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FITTED OPERATOR ON THE CRANK-NICOLSON SCHEME FOR SOLVING A SMALL TIME DELAYED CONVECTION-DIFFUSION EQUATIONS

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Abstract. This paper is concerned with singularly perturbed convection-diffusion parabolic partial differential equations which have time-delayed. We used the Crank-Nicolson(CN) scheme to build a fitted operator to solve the problem. The underling method's stability is investigated, and it is found to be unconditionally stable. We have shown graphically the unstableness of CN-scheme without fitting factor. The order of convergence of the present method is shown to be second order both in space and time in relation to the perturbation parameter. The efficiency of the scheme is demonstrated using model examples and the proposed technique is more accurate than the standard CN-method and some methods available in the literature, according to the findings.

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1. Introduction

A time delayed singularly perturbed differential equation is a differential equation that have at least one delay term in the time variable and the coefficient of highest order derivative is a small parameter. Delay differential equations are essential in modeling real-world phenomena and computational scientists can pay more attention to them. Due to the appearance of delay term in the mathematical modeling in different fields, they provide a better approximation of the observed phenomena, but computing their solutions has been major challenge. To explain the dynamics of certain biological systems, some singularly perturbed diffusive models have been developed in biology. In Many real-world applications they have small diffusion parameter's; for instant, Murray [11]. in

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blood the order of diffusion coefficient for hemoglobin molecules and oxygen are $10^{-7}cm^2s^{-1}$ and $10^{-5}cm^2s^{-1}$ respectively.

Boundary layers occur in the neighborhoods of the boundary of the domain, where the solution of singularly perturbed problems has a rapidly change. A boundary layer, either regular or parabolic form, may appear away from any domain corner. The boundary layer is parabolic type if the reduced problem's characteristics curve is parallel to the boundary, otherwise it is regular type Shishkin et al, [5].

Because of the rapidly changes (steep gradients) in the solution of SPP, the numerical schemes based on the classical method on uniform meshes are insufficient for solving boundary layer problems Kadalbajoo et al, [10]. Due to insufficient of the classical method the concept of an ϵ-uniform method are developed, which is a convergent numerical technique for solving a singularly perturbed problems independent of the size of the singular perturbation parameter ϵ.

If regular boundary layers are present, then applying an appropriate uniform mesh fitted finite difference method can often give ϵ- uniform method. According to Shishkin et al. [12], this method is not practical if a parabolic boundary layer exists.

Different Scholars have considered different numerical schemes to find the numerical solution of singularly perturbed parabolic problems. Such as, Using a fitted numerical system, numerical solutions for singularly perturbed parabolic reaction-diffusion problems are obtained [4, 7], numerical treatment for singularly perturbed parabolic convection-diffusion problems using average fitted operator finite difference technique Tesfaye et al, [2]; for singularly perturbed delay parabolic differential equation Chakravarthy et al, [1] developed based on uniform mesh and adaptive mesh a stable finite difference scheme; a numerical treatment for singularly perturbed delay parabolic partial differential equation using a fitted operator finite difference technique with second order [3, 8] and Kadalbajoo and Awasthi [9] examined the approximate solution for time-dependent singularly perturbed convection-diffusion equations exhibits a boundary layer on the right side of the domain using the midpoint upwind method on non-uniform mesh.

In this paper, a time delayed singularly perturbed parabolic partial differential equations with boundary layer are considered using the idea of fitted operator finite difference method. The fitted factor was derived by using the Crank-Nicolson finite difference method. The result is compeered to the classical Crank-Nicolson method and other method in the literature.

