ be the ring of integers and let $\mathbb{Z}_n$ be the ring of integers modulo n. For a commutative ring R, $R[X]\!]$ denotes either the polynomial ring $R[X]$ or the power series ring $R[\![X]\!]$. In [12, 13], the authors studied some properties of $\Gamma(\mathbb{Z}_n)$ and $\Gamma(\mathbb{Z}[X]\!])$. In fact, they completely characterized the diameter and the girth of $\Gamma(\mathbb{Z}_n)$ and $\Gamma(\mathbb{Z}[X]\!])$. Also, they calculated the chromatic number of $\Gamma(\mathbb{Z}_n)$ and $\Gamma(\mathbb{Z}[X]\!])$. The aim of this paper is to study some properties of $\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)$ and

$\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[X]\!])$. In Section 2, we completely characterize the diameter and the girth of $\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)$. We also calculate the chromatic number of $\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)$. In Section 3, we calculate the diameter and the girth of $\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[X]\!])$. We also calculate the chromatic number of $\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[X]\!])$.

Note that if $n = 1$, then $\mathbb{Z}(+)\mathbb{Z}_1$ is isomorphic to $\mathbb{Z}$; so $\Gamma(\mathbb{Z}(+)\mathbb{Z}_1)$ is the null graph. Therefore $\Gamma((\mathbb{Z}(+)\mathbb{Z}_1)[X]\!])$ is also the null graph (cf. [10, Theorem 2] and [7, Theorem 5]). Hence in this paper, we only consider the case that $n \geq 2$. Finally, we mention that all figures are drawn by using website http://graphonline.ru/en/.

2. The zero-divisor graph of $\mathbb{Z}(+)\mathbb{Z}_n$

We start this section with the characterization of $\mathrm{Z}(\mathbb{Z}(+)\mathbb{Z}_n)$.

Lemma 2.1. *Let $p_1, \ldots, p_r$ be distinct primes, $s_1, \ldots, s_r$ positive integers and $n = p_1^{s_1} \cdots p_r^{s_r}$. Then* $\mathrm{Z}(\mathbb{Z}(+)\mathbb{Z}_n) = \big\{(0,\alpha) \mid \alpha \in \mathbb{Z}_n \setminus \{0\}\big\} \cup \left(\bigcup_{i=1}^{r} \big\{(p_i k, \alpha) \mid k \in \mathbb{Z} \setminus \{0\} \text{ and } \alpha \in \mathbb{Z}_n\big\}\right)$.

Proof. Let $(0,\alpha)$ be a nonzero element of $\mathbb{Z}(+)\mathbb{Z}_n$. Then $(0,\alpha)(n,0) = (0,0)$; so $(0,\alpha) \in \mathrm{Z}(\mathbb{Z}(+)\mathbb{Z}_n)$. Let k be a nonzero integer and let $\alpha \in \mathbb{Z}_n$. Then for any $i \in \{1, \ldots, r\}$, $(p_i k, \alpha)\left(0, \frac{n}{p_i}\right) = (0,0)$; so $(p_i k, \alpha) \in \mathrm{Z}(\mathbb{Z}(+)\mathbb{Z}_n)$. For the reverse containment, let $(a,\alpha) \in \mathrm{Z}(\mathbb{Z}(+)\mathbb{Z}_n)$. Then $(a,\alpha)(b,\beta) = (0,0)$ for some $(b,\beta) \in \mathrm{Z}(\mathbb{Z}(+)\mathbb{Z}_n)$; so $ab = 0$ and $a\beta + b\alpha \equiv 0 \pmod{n}$. If $a = 0$, then $\alpha \not\equiv 0 \pmod{n}$; so we have nothing to prove. Suppose that $a \neq 0$. Then $b = 0$; so $\beta \not\equiv 0 \pmod{n}$ and $a\beta \equiv 0 \pmod{n}$. Therefore we can find an index $i \in \{1, \ldots, r\}$ such that a is divisible by p_i. Hence $(a,\alpha) = (p_i k, \alpha)$ for some nonzero integer k. Thus $\mathrm{Z}(\mathbb{Z}(+)\mathbb{Z}_n) = \big\{(0,\alpha) \mid \alpha \in \mathbb{Z}_n \setminus \{0\}\big\} \cup \left(\bigcup_{i=1}^{r} \big\{(p_i k, \alpha) \mid k \in \mathbb{Z} \setminus \{0\} \text{ and } \alpha \in \mathbb{Z}_n\big\}\right)$. □

