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NEW GENERALIZATION FAMILIES OF HIGHER ORDER DAEHEE NUMBERS AND POLYNOMIALS

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Abstract. In this paper, we present a new definition and generalization of the first and second kinds of Daehee numbers and polynomials with the higher order. Some new results for these polynomials and numbers are derived. Furthermore, some interesting special cases of the new generalized Daehee polynomials and numbers of higher order are deduced.

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1. Introduction

The first kind of Daehee polynomials are defined by the generating function, [8, 9, 10, 11, 12, 13, 14, 15, 16],

$$\left(\frac{\log(1+t)}{t}\right)(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}. \tag{1}$$

Th Daehee numbers, D_n, can be obtained, by setting $x = 0$ into (1),

$$\int_{\mathbb{Z}_p} (x)_n d\mu_0(x) = D_n. \tag{2}$$

For $k \in \mathbb{N}$ and $n \in \mathbb{Z} \geq 0$, the first kind of Daehee numbers of order k is given by, Kim [8]

$$D_n^{(k)} = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} (x_1 + x_2 + \cdots x_k)_n d\mu_0(x_1)d\mu_0(x_2)\cdots d\mu_0(x_k), \tag{3}$$

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The generating function of $D_n^{(k)}$ are given as

$$\sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} = \left(\frac{\log(1+t)}{t}\right)^k, \quad n \in \mathbb{Z} \geq 0, k \in \mathbb{N}. \tag{4}$$

The first kind Daehee polynomials of order k are given by

$$D_n^{(k)}(x) = \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + x_2 + \cdots x_k + x)_n d\mu_0(x_1) d\mu_0(x_2) \cdots d\mu_0(x_k). \tag{5}$$

For $k \in \mathbb{Z}$, the kth-order Bernoulli polynomials are defined by the generating function to be, [1, 2, 3, 8],

$$\left(\frac{t}{e^t - 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \tag{6}$$

Setting $x = 0$, into (6), the kth-order Bernoulli numbers, $B_n^{(k)}$, can be obtained. Kim proved the following relation

$$D_n^{(k)}(x) = \sum_{\ell=0}^{n} s(n,\ell) B_\ell^{(k)}(x), \tag{7}$$

and

$$B_n^{(k)}(x) = \sum_{\ell=0}^{n} S(n,\ell) D_\ell^{(k)}(x). \tag{8}$$

The kth-order of the Daehee numbers have the following formula

$$D_n^{(k)} = \frac{s(n+k,k)}{\binom{n+k}{k}}, \quad n \geq 0,\ k \geq 1, \tag{9}$$

where $s(n+k,k)$ are Stirling numbers of the first kind see [6, 7, 8].
Cauchy numbers of the first (second) kinds, C_n, $(\hat{C}_n)$, are introduced by Comtet [2] as follows

$$C_n = \int_0^1 (x)_n dx, \tag{10}$$

$$\hat{C}_n = \int_0^1 (-x)_n dx. \tag{11}$$

The first and second generalized Comtet numbers, $s_{\bar{\boldsymbol{\alpha}}}(n,i;\bar{\boldsymbol{r}})$, $S_{\bar{\boldsymbol{\alpha}}}(n,i;\bar{\boldsymbol{r}})$, respectively, are defined as follows, [2]

$$(x;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}})_n = \sum_{i=0}^{n} s_{\bar{\boldsymbol{\alpha}}}(n,i;\bar{\boldsymbol{r}}) x^i, \tag{12}$$

and

$$x^n = \sum_{i=0}^{n} S_{\bar{\boldsymbol{\alpha}}}(n,i;\bar{\boldsymbol{r}})(x;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}})_i, \tag{13}$$

where $(x;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}})_n = \prod_{i=0}^{n-1}(x-\alpha_i)^{r_i}$, $\bar{\boldsymbol{\alpha}} = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$ and $\bar{\boldsymbol{r}} = (r_0, r_1, \cdots, r_{n-1})$.

2. Main results

2.1. Generalized first order multiparameters Daehee polynomials and numbers of higher order. In this subsection, new definitions of the generalization families for the first kind of Daehee polynomials and numbers, $\check{D}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}(x)$, $\check{D}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}$ are introduced. Also, some new results and special cases are derived.