2. Method Formulation

Consider singularly perturbed delay parabolic partial differential equations (SPDPPDEs) in the form:

$$\begin{cases} -\epsilon\dfrac{\partial^2 u(x,t)}{\partial x^2} + a(x)\dfrac{\partial u(x,t)}{\partial x} + \dfrac{\partial u(x,t)}{\partial t} = b(x)u(x,t-\tau) + f(x,t), \\ (x,t) \in D \equiv (0,1)\times(0,T]. \end{cases} \tag{1}$$

with initial condition

$$u(x,t) = \psi(x,t), \quad (x,t) \in [0,1]\times[-\tau,0] \tag{1a}$$

and boundary condition

$$u(0,t) = S_1(t) \text{ and } u(1,t) = S_2(t), \tag{1b}$$

where $0 < \epsilon \ll 1$ is the singular perturbed parameter, $\tau \leq \epsilon$ is the delay parameter, $a(x) \geq \beta_0 > 0$, $b(x) \geq \beta_1 \geq 0$, $f(x,t)$, $\psi(x,t)$, $S_1(t)$ and $S_2(t)$ are bounded and smooth functions. The initial condition $u(x,t) = \psi(x,t)$ are also satisfied the compatibility conditions.

$$\psi(0,0) = S_1(0) \text{ and } \psi(1,0) = S_2(0) \tag{2}$$

$$\begin{cases} \dfrac{\partial\psi(0,0)}{\partial t} = \epsilon\dfrac{\partial^2\psi(0,0)}{\partial x^2} - a(0)\dfrac{\partial\psi(0,0)}{\partial x} + b(0)u(0,-\tau) + f(0,0) \\ \dfrac{\partial\psi(1,0)}{\partial t} = \epsilon\dfrac{\partial^2\psi(1,0)}{\partial x^2} - a(1)\dfrac{\partial\psi(1,0)}{\partial x} + b(1)u(1,-\tau) + f(1,0) \end{cases} \tag{3}$$

Using the assumption of equation (2) and (3) results problem (1) with the initial condition $(1a)$ and the boundary conditions $(1b)$ has a unique solution.

Now, consider equation (1) on a particular domain $(x,t) \in \overline{D} = [0,1]\times[0,T]$ together with (1a) and (1b). Because of the the sign of $a(x)$ and $b(x)$ the boundary layer of problem (1) occurs at $x = 1$. In order to apply fitted operator finite difference technique to solve equation (1) together with $(1a)$ and $(1b)$ first, approximate $u(x,t-\tau)$ using Taylor series expansion as follow.

$$u(x,t-\tau) = u(x,t) - \tau\frac{\partial u(x,t)}{\partial t} + O(\tau^2) \tag{4}$$

Substituting equation (4) into equation (1), we have

$$-\epsilon\frac{\partial^2 u(x,t)}{\partial x^2} + a(x)\frac{\partial u(x,t)}{\partial x} - b(x)u(x,t) + c(x)\frac{\partial u(x,t)}{\partial t} = f(x,t), \tag{5}$$

where $c(x) = 1 + \tau b(x)$.
The first procedure to apply finite difference technique is discretized the space and the time interval in to M and N subintervals respectively.

$$x_m = mh, \quad h = \frac{1}{M}, \quad m = 0,1,2,...,M.$$
$$t_n = nk, \quad k = \frac{T}{N}, \quad n = 0,1,2,...,N.$$

For simplicity, denote the approximate solution at (x_m, t_n) as $u_m^n = u(x_m, t_n)$. Applying the average of explicit and implicit finite difference method at (x_m, t_n) and (x_m, t_{n+1}) points respectively (Cranck-Nicolson finite difference method) for equation (5), we get.

$$\frac{-\epsilon}{2}\left(\frac{u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}}{h^2} + \frac{u_{m-1}^{n} - 2u_m^{n} + u_{m+1}^{n}}{h^2}\right) +$$
$$\frac{a_m}{2}\left(\frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2h} + \frac{u_{m+1}^{n} - u_{m-1}^{n}}{2h}\right) - \frac{b_m}{2}\Big(u_m^{n+1} + u_m^n\Big) + T_1$$
$$+ \frac{c_m}{2}\left(\frac{u_m^{n+1} - u_m^n}{k} + \frac{u_m^{n+1} - u_m^n}{k}\right) + T_2 = \frac{f_m^{n+1} + f_m^n}{2},$$

where $T_1 = h^2\Big(\frac{\epsilon}{12}\frac{\partial^4 u_m^{n+1}}{\partial x^4} - \frac{a_m^{n+1}}{6}\frac{\partial^4 u_m^{n+1}}{\partial x^4} + \frac{\epsilon}{12}\frac{\partial^4 u_m^{n}}{\partial x^4} - \frac{a_m^{n}}{6}\frac{\partial^4 u_m^{n}}{\partial x^4}\Big)$.
$T_2 = -c_m\frac{k^2}{6}\frac{\partial^3 u_m^n}{\partial t^3}$