Let $n = p_1^{s_1} \cdots p_r^{s_r}$ for some distinct primes $p_1, \ldots, p_r$ and some positive integers $s_1, \ldots, s_r$. From now on, let A_n denote the set $\big\{(0,\alpha) \mid \alpha \in \mathbb{Z}_n \setminus \{0\}\big\}$ and let B_n stand for the set $\bigcup_{i=1}^{r} \big\{(p_i k, \alpha) \mid k \in \mathbb{Z} \setminus \{0\} \text{ and } \alpha \in \mathbb{Z}_n\big\}$. It is obvious that $A_n \cap B_n = \emptyset$; so by Lemma 2.1, $\mathrm{Z}(\mathbb{Z}(+)\mathbb{Z}_n)$ is the disjoint union of A_n and B_n.

Remark 2.2. Let $n \geq 2$ be an integer.

(1) Let $(0,\alpha), (0,\beta) \in A_n$. Then $(0,\alpha)(0,\beta) = (0,0)$; so the induced subgraph of $\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)$ by the set A_n is the complete graph K_{n-1}.

(2) Write $n = p_1^{s_1} \cdots p_r^{s_r}$ for some distinct primes $p_1, \ldots, p_r$ and some positive integers $s_1, \ldots, s_r$. Let $(p_i k_1, \alpha_1), (p_j k_2, \alpha_2) \in B_n$. Then $(p_i k_1, \alpha_1)(p_j k_2, \alpha_2) \neq$

$(0,0)$; so the induced subgraph of $\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)$ induced by the set B_n is the countably infinite null graph $\overline{K}_\infty$.

Corollary 2.3. *Let $n \geq 2$ be an integer. Then the following assertions hold.*

(1) $\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)$ *is never a complete graph.*
(2) $\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)$ *is a star graph if and only if $n = 2$.*

Proof. (1) The result is an immediate consequence of Remark 2.2(2).

(2) Suppose that $n = 2$. Then $A_2 = \{(0,1)\}$ and $B_2 = \{(2k, \alpha) \mid k \in \mathbb{Z} \setminus \{0\}$ and $\alpha \in \mathbb{Z}_2\}$. Let $(2k_1, \alpha_1), (2k_2, \alpha_2)$ be two distinct elements of B_2. Then $(2k_1, \alpha_1) - (0,1) - (2k_2, \alpha_2)$ is a path in $\Gamma(\mathbb{Z}(+)\mathbb{Z}_2)$; so by Remark 2.2(2), $d\big((2k_1, \alpha_1), (2k_2, \alpha_2)\big) = 2$. Thus $\Gamma(\mathbb{Z}(+)\mathbb{Z}_2)$ is the star graph $K_{1,\infty}$.

For the converse, suppose that $\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)$ is a star graph. Then by Remark 2.2(2), there exists an element $(0,\alpha) \in A_n$ such that $(0,\alpha)(b,\beta) = (0,0)$ for all $(b,\beta) \in \mathrm{Z}(\mathbb{Z}(+)\mathbb{Z}_n) \setminus \{(0,\alpha)\}$. Note that the induced subgraph of $\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)$ induced by the set $\mathrm{Z}(\mathbb{Z}(+)\mathbb{Z}_n) \setminus \{(0,\alpha)\}$ is the null graph $\overline{K}_\infty$. If $n \geq 3$, then there exists an element $(0,\gamma) \in \mathrm{Z}(\mathbb{Z}(+)\mathbb{Z}_n) \setminus \{(0,\alpha)\}$. Note that $(n,0) \in B_n$ with $(0,\gamma)(n,0) = (0,0)$. This is a contradiction. Thus $n = 2$. $\square$

Remark 2.4. Let $n \geq 2$ be an integer. Suppose that $\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)$ is a bipartite graph. Then by Remark 2.2, $n = 2$ and the partite sets of $\Gamma(\mathbb{Z}(+)\mathbb{Z}_2)$ is A_2 and B_2. Thus $\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)$ is a (complete) bipartite graph if and only if $\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)$ is a star graph, if and only if $n = 2$.

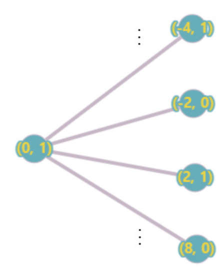


FIGURE 1. The star graph: $\Gamma(\mathbb{Z}(+)\mathbb{Z}_2)$

We now give the characterization of the diameters of $\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)$.