Definition 2.1. The generalized multi-parameters first order Daehee numbers of order k, $\check{D}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}$, are defined as follows

$$\check{D}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\prod_{i=0}^{n-1}(x_1x_2\cdots x_k-\alpha_i)^{r_i}d\mu_0(x_1)d\mu_0(x_2)\cdots d\mu_0(x_k). \quad (14)$$

Some new results for $\check{D}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}$ can be derived as follows.

Theorem 2.2. *For $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have*

$$\check{D}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \sum_{m=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}})\sum_{\ell_1=0}^{m}\cdots\sum_{\ell_k=0}^{m}\prod_{i=0}^{k}S(m,\ell_i)D_{\ell_i}. \quad (15)$$

Proof. From Eq. (12) and (14),

$$\begin{aligned}
\check{D}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{m=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}})(x_1x_2\cdots x_k)^m d\mu_0(x_1)\cdots d\mu_0(x_k)\\
&= \sum_{m=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}})\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1x_2\cdots x_k)^m d\mu_0(x_1)\cdots d\mu_0(x_k)\\
&= \sum_{m=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}})\int_{\mathbb{Z}_p}(x_1)^m d\mu_0(x_1)\cdots\int_{\mathbb{Z}_p}(x_k)^m d\mu_0(x_k)\\
&= \sum_{m=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}})\left[\sum_{\ell_1=0}^{m}S(m,\ell_1)\int_{\mathbb{Z}_p}(x_1)_{\ell_1}d\mu_0(x_1)\cdots\times\right.\\
&\qquad\left.\sum_{\ell_k=0}^{m}S(m,\ell_k)\int_{\mathbb{Z}_p}(x_k)_{\ell_k}d\mu_0(x_k)\right]\\
&= \sum_{m=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}})\left[\sum_{\ell_1=0}^{m}S(m,\ell_1)D_{\ell_1}\cdots\sum_{\ell_k=0}^{m}S(m,\ell_k)D_{\ell_k}\right]\\
&= \sum_{m=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}})\left[\sum_{\ell_1=0}^{m}\sum_{\ell_2=0}^{m}\cdots\sum_{\ell_k=0}^{m}\prod_{i=1}^{k}S(m,\ell_i)D_{\ell_i}\right]. \quad (16)
\end{aligned}$$

This completes the proof. □

Theorem 2.3. *For $n \in \mathbb{Z} \geq 0$, $k \in \mathbb{N}$, we have*

$$\check{D}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \sum_{m=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}}) \sum_{\ell_1=0}^{m} \cdots \sum_{\ell_k=0}^{m} \prod_{i=0}^{k} \frac{(-1)^{\ell_i}\ell_i! S(m,\ell_i)}{\ell_i+1}. \tag{17}$$

Proof. Substituting Eq. (9) into (15), we obtain (17). □

Remark 2.1.

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} \prod_{i=0}^{n-1}(x_1x_2\cdots x_k-\alpha_i)^{r_i}\, d\mu_0(x_1)d\mu_0(x_2)\cdots d\mu_0(x_k)$$
$$= \sum_{m=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}}) \sum_{\ell_1=0}^{m} \cdots \sum_{\ell_k=0}^{m} \prod_{i=0}^{k} \frac{(-1)^{\ell_i}\ell_i! S(m,\ell_i)}{\ell_i+1}. \tag{18}$$

Definition 2.4. The multi-parameter first order Poly-Cauchy numbers, $C^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}$ are defined by, [5]

$$C^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \int_0^{\ell_1}\int_0^{\ell_2}\cdots\int_0^{\ell_k} \prod_{i=0}^{n-1}(x_1x_2\cdots x_k-\alpha_i)^{r_i}\, dx_1dx_2\cdots dx_k. \tag{19}$$

Theorem 2.5. *For $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have*

$$C^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \sum_{i=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,i;\bar{\boldsymbol{r}}) \frac{(\ell_1\ell_2\cdots\ell_k)^{i+1}}{(i+1)^k}. \tag{20}$$