Simplifying and rearranging the above equation, we have

$$\begin{cases} 2c_m u_m^{n+1} - 2c_m u_m^n \\ +k\Big(-\epsilon\frac{u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}}{h^2} + a_m\frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2h} - b_m u_m^{n+1}\Big) \\ +k\Big(-\epsilon\frac{u_{m-1}^{n} - 2u_m^{n} + u_{m+1}^{n}}{h^2} + a_m\frac{u_{m+1}^{n} - u_{m-1}^{n}}{2h} - b_m u_m^{n}\Big) + T \\ = k\big(f_m^{n+1} + f_m^n\big), \end{cases} \tag{6}$$

where $T = T_1 + T_2$ is the total truncation error.
To obtain the more accurate numerical solution and ϵ- uniformly convergent numerical method, multiply the perturbation parameter of equation (6) by fitting factors σ_1 and σ_2.

$$\begin{cases} 2c_m u_m^{n+1} - 2c_m u_m^n \\ +k\Big(-\epsilon\sigma_1\frac{u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}}{h^2} + a_m\frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2h} - b_m u_m^{n+1}\Big) \\ +k\Big(-\epsilon\sigma_2\frac{u_{m-1}^{n} - 2u_m^{n} + u_{m+1}^{n}}{h^2} + a_m\frac{u_{m+1}^{n} - u_{m-1}^{n}}{2h} - b_m u_m^{n}\Big) \\ = k\big(f_m^{n+1} + f_m^n\big) \end{cases} \tag{7}$$

The fitting factors σ_1 and σ_2 can be evaluated such that the solution of equation (7) converges uniformly to the solution of equation (1). Using the theory of perturbation O'Malley [14] multiplying both side by h and then evaluating the limit of equation (7) as $h \longrightarrow 0$ gives:

$$\begin{cases} \lim\limits_{h\longrightarrow 0}\left(-\frac{\sigma_1}{\rho}(u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}) + \frac{a_m}{2}(u_{m+1}^{n+1} - u_{m-1}^{n+1})\right) + \\ \lim\limits_{h\longrightarrow 0}\left(-\frac{\sigma_2}{\rho}(u_{m+1}^{n} - 2u_m^{n} + u_{m-1}^{n}) + \frac{a_m}{2}(u_{m+1}^{n} - u_{m-1}^{n})\right) = 0, \end{cases} \tag{8}$$

where $\rho = \dfrac{h}{\epsilon}$.

Since, the finite difference approximation is evaluated at different time level, from equation (8), it follows that:

$$\begin{cases} \lim\limits_{h\longrightarrow 0}\left(-\frac{\sigma_1}{\rho}(u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}) + \frac{a_m}{2}(u_{m+1}^{n+1} - u_{m-1}^{n+1})\right) = 0 \\ \lim\limits_{h\longrightarrow 0}\left(-\frac{\sigma_2}{\rho}(u_{m+1}^{n} - 2u_m^{n} + u_{m-1}^{n}) + \frac{a_m}{2}(u_{m+1}^{n} - u_{m-1}^{n})\right) = 0 \end{cases} \tag{9}$$

Simplifying equation (9), we get

$$\frac{2\sigma_1}{\rho} = \frac{\lim\limits_{h\longrightarrow 0} a_m(u_{m+1}^{n+1} - u_{m-1}^{n+1})}{\lim\limits_{h\longrightarrow 0}(u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1})}, \quad and \tag{10}$$

$$\frac{2\sigma_2}{\rho} = \frac{\lim\limits_{h\longrightarrow 0} a_m(u_{m+1}^{n} - u_{m-1}^{n})}{\lim\limits_{h\longrightarrow 0}(u_{m+1}^{n} - 2u_m^{n} + u_{m-1}^{n})} \tag{11}$$

The solution of equation (5) based on matched asymptotic expansion method O'Malley and Roos et al., [13, 14] is given as:

Let $u_0(x,t)$ is outer solution and to obtain the inner solution assume $y = \frac{1-x}{\epsilon}$ and $U(y,t) = u(x,t)$.