Theorem 2.5. *Let $n \geq 2$ be an integer. Then the following statements hold.*

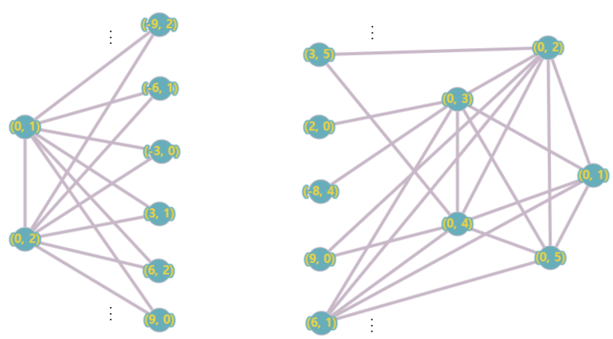
(1) $\mathrm{diam}(\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)) = 2$ *if (and only if) $n = p^s$ for some prime p and some integer $s \geq 1$.*
(2) $\mathrm{diam}(\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)) = 3$ *if (and only if) $n = p_1^{s_1} \cdots p_r^{s_r}$ for some distinct primes $p_1, \ldots, p_r$ $(r \geq 2)$ and some positive integers $s_1, \ldots, s_r$.*

Proof. (1) Suppose that $n = p$ for some prime p. If $p = 2$, then by Corollary 2.3(2), $\Gamma(\mathbb{Z}(+)\mathbb{Z}_2)$ is a star graph; so $\mathrm{diam}(\Gamma(\mathbb{Z}(+)\mathbb{Z}_2)) = 2$. If $p \geq 3$, let $(0,\alpha) \in$

A_p and let $(pk_1, \beta_1), (pk_2, \beta_2)$ be distinct elements of B_p. Then $(0, \alpha)(pk_1, \beta_1) = (0, 0) = (0, \alpha)(pk_2, \beta_2)$; so by Remark 2.2(2), $d\big((pk_1, \beta_1), (pk_2, \beta_2)\big) = 2$. Note that by Remark 2.2(1), the induced subgraph of $\Gamma(\mathbb{Z}(+)\mathbb{Z}_p)$ induced by the set A_p is the complete graph K_{p-1}. Hence $\operatorname{diam}(\Gamma(\mathbb{Z}(+)\mathbb{Z}_p)) = 2$.

We next suppose that $n = p^s$ for some prime p and some integer $s \geq 2$. Then by Remark 2.2(1), the induced subgraph of $\Gamma(\mathbb{Z}(+)\mathbb{Z}_{p^s})$ induced by the set A_{p^s} is the complete graph K_{p^s-1}. Let $(pk_1, \alpha_1), (pk_2, \alpha_2)$ be distinct elements in B_{p^s}. Then $(pk_1, \alpha_1)(0, p^{s-1}) = (0, 0) = (pk_2, \alpha_2)(0, p^{s-1})$; so by Remark 2.2(2), $d\big((pk_1, \alpha_1), (pk_2, \alpha_2)\big) = 2$. Also, by Remark 2.2(1), $d\big((0, \beta), (pk_1, \alpha_1)\big) \leq 2$ for all $(0, \beta) \in A_{p^s}$. Hence $\operatorname{diam}(\Gamma(\mathbb{Z}(+)\mathbb{Z}_{p^s})) = 2$.

(2) Suppose that $n = p_1^{s_1} \cdots p_r^{s_r}$ for some distinct primes $p_1, \ldots, p_r$ $(r \geq 2)$ and some positive integers $s_1, \ldots, s_r$. Let $(p_i, 0), (p_j, 0) \in B_n$ with $i \neq j$. Then by Remark 2.2(2) and [2, Theorem 2.3], $2 \leq d\big((p_i, 0), (p_j, 0)\big) \leq 3$. Suppose to the contrary that there exists an element $(a, \alpha) \in Z(\mathbb{Z}(+)\mathbb{Z}_n) \setminus \{(p_i, 0), (p_j, 0)\}$ such that $(p_i, 0) - (a, \alpha) - (p_j, 0)$ is a path in $\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)$. Then by Remark 2.2(2), $(a, \alpha) \in A_n$; so $a = 0$ and $\alpha \not\equiv 0 \pmod{n}$. Now, $p_i\alpha \equiv 0 \pmod{n}$ and $p_j\alpha \equiv 0 \pmod{n}$; so α is a multiple of both $\frac{n}{p_i}$ and $\frac{n}{p_j}$. Therefore α is divisible by n. This is absurd. Hence $d\big((p_i, 0), (p_j, 0)\big) = 3$. Thus $\operatorname{diam}(\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)) = 3$. □