Proof. From Eq. (19), we have

$$\begin{aligned}
C^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} &= \int_0^{\ell_1}\int_0^{\ell_2}\cdots\int_0^{\ell_k} \sum_{m=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}})(x_1x_2\cdots x_k)^m dx_1dx_2\cdots dx_k \\
&= \sum_{m=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}}) \int_0^{\ell_1}\int_0^{\ell_2}\cdots\int_0^{\ell_k} (x_1x_2\cdots x_k)^m dx_1dx_2\cdots dx_k \\
&= \sum_{m=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}}) \int_0^{\ell_1}(x_1)^m dx_1 \cdots \int_0^{\ell_k}(x_k)^m dx_k, \qquad (21)
\end{aligned}$$

then, (20) is obtained. □

Definition 2.6. The generalized first order Daehee numbers of order k, $\check{D}^{(k)}_n$ are defined by

$$\check{D}^{(k)}_n = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} (x_1x_2\cdots x_k)_n d\mu_0(x_1)d\mu_0(x_2)\cdots d\mu_0(x_k). \tag{22}$$

Theorem 2.7. *For $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have*

$$\check{D}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \sum_{\ell=0}^{|r|} S(n,\ell;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}) \check{D}^{(k)}_{\ell}. \tag{23}$$

Proof. Using Eq. (22), we have

$$\begin{aligned}
\check{D}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{m=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}})(x_1x_2\cdots x_k)^m d\mu_0(x_1)\cdots d\mu_0(x_k)\\
&= \sum_{m=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}})\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1x_2\cdots x_k)^m d\mu_0(x_1)\cdots d\mu_0(x_k)\\
&= \sum_{m=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}})\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{\ell=0}^{m} S(m,\ell)(x_1\cdots x_k)_\ell d\mu_0(x_1)\cdots d\mu_0(x_k)\\
&= \sum_{m=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}})\sum_{\ell=0}^{m} S(m,\ell)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1\cdots x_k)_\ell d\mu_0(x_1)\cdots d\mu_0(x_k).
\end{aligned} \tag{24}$$

Substituting from Eq. (22) into Eq. (24), we have

$$\check{D}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \sum_{m=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}})\sum_{\ell=0}^{m} S(m,\ell)\check{D}^{(k)}_{\ell} = \sum_{\ell=0}^{|r|}\sum_{m=\ell}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}})S(m,\ell)\check{D}^{(k)}_{\ell}.$$

But,

$$\sum_{m=\ell}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}})S(m,\ell) = S(n,\ell;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}),$$

where $S(n,\ell;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}})$ are the generalized second kind multiparameter non-central Stirling numbers, see [2, Eq. (4.4)], hence, we obtain Eq. (23). □

The multiparameters first order Daehee polynomials of order k, $\check{D}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}(x)$, can be defined as follows.

Definition 2.8. For $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, $\check{D}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}(x)$ are defined by

$$\check{D}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}(x) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\prod_{i=0}^{n-1}(x_1x_2\cdots x_k x-\alpha_i)^{r_i} d\mu_0(x_1)\cdots d\mu_0(x_k). \tag{25}$$

Definition 2.9. The generalized first kind Daehee polynomials of order k, $\check{D}^{(k)}_{n}(x)$ are given as

$$\check{D}^{(k)}_{n}(x) = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1x_2\cdots x_k x)_n d\mu_0(x_1)d\mu_0(x_2)\cdots d\mu_0(x_k). \tag{26}$$

Theorem 2.10. *For $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have*

$$\check{D}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}(x) = \sum_{\ell=0}^{|r|} S(n,\ell;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}})\check{D}^{(k)}_{\ell}(x). \tag{27}$$

Proof. From Eq. (25), we have

$$\check{D}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}(x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{m=0}^{|r|}s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}})(x_1x_2\cdots x_kx)^m d\mu_0(x_1)\cdots d\mu_0(x_k).$$

Using Theorem 2.7, hence, we obtain Eq. (27). □

Some special cases: The first kind Daehee polynomials and numbers, [7, 8, 11], can be obtained from the new definition as a special cases.

Case 1: (i) Setting $r_i=r,\alpha_i=i$ in Eq. (25), we have

$$\begin{aligned}\check{D}^{(k)}_{n;i,r}(x) &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\prod_{i=0}^{n-1}(x_1x_2\cdots x_kx-i)^r d\mu_0(x_1)\cdots d\mu_0(x_k)\\ &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1x_2\cdots x_kx)_{nr}\, d\mu_0(x_1))\cdots d\mu_0(x_k). \qquad (28)\end{aligned}$$

Replacing nr by n, the generalized first kind Daehee polynomials of order k, can be obtained.