Using chain rule, we have

$$\frac{\partial u}{\partial x} = \frac{\partial U}{\partial y}\frac{dy}{dx} = -\frac{1}{\epsilon}\frac{\partial U}{\partial y}$$

similarly

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\epsilon^2}\frac{\partial^2 U}{\partial y^2}$$

equation (5) becomes:

$$-\frac{1}{\epsilon}\frac{\partial^2 U(y,t)}{\partial y^2} - \frac{1}{\epsilon}a(1-\epsilon y)\frac{\partial U(y,t)}{\partial y} - b(1-\epsilon y)U(y,t) + c(1-\epsilon y)\frac{\partial U(y,t)}{\partial t} = f(1-\epsilon y, t).$$

Multiplying both sides by ϵ and taking $\epsilon = 0$, we get

$$-\frac{\partial^2 U(y,t)}{\partial y^2} - a(1)\frac{\partial U(y,t)}{\partial y} = 0$$

The inner solution becomes:

$$u_{in}(x,t) = c_1 + c_2 e^{-a(1)\frac{1-x}{\epsilon}}, \quad u(1,t) = S_2(t))$$

Thus, the composite solution becomes:

$$u(x,t) = u_0(x,t) + Ae^{-a(1)\frac{1-x}{\epsilon}} \tag{12}$$

Equation (12) at a point $t = t_n$ becomes:

$$u^n(x) = u_0^n(x) + Ae^{-a(1)\frac{1-x}{\epsilon}} \tag{13}$$

Since $x_m = mh$, then the limit of equation (13) at x_m becomes:

$$\lim_{h\longrightarrow 0} u_m^n = u_0^n(0) + Ae^{-\frac{a(1)}{\epsilon}} e^{a(1)m\rho} \tag{14}$$

From equation (14), we have

$$\begin{cases} \lim_{h\longrightarrow 0} \left(u_{m+1}^n - u_{m-1}^n\right) = Ae^{-\frac{a^n(1)}{\epsilon}} e^{a(1)m\rho}\Big(e^{a(1)\rho} - e^{-a(1)\rho}\Big) \\ \lim_{h\longrightarrow 0} \left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right) = Ae^{-\frac{a(1)}{\epsilon}} e^{a(1)m\rho}\Big(e^{a(1)\rho} - 2 + e^{-a(1)\rho}\Big) \end{cases} \tag{15}$$

substituting equation (15) into equation (10) and (11) and simplifying , we get

$$\sigma_1 = \frac{\rho a^{n+1}(1)}{2} coth\Big[\frac{\rho a^{n+1}(1)}{2}\Big] \tag{16}$$

and

$$\sigma_2 = \frac{\rho a^{n}(1)}{2} coth\Big[\frac{\rho a^{n}(1)}{2}\Big] \tag{17}$$

After substituting the values of σ_1 and σ_2 in the scheme (7) and simplifying ,we get the three term recurrence relation: system that can solved using Thomas algorithm.

$$-E_m^{n+1}u_{m-1}^{n+1} + F_m^{n+1}u_m^{n+1} - G_m^{n+1}u_{m+1}^{n+1} = H_m^{n+1} \tag{18}$$

for $n = 0, 1, 2, ..., N$ and $m = 1, 2, ..., M$, where

$$E_m^{n+1} = k\left(\frac{\epsilon\sigma_1}{h^2} + \frac{a_m}{2h}\right) \tag{18a}$$

$$F_m^{n+1} = 2c_m + k\left(\frac{2\epsilon\sigma_1}{h^2} - b_m\right) \tag{18b}$$

$$G_m^{n+1} = k\left(\frac{\epsilon\sigma_1}{h^2} - \frac{a_m}{2h}\right) \tag{18c}$$

$$\begin{aligned} H_m^{n+1} &\\ &= k\left(\frac{\epsilon\sigma_2}{h^2} + \frac{a_m}{2h}\right)u_{m-1}^n + \left[2c_m - k\left(\frac{2\epsilon\sigma_2}{h^2} - b_m\right)\right]u_m^n \\ &+ k\left(\frac{\epsilon\sigma_2}{h^2} - \frac{a_m}{2h}\right)u_{m+1}^n + k\Big(f_m^{n+1} + f_m^n\Big). \end{aligned} \tag{18d}$$

Equation (18) is a tridiagonal system that can solved using Thomas algorithm. Since c_m and a_m are non-negative, then equation (18) becomes diagonally dominant.

$$i.e. \ \mid F_m^{n+1} \mid > \mid E_m^{n+1} \mid + \mid G_m^{n+1} \mid .$$

Thus, the present method have convergent solution.