$\operatorname{diam}(\Gamma(\mathbb{Z}(+)\mathbb{Z}_3)) = 2$ $\qquad$ $\operatorname{diam}(\Gamma(\mathbb{Z}(+)\mathbb{Z}_6)) = 3$

FIGURE 2. The diameter of some zero-divisor graphs

Next, we study the girth of $\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)$.

Theorem 2.6. *Let $n \geq 2$ be an integer. Then the following statements hold.*

(1) $g(\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)) = 3$ *if (and only if) $n \geq 3$.*
(2) $g(\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)) = \infty$ *if (and only if) $n = 2$.*

Proof. (1) Let $n \geq 3$ be an integer. Note that $(0, 1) - (0, 2) - (n, 1) - (0, 1)$ is a cycle of length 3 in $\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)$. Thus $g(\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)) = 3$.

(2) Note that $A_2 = \{(0, 1)\}$. If there exists a cycle in $\Gamma(\mathbb{Z}(+)\mathbb{Z}_2)$, then we can find two distinct elements $(2k_1, \alpha_1), (2k_2, \alpha_2) \in B_2$ such that $(2k_1, \alpha_1)$ and

$(2k_2, \alpha_2)$ are adjacent. However, this is impossible because of Remark 2.2(2). Hence $\Gamma(\mathbb{Z}(+)\mathbb{Z}_2)$ has no cycles. Thus $\mathrm{g}(\Gamma(\mathbb{Z}(+)\mathbb{Z}_2)) = \infty$. □

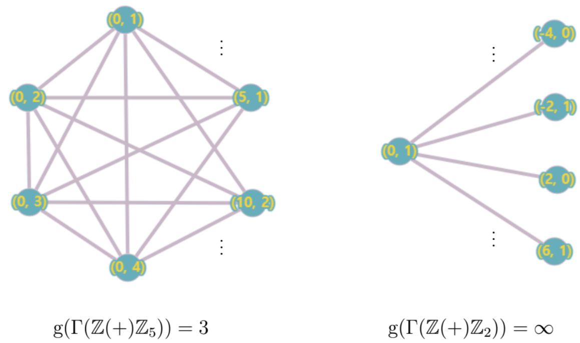


FIGURE 3. The girth of some zero-divisor graphs

The final study in this section is to calculate the chromatic number of $\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)$. To do this, we need the following lemma.

Lemma 2.7. *Let $n \geq 2$ be an integer and let $C = A_n \cup \{(n, 0)\}$. Then C is a maximal clique of $\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)$.*

Proof. Note that the product of any two distinct elements of C is $(0, 0)$; so C is a clique of $\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)$. Suppose to the contrary that there exists an element $(a, \alpha) \in \mathrm{Z}(\mathbb{Z}(+)\mathbb{Z}_n) \setminus C$ such that $(a, \alpha)(b, \beta) = (0, 0)$ for all $(b, \beta) \in C$. Then $(a, \alpha)(n, 0) = (0, 0)$. Therefore $a = 0$, which implies that $\alpha \not\equiv 0 \pmod{n}$. Hence $(a, \alpha) \in C$. This is a contradiction to the choice of (a, α). Thus C is a maximal clique of $\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)$. □

Theorem 2.8. *If $n \geq 2$ is an integer, then $\chi(\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)) = n$.*

Proof. Let $C = A_n \cup \{(n, 0)\}$. Then by Lemma 2.7, C is a maximal clique of $\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)$. For each $i \in \{1, \ldots, n-1\}$, let $\bar{i}$ be the color of $(0, i)$ and let $\overline{n}$ be the color of $(n, 0)$. Note that $\mathrm{Z}(\mathbb{Z}(+)\mathbb{Z}_n) \setminus C$ is a nonempty set. Let $(a, \alpha) \in \mathrm{Z}(\mathbb{Z}(+)\mathbb{Z}_n) \setminus C$. Then by Lemma 2.7, there exists an element $(b, \beta) \in C$ such that (a, α) and (b, β) are not adjacent. In this case, we color (a, α) with the color of (b, β). Note that by Lemma 2.1, $\mathrm{Z}(\mathbb{Z}(+)\mathbb{Z}_n) \setminus C \subsetneq B_n$; so by Remark 2.2(2), any two vertices in $\mathrm{Z}(\mathbb{Z}(+)\mathbb{Z}_n) \setminus C$ are not adjacent. Thus $\chi(\Gamma(\mathbb{Z}(+)\mathbb{Z}_n)) = n$. □