(ii) Setting $r_i=r,\alpha_i=i$ in Eq. (14), we obtain

$$\check{D}^{(k)}_{n;i,r}=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1x_2\cdots x_k)_{nr}\, d\mu_0(x_1)\cdots d\mu_0(x_k). \qquad (29)$$

Replacing nr by n, the first kind Daehee numbers of order k can be obtained.

Case 2: (i) Setting $r_i=r,\alpha_i=\alpha$ in Eq. (25), we obtain

$$\begin{aligned}\check{D}^{(k)}_{n;\alpha,r}(x) &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1\cdots x_kx-\alpha)^{nr} d\mu_0(x_1)\cdots d\mu_0(x_k)\\ &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{\ell=0}^{nr}S(nr,\ell)(x_1\cdots x_kx-\alpha)_\ell d\mu_0(x_1)\cdots d\mu_0(x_k)\\ &= \sum_{\ell=0}^{nr}S(nr,\ell)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1\cdots x_kx-\alpha)_\ell d\mu_0(x_1)\cdots d\mu_0(x_k)\\ &= \sum_{\ell=0}^{nr}S(nr,\ell)\check{D}^{(k)}_{\ell,\alpha}(x). \qquad (30)\end{aligned}$$

(ii) Setting $r_i=r,\alpha_i=\alpha$ in Eq. (14), we obtain

$$\begin{aligned}\check{D}^{(k)}_{n;\alpha,r} &= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1x_2\cdots x_k-\alpha)^{nr} d\mu_0(x_1)d\mu_0(x_2)\cdots d\mu_0(x_k)\\ &= \sum_{\ell=0}^{nr}S(nr,\ell)\check{D}^{(k)}_{\ell,\alpha}. \qquad (31)\end{aligned}$$

At $\alpha_i = 0$ in Eq. (30), we obtain

$$\check{D}^{(k)}_{n;0,r}(x) = \sum_{\ell=0}^{nr} S(nr,\ell)\check{D}^{(k)}_{\ell}(x). \tag{32}$$

At $\alpha_i = 0$ in Eq. (31), we obtain

$$\check{D}^{(k)}_{n;0,r} = \sum_{\ell=0}^{nr} S(nr,\ell)\check{D}^{(k)}_{\ell}. \tag{33}$$

Case 3: (i) Setting $r_i = 1, \alpha_i = \alpha$ in Eq. (30), we have

$$\check{D}^{(k)}_{n;\alpha,1}(x) = \sum_{\ell=0}^{n} S(n,\ell)\check{D}^{(k)}_{\ell,\alpha}(x). \tag{34}$$

(ii) Setting $r_i = 1, \alpha_i = \alpha$ in Eq. (31), we obtain

$$\check{D}^{(k)}_{n;\alpha,1} = \sum_{\ell=0}^{n} S(n,\ell)\check{D}^{(k)}_{\ell,\alpha}. \tag{35}$$

(iii) Setting $r_i = 1, \alpha_i = 1$ in Eq. (25), we obtain

$$\begin{aligned}\check{D}^{(k)}_{n;1,1}(x) &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1x_2\cdots x_kx-1)^n d\mu_0(x_1)d\mu_0(x_2)\cdots d\mu_0(x_k)\\ &= \sum_{\ell=0}^{n} S(n,\ell)\check{D}^{(k)}_{\ell,1}(x).\end{aligned} \tag{36}$$

(iv) Setting $r_i = 1, \alpha_i = 1$ in Eq. (14), we obtain

$$\begin{aligned}\check{D}^{(k)}_{n;1,1} &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1x_2\cdots x_k-1)^n d\mu_0(x_1)d\mu_0(x_2)\cdots d\mu_0(x_k)\\ &= \sum_{\ell=0}^{n} S(n,\ell)\check{D}^{(k)}_{\ell,1}.\end{aligned} \tag{37}$$

Case 4: (i) Setting $r_i = 1, \alpha_i = 0$ in Eq. (25), we obtain

$$\begin{aligned}\check{D}^{(k)}_{n;0,1}(x) &= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1x_2\cdots x_kx)^n d\mu_0(x_1)d\mu_0(x_2)\cdots d\mu_0(x_k)\\ &= \sum_{\ell=0}^{n} S(n,\ell)\check{D}^{(k)}_{\ell}(x).\end{aligned} \tag{38}$$