3. Convergence Analysis

To show the convergence of the method we show by using Lax's equivalent theorem.

3.1. Consistency of the method. The consistency of the original equation (1) induced by the consistency of the transformed equation (5). From equation (6) the total local truncation error of the present scheme at a point $u(x_m, t_n)$, we have

$$T_m^n = T_1 + T_2,$$

where $T_1 = h^2\Big(\frac{\epsilon}{12}\frac{\partial^4 u_m^{n+1}}{\partial x^4} - \frac{a_m}{6}\frac{\partial^4 u_m^{n+1}}{\partial x^4} + \frac{\epsilon}{12}\frac{\partial^4 u_m^{n}}{\partial x^4} - \frac{a_m}{6}\frac{\partial^4 u_m^{n}}{\partial x^4}\Big)$,
$T_2 = -c_m\frac{k^2}{3}\frac{\partial^3 u_m^n}{\partial t^3}$.

And, we have that $\lim\limits_{(h,k)\longrightarrow(0,0)} T_m^n = 0$.
Thus, the present method is consistent and the method has $O(h^2 + k^2)$ order of convergence.

3.2. Stability of the method. Using Von Neumann stability technique, we have

$$u_m^n = \xi^n e^{im\theta}, \tag{19}$$

where $i = \sqrt{-1}$, θ is real number and ξ is amplitude factor.
Substituting equation (19) into equation (18), gives

$$\begin{aligned}&\xi(-E_m^{n+1}e^{-i\theta} + F_m^{n+1} - G_m^{n+1}e^{i\theta})\\ &= k\left(\frac{\epsilon\sigma_2}{h^2} + \frac{a_m}{2h}\right)e^{-i\theta} + \left[2c_m - k\left(\frac{2\epsilon\sigma_2}{h^2} - b_m\right)\right] + k\left(\frac{\epsilon\sigma_2}{h^2} - \frac{a_m}{2h}\right)e^{i\theta}\end{aligned}$$

Substituting the value of E_m^{n+1}, F_m^{n+1} and G_m^{n+1} into the above equation and simplifying, we get

$$\xi = \frac{2c_m - 4k\frac{\epsilon\sigma_2}{h^2}sin^2\dfrac{\theta}{2} - iksin\theta\frac{a_m}{h} + kb_m}{2c_m + 4k\frac{\epsilon\sigma_1}{h^2}sin^2\dfrac{\theta}{2} + iksin\theta\frac{a_m}{h} + kb_m}.$$

Let $d_m = 2c_m + kb_m$, then the above equation becomes:

$$\xi \leq \frac{d_m - 4k\frac{\epsilon\sigma_2}{h^2} sin^2\dfrac{\theta}{2} - iksin\theta\frac{a_m}{h}}{d_m + 4k\frac{\epsilon\sigma_1}{h^2} sin^2\dfrac{\theta}{2} + iksin\theta\frac{a_m}{h}}.$$

For any mesh size in both x and t variables, we have $|\xi| \leq 1$.
Thus, the present method is unconditionally stable.

From Lax's equivalent theorem, a stable and consistent method is convergent and hence the present method is convergent.