Remark 2.9. Let $n \geq 2$ be an integer and let $C = A_n \cup \{(n, 0)\}$. Take any element $(a, \alpha) \in \mathrm{Z}(\mathbb{Z}(+)\mathbb{Z}_n) \setminus C$. Then $(a, \alpha) \in B_n$; so by Remark 2.2(2), (a, α) and $(n, 0)$ are not adjacent. Hence we can always color (a, α) with $\overline{n}$ in the proof of Theorem 2.8.

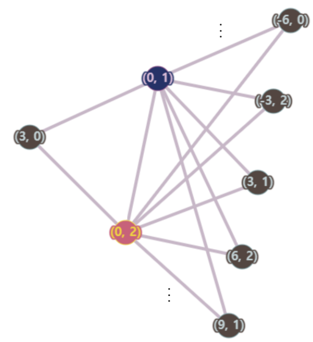


FIGURE 4. The coloring of some zero-divisor graphs

3. The zero-divisor graph of $(\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!]$

Let R be a commutative ring with identity, $R[X]$ the polynomial ring over R and $R[\![X]\!]$ the power series ring over R. Let $R[X]\!]$ denote either the polynomial ring or the power series ring. Recall that R is a *Noetherian ring* if it satisfies the ascending chain condition on ideals of R (or equivalently, every ideal of R is finitely generated.) In order to study the zero-divisor graph of $(\mathbb{Z}(+)\mathbb{Z}_n)[X]\!]$, we need the following lemma which is well known as McCoy's theorem.

Lemma 3.1. ([10, Theorem 2] and [7, Theorem 5]) *Let R be a commutative ring with identity. Then the following assertions hold.*

(1) *If $f \in \mathrm{Z}(R[X])$, then there exists a nonzero element $r \in R$ such that $rf = 0$.*
(2) *If R is a Noetherian ring and $f \in \mathrm{Z}(R[\![X]\!])$, then there exists a nonzero element $r \in R$ such that $rf = 0$.*

At this time, we should note that $\mathbb{Z}$ is a Noetherian ring and for any integer $n \geq 2$, $\mathbb{Z}_n$ is a finitely generated $\mathbb{Z}$-module; so $\mathbb{Z}(+)\mathbb{Z}_n$ is a Noetherian ring [4, Theorem 4.8] (or [9, Corollary 3.9]).

Lemma 3.2. *Let $p_1, \ldots, p_r$ be distinct primes, $s_1, \ldots, s_r$ positive integers and $n = p_1^{s_1} \cdots p_r^{s_r}$. Then* $\mathrm{Z}((\mathbb{Z}(+)\mathbb{Z}_n)[X]\!]) = \left\{ \sum_{m\geq 0} (0, b_m)X^m \mid b_m \neq 0 \text{ for some } m \in \mathbb{N}_0 \right\} \cup \left(\bigcup_{\ell=1}^{r} \left\{ \sum_{m\geq 0} (p_\ell k_m, b_m)X^m \mid k_m \neq 0 \text{ for some } m \in \mathbb{N}_0 \text{ and } b_m \in \mathbb{Z}_n \right\} \right)$.

Proof. Let $f = \sum_{m\geq 0} (0, b_m)X^m$ be a nonzero element of $(\mathbb{Z}(+)\mathbb{Z}_n)[X]\!]$. Then $(n, 0)f = (0, 0)$; so $f \in \mathrm{Z}((\mathbb{Z}(+)\mathbb{Z}_n)[X]\!])$. Fix an index $\ell \in \{1, \ldots, r\}$, and let

$g = \sum_{m\geq 0}(p_\ell k_m, b_m)X^m$ be an element of $(\mathbb{Z}(+)\mathbb{Z}_n)[X]\!]$, where $k_m \neq 0$ for some $m \in \mathbb{N}_0$. Then $\left(0, \frac{n}{p_\ell}\right) g = (0,0)$; so $g \in Z((\mathbb{Z}(+)\mathbb{Z}_n)[X]\!])$.