(ii) Setting $r_i = 1, \alpha_i = 0$ in Eq. (14), we obtain

$$\check{D}^{(k)}_{n;0,1} = \sum_{\ell=0}^{n} S(n,\ell)\check{D}^{(k)}_{\ell}. \tag{39}$$

Case 5: (i) Setting $r_i = 1, \alpha_i = i,\ i = 0, 1, \cdots, n-1$ in Eq. (25), we have

$$\begin{aligned}\check{D}^{(k)}_{n;i,1}(x) &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\prod_{i=0}^{n-1}(x_1x_2\cdots x_kx - i)d\mu_0(x_1)\cdots d\mu_0(x_k)\\ &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1x_2\cdots x_kx)_n d\mu_0(x_1)\cdots d\mu_0(x_k)\\ &= \check{D}^{(k)}_n(x), \end{aligned}\tag{40}$$

the Daehee polynomials of order k, which defined by Kim [8], is obtained.

(ii) Setting $r_i = 1, \alpha_i = i,\ i = 0, 1, \cdots, n-1$ in Eq. (14), we have

$$\begin{aligned}\check{D}^{(k)}_{n;i,1} &= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\prod_{i=0}^{n-1}(x_1x_2\cdots x_k - i)d\mu_0(x_1)d\mu_0(x_2)\cdots d\mu_0(x_k)\\ &= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1x_2\cdots x_k)_n d\mu_0(x_1)d\mu_0(x_2)\cdots d\mu_0(x_k)\\ &= \check{D}^{(k)}_n, \end{aligned}\tag{41}$$

the first kind Daehee numbers of order k, see [8], is obtained.

Case 6: Setting $x_1x_2\cdots x_k = x$ in Eq. (14), we obtain

$$\check{D}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \int_{\mathbb{Z}_p}(x-\alpha_0)^{r_0}(x-\alpha_1)^{r_1}\cdots(x-\alpha_{n-1})^{r_{n-1}}d\mu_0(x). \tag{42}$$

Case 7: Setting $r_i = 1$ in Eq. (42), we obtain

$$\check{D}_{n;\bar{\boldsymbol{\alpha}}} = \int_{\mathbb{Z}_p}(x-\alpha_0)(x-\alpha_1)\cdots(x-\alpha_{n-1})d\mu_0(x). \tag{43}$$

Which we define $\check{D}_{n;\bar{\boldsymbol{\alpha}}}$ by generalized first kind Daehee numbers.

Theorem 2.11.

$$\int_{\mathbb{Z}_p}(x-\alpha_0)(x-\alpha_1)\cdots(x-\alpha_{n-1})d\mu_0(x) = \sum_{i=0}^{n} S(n,i;\bar{\boldsymbol{\alpha}})\frac{s(n+i,i)}{\binom{n+i}{i}}. \tag{44}$$

Proof. Substituting Eq. (9) into Eq. (43), we obtain Eq. (44). □

Case 8: Setting $\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} = \int_0^{\ell_1}\int_0^{\ell_2}\cdots\int_0^{\ell_k}$ in Eq. (14), the multiparameter Poly-Cauchy numbers of the first kind $C^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}$ are obtained, see Eq. (19).

2.2. Generalized second kind Daehee polynomials and numbers. In this subsection, new definition of the generalized multiparameters of the second kind Daehee polynomials and numbers with order k, $\widehat{\tilde{D}}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}$, are established. New results are derived. Also, some special cases are introduced.

Definition 2.12. For $n \in \mathbb{Z}$, $k \in \mathbb{N}$, $\widehat{\tilde{D}}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}$ are defined by

$$\widehat{\tilde{D}}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \prod_{i=0}^{n-1} (-x_1 x_2 \cdots x_k - \alpha_i)^{r_i} d\mu_0(x_1) \cdots d\mu_0(x_k). \tag{45}$$

Theorem 2.13. *For $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have*

$$\widehat{\tilde{D}}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \sum_{m=0}^{|r|} S_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}}) \sum_{\ell=0}^{m} L(m,\ell) \sum_{\ell_1=0}^{\ell} \cdots \sum_{\ell_k=0}^{\ell} \prod_{i=0}^{k} S(\ell,\ell_i) D_{\ell_i}. \tag{46}$$