4. Examples and Solutions

To examine the performance of the present scheme, we considered model problems which appear in literature and their approximate solutions are available for comparison.
We considered the double mesh principle to compute the absolute maximum error and the rate of convergence of the present method, if the exact solution for the given problem is unknown.
To calculate the absolute maximum error at the specified mesh points, we used the following formula:
Case 1:- If the exact solution is known

$$E_\epsilon^{M,N} = \max_{(x_m,t_n)\in D} \left| u(x_m, t_n) - u_m^n \right|$$

Case 2:- If the exact solution is unknown

$$E_\epsilon = \max_{(x_m,t_n)\in D} \left| \left(u_m^n\right)^{M,N} - \left(u_{2m-1}^{2n-1}\right)^{2M,2N} \right|$$

In addition, in order to determine the corresponding rate of convergence we evaluate using the formula:

$$R_\epsilon^{M,N} = \frac{log(E_\epsilon^{M,N}) - log(E_\epsilon^{2M,2N})}{log(2)}.$$

Example 4.1. Consider the following time delayed convection-diffusion problem:

$$\begin{cases} -\epsilon\dfrac{\partial^2 u}{\partial x^2} + (2 - x^2)\dfrac{\partial u}{\partial x} + (x+1)(t+1)u(x,t) + \dfrac{\partial u}{\partial t} \\ = u(x, t-\tau) + 10t^2 exp(-t)x(1-x) \\ (x,t) \in [0,1] \times [0,2] \end{cases}$$

subject to initial condition

$$u(x,t) = 0, \quad (x,t) \in [0,1] \times [-\tau, 0]$$

and boundary condition

$$u(0,t) = 0 \quad and \quad u(1,t) = 0, \quad t \in (0,2]$$

The maximum absolute point-wise error and rate of convergence are shown in Tables 1, 2 and 3 below for the present method and Crank-Nicolson method are presented, together with different values of perturbation parameter ϵ and number of subinterval of x and t. Figure 1 represent the computed solution's physical behavior. Figure 2 indicate the position of boundary layer using point-wise error
.

TABLE 1. Comparison of absolute maximum errors for $\tau = 0.8\epsilon$ for Example 4.1.

$\epsilon \downarrow$ M,N $\longrightarrow$	32,32	64,64	128,128	256,256
Present method				
2^{-4}	3.3653e-04	8.4512e-05	2.1150e-05	5.2889e-06
2^{-6}	9.6759e-04	2.5955e-04	6.6543e-05	1.6725e-05
2^{-8}	2.9810e-03	1.2394e-03	3.7914e-04	1.0093e-04
2^{-10}	3.0972e-03	1.7501e-03	8.8923e-04	3.5031e-04
2^{-12}	3.0972e-03	1.7512e-03	9.2352e-04	4.7292e-04
2^{-14}	3.0972e-03	1.7512e-03	9.2352e-04	4.7324e-04
CN-method				
2^{-4}	1.1099e-03	2.6934e-04	6.7037e-05	1.6741e-05
2^{-6}	1.8031e-02	4.5114e-03	9.9789e-04	2.4262e-04
2^{-8}	1.0872e-01	5.3821e-02	1.8443e-02	4.4862e-03
2^{-10}	2.2302e-01	1.8085e-01	1.1657e-01	5.5492e-02
2^{-12}	2.7671e-01	2.7184e-01	2.4278e-01	1.8875e-01
2^{-14}	2.9290e-01	3.0466e-01	3.0200e-01	2.8467e-01

TABLE 2. Comparison of rate of convergence for $\tau = 0.8\epsilon$ for Example 4.1

$\epsilon \downarrow$ M,N $\longrightarrow$	32,32	64,64	128,128
present method			
2^{-4}	1.9935	1.9985	1.9996
2^{-6}	1.8984	1.9636	1.9922
2^{-8}	1.2662	1.7088	1.9093
2^{-10}	0.8236	0.9768	1.3439
2^{-12}	0.8226	0.9232	0.9656
2^{-14}	0.8226	0.9232	0.9646
CN- method			
2^{-4}	2.0429	2.0064	2.0015
2^{-6}	1.9988	2.1766	2.0402
2^{-8}	1.0143	1.5451	2.0395
2^{-10}	0.3024	0.6336	1.0708
2^{-12}	0.0256	0.1631	0.3632
2^{-14}	-0.0568	0.0126	0.0853