For the reverse containment, let $f = \sum_{m\geq 0}(a_m, b_m)X^m \in Z((\mathbb{Z}(+)\mathbb{Z}_n)[X]\!])$. If $a_m = 0$ for all $m \in \mathbb{N}_0$, then the proof is done; so we next suppose that $a_m \neq 0$ for some $m \in \mathbb{N}_0$. Now, by Lemma 3.1, there exists an element $(r,s) \in Z(\mathbb{Z}(+)\mathbb{Z}_n)$ such that $(r,s)f = (0,0)$; so $(r,s)(a_m,b_m) = (0,0)$ for all $m \in \mathbb{N}_0$. Therefore $r = 0$ and $a_m s \equiv 0 \pmod{n}$ for all $m \in \mathbb{N}_0$. Since $s \not\equiv 0 \pmod{n}$, we can find an index $\ell \in \{1,\dots,r\}$ such that s is not divisible by $p_\ell^{s_\ell}$; so a_m is divisible by p_ℓ for all $m \in \mathbb{N}_0$. Hence $f = \sum_{m\geq 0}(p_\ell k_m, b_m)X^m$, where $k_m \neq 0$ for some $m \in \mathbb{N}_0$. Thus $Z((\mathbb{Z}(+)\mathbb{Z}_n)[X]\!]) = \left\{\sum_{m\geq 0}(0,b_m)X^m \,|\, b_m \neq 0 \text{ for some } m \in \mathbb{N}_0\right\} \cup \left(\bigcup_{\ell=1}^{r}\left\{\sum_{m\geq 0}(p_\ell k_m, b_m)X^m \,|\, k_m \neq 0 \text{ for some } m \in \mathbb{N}_0 \text{ and } b_m \in \mathbb{Z}_n\right\}\right)$. ☐

Let $n = p_1^{s_1}\cdots p_r^{s_r}$ for some distinct primes $p_1,\dots,p_r$ and some positive integers $s_1,\dots,s_r$. From now on, let $C_n = \left\{\sum_{m\geq 0}(0,b_m)X^m \,|\, b_m \neq 0 \text{ for some } m \in \mathbb{N}_0\right\}$ and let $D_n = \bigcup_{\ell=1}^{r}\left\{\sum_{m\geq 0}(p_\ell k_m, b_m)X^m \,|\, k_m \neq 0 \text{ for some } m \in \mathbb{N}_0 \text{ and } b_m \in \mathbb{Z}_n\right\}$. It is obvious that $C_n \cap D_n = \emptyset$; so by Lemma 3.2, $Z((\mathbb{Z}(+)\mathbb{Z}_n)[X]\!])$ is the disjoint union of C_n and D_n.

Remark 3.3. Let $n \geq 2$ be an integer.

(1) Let $\sum_{m\geq 0}(0,a_m)X^m$ and $\sum_{m\geq 0}(0,b_m)X^m$ be two elements of C_n. Then $\left(\sum_{m\geq 0}(0,a_m)X^m\right)\left(\sum_{m\geq 0}(0,b_m)X^m\right) = (0,0)$. Thus the induced subgraph of $\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[X]\!])$ by the set C_n is the complete graph K_∞. In fact, the induced subgraph of $\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[X])$ by the set C_n is the countably infinite complete graph. Also, note that $|C_n| = c$ in $\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!])$, where c is the cardinality of the set of real numbers. Hence the induced subgraph of $\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!])$ by the set C_n is the uncountably infinite complete graph.

(2) Write $n = p_1^{s_1}\cdots p_r^{s_r}$ for some distinct primes $p_1,\dots,p_r$ and some positive integers $s_1,\dots,s_r$. Let $\sum_{m\geq 0}(p_i k_m, d_m)X^m$, $\sum_{m\geq 0}(p_j h_m, e_m)X^m \in D_n$. Then $\left(\sum_{m\geq 0}(p_i k_m, d_m)X^m\right)\left(\sum_{m\geq 0}(p_j h_m, e_m)X^m\right) \neq (0,0)$. Hence the induced subgraph of $\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[X]\!])$ by the set D_n is the infinite null graph $\overline{K}_\infty$. More precisely, $|D_n| = \aleph_0$ in $\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[X])$ and $|D_n| = c$ in $\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!]))$; so

the induced subgraph of $\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[X])$ (resp., $\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!])$) by the set D_n is the countably (resp., uncountably) infinite null graph.