Proof. Using (45), we have

$$\widehat{\tilde{D}}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{m=0}^{|r|} S_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}}) (-x_1 x_2 \cdots x_k)_m d\mu_0(x_1) \cdots d\mu_0(x_k).$$

From the definition of Lah numbers,

$$(-x_1 x_2 \cdots x_k)_m = \sum_{\ell=0}^{m} L(m,\ell)(x_1 x_2 \cdots x_k)_\ell,$$

hence

$$\widehat{\tilde{D}}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{m=0}^{|r|} S_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}}) \sum_{\ell=0}^{m} L(m,\ell)(x_1 \cdots x_k)_\ell d\mu_0(x_1) \cdots d\mu_0(x_k). \tag{47}$$

Substituting from Eq. (15) into (47), we obtain (46). □

Definition 2.14. The multiparameter of the second kind Poly-Cauchy numbers $\hat{C}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}$ are defined by

$$\hat{C}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \int_0^{\ell_1} \int_0^{\ell_2} \cdots \int_0^{\ell_k} \prod_{i=0}^{n-1} (-x_1 x_2 \cdots x_k - \alpha_i)^{r_i} dx_1 dx_2 \cdots dx_k. \tag{48}$$

Theorem 2.15. *For $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have*

$$\hat{C}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \sum_{i=0}^{|r|} \sum_{\ell=0}^{m} s_{\bar{\boldsymbol{\alpha}}}(n,i;\bar{\boldsymbol{r}}) L(m,\ell) \frac{(\ell_1 \ell_2 \cdots \ell_k)^{i+1}}{(i+1)^k}. \tag{49}$$

Proof. From Eq. (47), we have

$$\hat{C}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \int_0^{\ell_1} \cdots \int_0^{\ell_k} \sum_{m=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}}) (-x_1 x_2 \cdots x_k)^m dx_1 \cdots dx_k$$

$$= \int_0^{\ell_1} \cdots \int_0^{\ell_k} \sum_{m=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}}) \sum_{\ell=0}^{m} L(m,\ell)(x_1 x_2 \cdots x_k)_\ell dx_1 \cdots dx_k.$$

Then, we obtain (49).

□

Definition 2.16. For $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, $\widehat{\check{D}}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}(x)$ are defined by

$$\widehat{\check{D}}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \prod_{i=0}^{n-1} (-x_1 x_2 \cdots x_k x - \alpha_i)^{r_i} d\mu_0(x_1) \cdots d\mu_0(x_k). \tag{50}$$

Theorem 2.17. *For $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have*

$$\widehat{\check{D}}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}(x) = \sum_{m=0}^{|r|} S_{\bar{\boldsymbol{\alpha}}}(n,m,\bar{\boldsymbol{r}}) \sum_{\ell=0}^{m} L(m,\ell) \sum_{\ell_1=0}^{\ell} \cdots \sum_{\ell_k=0}^{\ell} \prod_{i=0}^{k} S(\ell,\ell_i) \check{D}_{\ell_i}(x). \tag{51}$$

Proof. From Eq. (50), we have

$$\begin{aligned} \widehat{\check{D}}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}(x) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{m=0}^{|r|} S_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}})(-x_1 \cdots x_k)_m d\mu_0(x_1) \cdots d\mu_0(x_k) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{m=0}^{|r|} S_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}}) \sum_{\ell=0}^{m} L(m,\ell)(x_1 \cdots x_k)_\ell d\mu_0(x_1) \cdots d\mu_0(x_k). \end{aligned}$$

By using Theorem 2.13, the proof is completed. □

Definition 2.18. The second kind of generalized Dahee polynomials with order k, $\widehat{\check{D}}^{(k)}_n(x)$ are defined by

$$\widehat{\check{D}}^{(k)}_n(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 x_2 \cdots x_k x)_n d\mu_0(x_1) \cdots d\mu_0(x_k). \tag{52}$$

Theorem 2.19. *For $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have*

$$\widehat{\check{D}}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}(x) = \sum_{\ell=0}^{|r|} (-1)^\ell S(n,\ell;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}) \check{D}^{(k)}_\ell(x). \tag{53}$$

Proof. From Eq. (50) and using Eq. (12) and Eq. (16), we have

$$\widehat{\check{D}}^{(k)}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{m=0}^{|r|} (-1)^m s_{\bar{\boldsymbol{\alpha}}}(n,m;\bar{\boldsymbol{r}})(x_1 x_2 \cdots x_k x)^m d\mu_0(x_1) \cdots d\mu_0(x_k).$$

By using Theorem 2.15, the proof is completed. □

Some special cases: The second kind Daehee polynomials and numbers, [7, 8, 11], can be obtained from the new definition as a special cases.