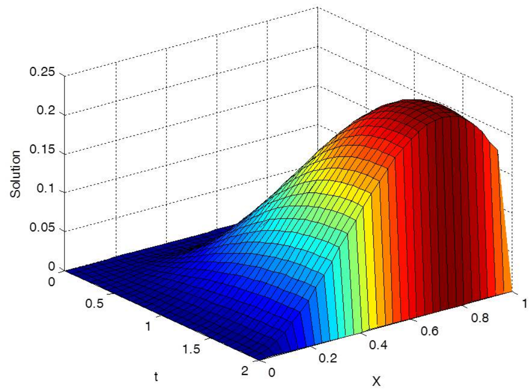

FIGURE 1. The behavior of the solutions for M=N=32, $\epsilon = 10^{-10}$ and $\tau = 0.8\epsilon$ for Example 4.1

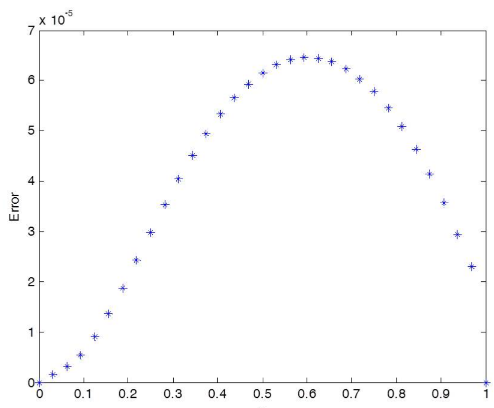

FIGURE 2. The pointwise absolute errors for M=N=32, $\epsilon = 10^{-10}$ and $\tau = 0.8\epsilon$ for Example 4.1.

TABLE 3. Comparison of absolute maximum errors for $\tau = 0.2\epsilon$ for Example 4.1 .

$\epsilon \downarrow$ M,N $\longrightarrow$	32,32	64,64	128,128	256,256
present method				
10^{-2}	1.6745e-03	4.8597e-04	1.2709e-04	3.2143e-05
10^{-4}	3.0972e-03	1.7512e-03	9.2352e-04	4.7324e-04
10^{-6}	3.0972e-03	1.7512e-03	9.2352e-04	4.7324e-04
10^{-8}	3.0972e-03	1.7512e-03	9.2352e-04	4.7324e-04
CN-Method				
10^{-2}	3.7099e-02	1.1472e-02	2.5449e-03	5.8998e-04
10^{-4}	2.8937e-01	2.9715e-01	2.8781e-01	2.5948e-01
10^{-6}	2.9840e-01	3.1689e-01	3.2698e-01	3.3180e-01
10^{-8}	2.9849e-01	3.1709e-01	3.2744e-01	3.3288e-01

Example 4.2. Consider the one-dimensional singularly perturbed parabolic partial differential equation with delay term [3].

$$\begin{cases} -\epsilon \dfrac{\partial^2 u(x,t)}{\partial x^2} + (2-x^2)\dfrac{\partial u(x,t)}{\partial x} + xu(x,t) + \dfrac{\partial u(x,t)}{\partial t} - u(x,t-\tau) \\ = 10t^2 exp(-t)x(1-x), \\ (x,t) \in (0,1) \times (0,2], \\ \text{subject to initial condition } u(x,t) = 0, \quad (x,t) \in [0,1] \times [-\tau, 0,] \\ \text{and boundery condition } u(0,t) = u(1,t) = 0, \quad t \in (0,2], \end{cases} \tag{20}$$

In Table 4 comparison of absolute maximum point wise error and rate of convergence using present method and CN- method are presented below . Figure 3(a) represent the computed solution's physical behavior and Figure 3(b) to indicates the position of boundary layer using different time level. Figure 4 indicate the physical behavior of the model using without fitting factor (uniform mesh CN-method) and the oscillation indicates the unstableness of the method or the method have a disturbance on the solution.

Table 4. Comparison of absolute maximum errors and rate of convergence for $\tau = 0.5\epsilon$ for Example 4.2.