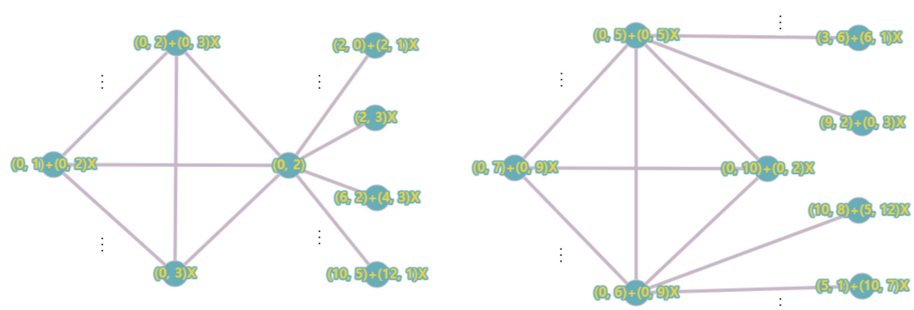
Theorem 3.4. *Let $n \geq 2$ be an integer. Then the following statements hold.*

(1) $\mathrm{diam}(\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!])) = 2$ *if (and only if) $n = p^s$ for some prime p and some integer $s \geq 1$.*

(2) $\mathrm{diam}(\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!])) = 3$ *if (and only if) $n = p_1^{s_1} \cdots p_r^{s_r}$ for some distinct primes $p_1, \ldots, p_r$ $(r \geq 2)$ and some positive integers $s_1, \ldots, s_r$.*

Proof. (1) Suppose that $n = p^s$ for some prime p and some integer $s \geq 1$. Let f and g be two distinct elements of $\mathrm{Z}((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!])$. If $f, g \in C_n$, then f and g are adjacent by Remark 3.3(1). Suppose that at least one of f and g belongs to D_n. Then $f - (0, p^{s-1}) - g$ is a path in $\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!])$; so $d(f, g) \leq 2$. Hence $\mathrm{diam}(\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!])) \leq 2$. Note that by Remark 3.3(2), $\mathrm{diam}(\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!])) \geq 2$. Thus $\mathrm{diam}(\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!])) = 2$.

(2) Suppose that $n = p_1^{s_1} \cdots p_r^{s_r}$ for some distinct primes $p_1, \ldots, p_r$ $(r \geq 2)$ and some positive integers $s_1, \ldots, s_r$. Let $(p_i, 0), (p_j, 0) \in D_n$ with $i \neq j$. Then by Remark 3.3(2), $d((p_i, 0), (p_j, 0)) \geq 2$. Suppose to the contrary that there exists an element $f = \sum_{m \geq 0} (a_m, b_m) X^m \in \mathrm{Z}((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!]) \setminus \{(p_i, 0), (p_j, 0)\}$ such that $(p_i, 0) f = (0, 0) = (p_j, 0) f$. Then by Remark 3.3(2), $f \in C_n$; so for all $m \in \mathbb{N}_0$, $a_m = 0$, $p_i b_m \equiv 0 \equiv p_j b_m \pmod{n}$. Therefore b_m is a multiple of both $\frac{n}{p_i}$ and $\frac{n}{p_j}$ for all $m \in \mathbb{N}_0$, which implies that $b_m \equiv 0 \pmod{n}$ for all $m \in \mathbb{N}_0$. This is absurd. Hence $d((p_i, 0), (p_j, 0)) \geq 3$. Thus $\mathrm{diam}(\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!])) = 3$ [2, Theorem 2.3]. □



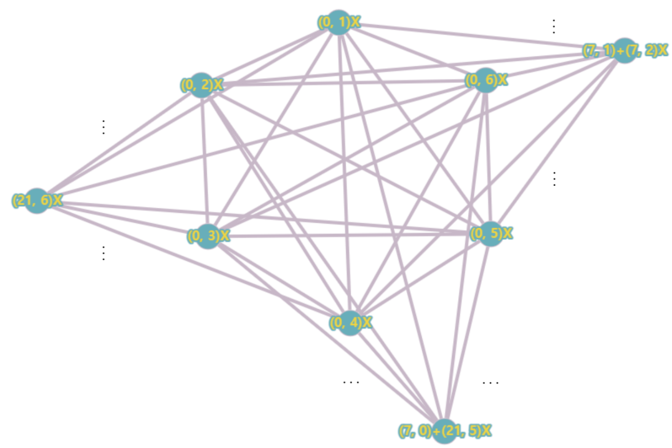
$\mathrm{diam}(\Gamma((\mathbb{Z}(+)\mathbb{Z}_4)[\![X]\!])) = 2$ $\mathrm{diam}(\Gamma((\mathbb{Z}(+)\mathbb{Z}_{15})[\![X]\!])) = 3$