Case 1: (i) Setting $r_i = r, \alpha_i = i$ in Eq. (50), we have

$$\begin{aligned}\widehat{\tilde{D}}^{(k)}_{n;i,\bar{\boldsymbol{r}}}(x) &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\prod_{i=0}^{n-1}(-x_1x_2\cdots x_kx-i)^r d\mu_0(x_1)\cdots d\mu_0(x_k)\\ &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1x_2\cdots x_kx)_{nr}d\mu_0(x_1)\cdots d\mu_0(x_k). \qquad (54)\end{aligned}$$

The second kind of the generalized Daehee polynomials, can be obtained by replacing nr by n.

(ii) Setting $r_i = r, \alpha_i = i$, in Eq. (45), we obtain

$$\widehat{\tilde{D}}^{(k)}_{n;i,\bar{\boldsymbol{r}}} = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1x_2\cdots x_k)_{nr}d\mu_0(x_1)d\mu_0(x_2)\cdots d\mu_0(x_k). \qquad (55)$$

Replacing nr by n, the second kind of Daehee numbers with order k can be obtained.

Case 2: (i) Setting $r_i = r, \alpha_i = \alpha$ in Eq. (50), we obtain

$$\begin{aligned}\widehat{\tilde{D}}^{(k)}_{n;\alpha,r}(x) &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1\cdots x_kx-\alpha)^{nr}d\mu_0(x_1)\cdots d\mu_0(x_k)\\ &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{\ell=0}^{nr}S(nr,\ell)(-x_1\cdots x_kx-\alpha)_\ell d\mu_0(x_1)\cdots d\mu_0(x_k)\\ &= \sum_{\ell=0}^{nr}S(nr,\ell)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1\cdots x_kx-\alpha)_\ell d\mu_0(x_1)\cdots d\mu_0(x_k)\\ &= \sum_{\ell=0}^{nr}S(nr,\ell)\widehat{\tilde{D}}^{(k)}_{\ell,\alpha}(x). \qquad (56)\end{aligned}$$

(ii) Setting $r_i = r, \alpha_i = \alpha$ in Eq. (45), we obtain

$$\begin{aligned}\widehat{\tilde{D}}^{(k)}_{n;\alpha,r} &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1x_2\cdots x_k-\alpha)^{nr}d\mu_0(x_1)\cdots d\mu_0(x_k)\\ &= \sum_{\ell=0}^{nr}S(nr,\ell)\widehat{\tilde{D}}^{(k)}_{\ell,\alpha}. \qquad (57)\end{aligned}$$

At $\alpha_i = 0$ in Eq. (56), we obtain

$$\widehat{\tilde{D}}^{(k)}_{n;0,r}(x) = \sum_{\ell=0}^{nr}S(nr,\ell)\widehat{\tilde{D}}^{(k)}_{\ell}(x). \qquad (58)$$

At $\alpha_i = 0$ in Eq. (57), we obtain

$$\widehat{\check{D}}^{(k)}_{n;0,r} = \sum_{\ell=0}^{nr} S(nr,\ell)\widehat{\check{D}}^{(k)}_{\ell}. \tag{59}$$

Case 3: (i) Setting $r_i = 1$ and $\alpha_i = \alpha$ in Eq. (50), we obtain

$$\widehat{\check{D}}^{(k)}_{n;\alpha,1}(x) = \sum_{\ell=0}^{n} S(n,\ell)\widehat{\check{D}}_{\ell,\alpha}(x). \tag{60}$$

(ii) Setting $r_i = 1, \alpha_i = \alpha$ in Eq. (45), we obtain

$$\widehat{\check{D}}^{(k)}_{n;\alpha,1} = \sum_{\ell=0}^{n} S(n,\ell)\widehat{\check{D}}^{(k)}_{\ell,\alpha}. \tag{61}$$