$\epsilon \downarrow$ M,N $\longrightarrow$	64,32	128,64	256,128	512,256
present method				
10^{-2}	2.7417e-03	5.7576e-04	1.4326e-04	3.5509e-05
10^{-4}	7.1656e-03	3.6211e-03	1.8200e-03	9.1225e-04
10^{-6}	7.1657e-03	3.6212e-03	1.8200e-03	9.1234e-04
10^{-8}	7.1657e-03	3.6212e-03	1.8200e-03	9.1234e-04
Rate	0.9847	0.9925	0.9963	
CN-Method				
10^{-2}	4.1219e-02	8.9695e-03	2.1048e-03	5.2720e-04
10^{-4}	1.1685e+00	1.0011e+00	8.8627e-01	7.0904e-01
10^{-6}	1.4293e+00	1.4882e+00	1.4765e+00	1.3492e+00
10^{-8}	1.4327e+00	1.5038e+00	1.5389e+00	1.5544e+00
Rate	-0.0698	-0.0333	-0.0145	

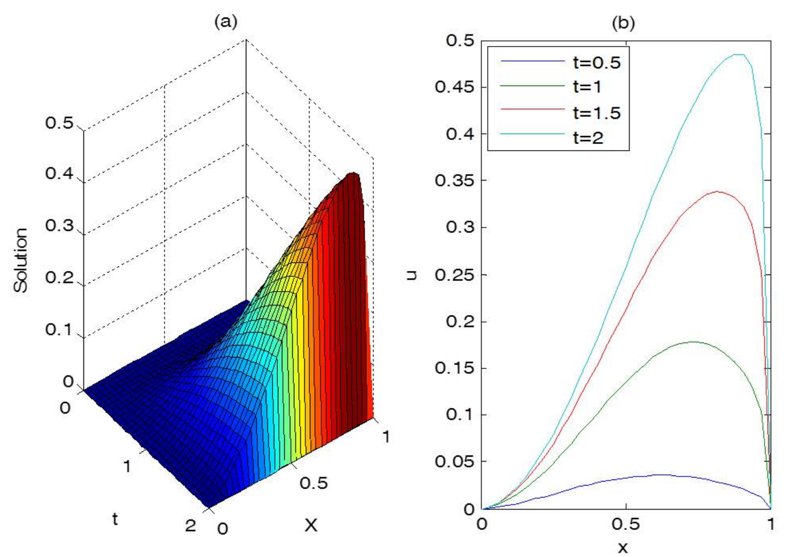

Figure 3. Numerical solution profile for present method (a) and numerical results at different time (b) for Example 4.2 for M=N=32, $\epsilon = 10^{-2}$ and $\tau = 0.5\epsilon$.

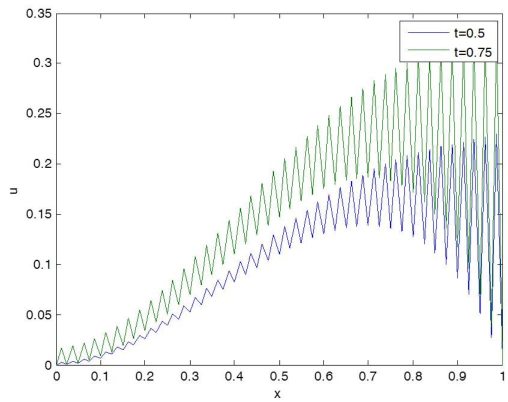

FIGURE 4. Numerical solution profile without fitting factor for Example 4.2 at M=N=80, $\epsilon = 10^{-4}$ and $\tau = 0.5\epsilon$

5. Results and Discussion

In this study we have discussed the approximate solution for a time delayed singularly perturbed parabolic partial differential convection-diffusion problems using fitted operator on Crank-Nicolson's scheme. The basic mathematical procedures are defining the model problem, discretizing the solution domain uniformly, applying Crank-Nicolson' method and introducing a fitting parameter which is determined using theory of perturbation. Simplifying the developed scheme, we get a diagonal dominant three-term recurrence relations at each time level which is solved by using Thomas algorithm.

The model examples are used to exemplify the proposed method's efficiency and effectiveness. The current method is second order convergent with respect to time and space variables and almost first order rate of convergence. The stability and consistency have been established very well to guarantee the convergence of the proposed method.

6. Conclusions

The numerical results, maximum point wise error, and rate of convergence for the test problems are presented. The solution and error graphs are also displayed. The findings show that the proposed method outperforms Crank-Nicolson's method as well as some earlier methods in the literature. Further based on the result we can generalize that the proposed method is convergent independent to the perturbation parameter.

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