FIGURE 5. The diameter of some zero-divisor graphs

The girth of $\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!])$ can be easily characterized as follows:

Theorem 3.5. *For any integer $n \geq 2$,* $\mathrm{g}(\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!])) = 3$.

Proof. Fix an integer $n \geq 2$. Note that $(0, 1) - (0, 1)X - (0, 1)X^2 - (0, 1)$ is a cycle of length 3 in $\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!])$. Thus $\mathrm{g}(\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!])) = 3$. □



g(Γ((ℤ(+)ℤ$_7$)[X⟧)) = 3

Figure 6. The girth of some zero-divisor graphs

Lemma 3.6. *Let $n \geq 2$ be an integer and let $C = C_n \cup \{(n, 0)\}$. Then C is a maximal clique of $\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[X\rrbracket)$.*

Proof. Note that any two distinct elements of C are adjacent; so C is a clique of $\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[X\rrbracket)$. Suppose to the contrary that there exists an element $f = \sum_{m\geq 0}(a_m, b_m)X^m \in Z((\mathbb{Z}(+)\mathbb{Z}_n)[X\rrbracket) \setminus C$ such that f is adjacent to all elements in C. Then $(n, 0)f = (0, 0)$; so $a_m = 0$ for all $m \in \mathbb{N}_0$. Hence $f = \sum_{m\geq 0}(0, b_m)X^m \in C$. This is a contradiction to the choice of f. Thus C is a maximal clique of $\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[X\rrbracket)$. □

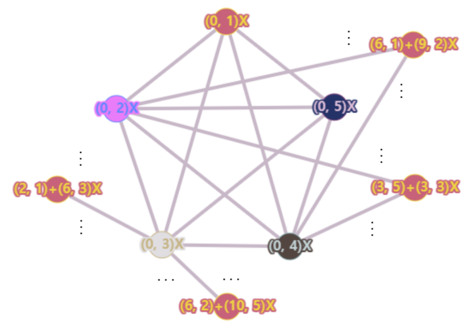
Theorem 3.7. *For an integer $n \geq 2$, the following statements hold.*

(1) $\chi(\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[X])) = \aleph_0$.
(2) $\chi(\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!])) = c$.

Proof. (1) Let C be a maximal clique of $\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[X])$ as in Lemma 3.6. Then by Remark 3.3(1) and Lemma 3.6, the chromatic number of the induced subgraph of $\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[X])$ by the set C is $\aleph_0$. Let $\overline{n}$ be the color of $(n, 0)$ and take any element $f = \sum_{i=0}^{m}(a_i, b_i)X^i \in Z((\mathbb{Z}(+)\mathbb{Z}_n)[X]) \setminus C$. Then $f \in D_n$ by the paragraph just after Lemma 3.2; so by Remark 3.3(2), f and $(n, 0)$ are not adjacent. Hence we color f with $\overline{n}$. Thus $\chi(\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[X])) = \aleph_0$.

(2) Let C be a maximal clique of $\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!])$ as in Lemma 3.6. Then by Remark 3.3(1) and Lemma 3.6, the chromatic number of the induced subgraph of $\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!])$ by the set C is c. Let $\overline{n}$ be the color of $(n, 0)$ and choose

any element $f = \sum_{i=0}^{\infty}(a_i, b_i)X^i \in Z((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!]) \setminus C$. Then $f \in D_n$ by the paragraph after Lemma 3.2; so by Remark 3.3(2), f and $(n, 0)$ are not adjacent. Hence we color f with $\overline{n}$. Thus $\chi(\Gamma((\mathbb{Z}(+)\mathbb{Z}_n)[\![X]\!])) = c$. □



$\chi(\Gamma((\mathbb{Z}(+)\mathbb{Z}_6)[\![X]\!])) = \infty$

FIGURE 7. The coloring of some zero-divisor graphs

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