(iii) Setting $r_i = 1, \alpha_i = 1$ in Eq. (50), we obtain

$$\begin{aligned}\widehat{\check{D}}^{(k)}_{n;1,1}(x) &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1x_2\cdots x_kx-1)^n d\mu_0(x_1)\cdots d\mu_0(x_k)\\ &= \sum_{\ell=0}^{n} S(n,\ell)\widehat{\check{D}}^{(k)}_{\ell,1}(x).\end{aligned} \tag{62}$$

(iv) Setting $r_i = 1, \alpha_i = 1$ in Eq. (45), we obtain

$$\begin{aligned}\widehat{\check{D}}^{(k)}_{n;1,1} &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1x_2\cdots x_k-1)^n d\mu_0(x_1)\cdots d\mu_0(x_k)\\ &= \sum_{\ell=0}^{n} S(n,\ell)\widehat{\check{D}}^{(k)}_{\ell,1}.\end{aligned} \tag{63}$$

Case 4: (i) Setting $r_i = 1, \alpha_i = 0$ in Eq. (50), we obtain

$$\begin{aligned}\widehat{\check{D}}^{(k)}_{n;0,1}(x) &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1x_2\cdots x_kx)^n d\mu_0(x_1)\cdots d\mu_0(x_k)\\ &= \sum_{\ell=0}^{n} S(n,\ell)\widehat{\check{D}}^{(k)}_{\ell}(x).\end{aligned} \tag{64}$$

(ii) Setting $r_i = 1, \alpha_i = 0$ in Eq. (45), we obtain

$$\widehat{\check{D}}^{(k)}_{n;0,1} = \sum_{\ell=0}^{n} S(n,\ell)\check{D}^{(k)}_{\ell}. \tag{65}$$

Case 5: (i) Setting $r_i = 1$, $\alpha_i = i, i = 0, 1, \cdots, n-1$ in Eq. (50), we have

$$\begin{aligned}\widehat{\check{D}}^{(k)}_{n;i,1}(x) &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\prod_{i=0}^{n-1}(-x_1x_2\cdots x_kx-i) d\mu_0(x_1)\cdots d\mu_0(x_k)\\ &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1x_2\cdots x_kx)_n d\mu_0(x_1)\cdots d\mu_0(x_k)\end{aligned}$$

$$= \hat{\tilde{D}}_n^{(k)}(x), \tag{66}$$

The Daehee polynomials of order k, which defined by Kim [8] are obtained.

(ii) Setting $r_i = 1, \alpha_i = i, i = 0, 1, \cdots, n-1$ in Eq. (45), we have

$$\begin{aligned} \hat{\tilde{D}}_{n;i,1}^{(k)} &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \prod_{i=0}^{n-1} (-x_1 x_2 \cdots x_k - i) d\mu_0(x_1) \cdots d\mu_0(x_k) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 x_2 \cdots x_k)_n d\mu_0(x_1) \cdots d\mu_0(x_k) \\ &= \hat{\tilde{D}}_n^{(k)}, \end{aligned} \tag{67}$$

The second kind Daehee numbers with order k, see [8], is obtained.

Case 6: Setting $x_1 x_2 \cdots x_k = x$ in Eq. (45), we obtain

$$\hat{\tilde{D}}_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \int_{\mathbb{Z}_p} (-x-\alpha_0)^{r_0}(-x-\alpha_1)^{r_1} \cdots (-x-\alpha_{n-1})^{r_{n-1}} d\mu_0(x). \tag{68}$$

Case 7: Setting $r_i = 1$ in Eq. (68), we obtain

$$\hat{\tilde{D}}_{n;\bar{\boldsymbol{\alpha}}} = \int_{\mathbb{Z}_p} (-x-\alpha_0)(-x-\alpha_1) \cdots (-x-\alpha_{n-1}) d\mu_0(x). \tag{69}$$

Which is defined $\hat{\tilde{D}}_{n;\bar{\boldsymbol{\alpha}}}$ by generalized Daehee numbers of the second kind.

Case 8: Setting $\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} = \int_0^{\ell_1}\int_0^{\ell_2}\cdots\int_0^{\ell_k}$ in Eq. (45), the multiparameter Poly-Cauchy numbers of the second kind $C_{n;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}^{(k)}$ are obtained, see Eq. (48).

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