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EXPLICIT IDENTITIES INVOLVING GEOMETRIC POLYNOMIALS ARISING FROM DIFFERENTIAL EQUATIONS AND THEIR ZEROS†

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ABSTRACT. In this paper, we study differential equations arising from the generating functions of the geometric polynomials. We give explicit identities for the geometric polynomials. Finally, we investigate the zeros of the geometric polynomials by using computer.

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1. Introduction

Recently, many mathematicians have studied in the area of the Bernoulli numbers, Euler numbers, Genocchi numbers, tangent numbers, exponential polynomials, and special polynomials(see [1-18]). The moments of the Poisson distribution are well-known to be connected to the combinatorics of the Bell and Stirling numbers. The classical Stirling numbers of the second kind $S_2(n,k)$ is the number of partitions of a set of n elements into k disjoint nonempty subsets and is defined as coefficients of the relation

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

where $(x)_k = x(x-1)\cdots(x-n+1)$ is the falling factorial of x degree k and $(x)_0 = 1$. Further study of $S_2(n,k)$ leads us to the Bell numbers(see [4, 11, 17]).

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As is well known, the Bell numbers B_n are given by the generating function

$$e^{(e^t-1)} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{1.1}$$

Note that $B_n = \sum_{k=0}^{n} S_2(n,k)$. The Bell polynomials $B_n(\lambda)$ are given by the generating function

$$e^{\lambda(e^t-1)} = \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}. \tag{1.2}$$

The geometric polynomials $\omega_n(x)$ are defined by the generating function:

$$\sum_{n=0}^{\infty} \omega_n(x) \frac{t^n}{n!} = \frac{1}{1-x(e^t-1)}, \text{ (see [4])}. \tag{1.3}$$

The Bell and geometric polynomials are connected by the relation

$$\omega_n(x) = \int_0^{\infty} B_n(x\lambda) e^{-\lambda} d\lambda. \tag{1.4}$$

In view of (1.4) this gives

$$\sum_{n=0}^{\infty} \omega_n(x) \frac{t^n}{n!} = \int_0^{\infty} e^{-\lambda(1-x(e^t-1))} d\lambda.$$

The numbers $\omega_n(1) = \sum_{k=0}^{n} S_2(n,k)k!$ are known as the preferential arrangement numbers(see [4, 11]). For $k \in \mathbb{N}$, the geometric polynomials $\omega_n^{(k)}(x)$ of order k are defined by the generating function:

$$\sum_{n=0}^{\infty} \omega_n^{(k)}(x) \frac{t^n}{n!} = \left(\frac{1}{1-x(e^t-1)} \right)^k. \tag{1.5}$$

We can naturally define a 2-variable polynomials $\omega_n(x,y)$ by multiplying e^{yt} on the right side of the Eq. (1.3) as follows:

$$\sum_{n=0}^{\infty} \omega_n(x,y) \frac{t^n}{n!} = \left(\frac{1}{1-x(e^t-1)} \right) e^{yt}. \tag{1.6}$$

Again, for $k \in \mathbb{N}$, the 2-variable geometric polynomials $\omega_n^{(k)}(x,y)$ of order k are defined by the generating function

$$\sum_{n=0}^{\infty} \omega_n^{(k)}(x,y) \frac{t^n}{n!} = \left(\frac{1}{1-x(e^t-1)} \right)^k e^{yt}. \tag{1.7}$$

The first few examples of geometric polynomials are

$$\begin{aligned}
\omega_0^{(k)}(x,y) &= 1,\\
\omega_1^{(k)}(x,y) &= kx + y,\\
\omega_2^{(k)}(x,y) &= kx + kx^2 + k^2x^2 + 2kxy + y^2,\\
\omega_3^{(k)}(x,y) &= kx + 3kx^2 + 3k^2x^2 + 2kx^3 + 3k^2x^3 + k^3x^3 + 3kxy + 3kx^2y\\
&\quad + 3k^2x^2y + 3kxy^2 + y^3.
\end{aligned}$$

When $k = 1$, above (1.5) and (1.7) will become the corresponding definitions of the geometric polynomials $\omega_n(x)$ and the 2-variable geometric polynomials $\omega_n(x,y)$. Note that $\omega_n^{(k)}(x,0) = \omega_n^{(k)}(x)$ and $\omega_n(x,0) = \omega_n(x)$. From (1.3) and (1.6), we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} \omega_n(x,y)\frac{t^n}{n!} &= \left(\frac{1}{1-x(e^t-1)}\right)e^{yt}\\
&= \left(\sum_{n=0}^{\infty}\omega_n(x)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} y^m\frac{t^m}{m!}\right) \qquad (1.8)\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\omega_k(x)y^{n-k}\right)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients on both sides of (1.8), we obtain

$$\omega_n(x,y) = \sum_{k=0}^{n}\binom{n}{k}\omega_k(x)y^{n-k}. \qquad (1.9)$$

By (1.5), we have

$$\begin{aligned}
\sum_{l=0}^{\infty}\omega_l^{(k)}(x)\frac{t^l}{l!} &= \sum_{n=0}^{\infty}\binom{n+k-1}{n}x^n n!\frac{(e^t-1)^n}{n!}\\
&= \sum_{n=0}^{\infty}\binom{n+k-1}{n}x^n n!\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!} \qquad (1.10)\\
&= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{n+k-1}{n}n!x^nS_2(l,n)\right)\frac{t^l}{l!}.
\end{aligned}$$

Again, by comparing the coefficients of $\frac{t^l}{l!}$ on the both sides of (1.10), we get

$$\omega_l^{(k)}(x) = \sum_{n=0}^{l}\binom{n+k-1}{n}n!x^nS_2(l,n). \qquad (1.11)$$

From (1.7), we get

$$
\begin{aligned}
\sum_{l=0}^{\infty}\omega_l^{(k)}(x,y)\frac{t^l}{l!} &= \left(\frac{1}{1-x(e^t-1)}\right)^k e^{yt}\\
&= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{n+k-1}{n}n!x^nS_2(l,n)\right)\frac{t^l}{l!}\\
&= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{n+k-1}{n}n!x^nS_2(l,n)\right)\frac{t^l}{l!}\sum_{l=0}^{\infty}y^l\frac{t^l}{l!}\\
&= \sum_{l=0}^{\infty}\left(\sum_{j=0}^{l}\sum_{n=0}^{j}\binom{l}{j}\binom{n+k-1}{n}n!x^nS_2(j,n)y^{l-j}\right)\frac{t^l}{l!}.
\end{aligned}
\tag{1.12}
$$

We get

$$
\omega_l^{(k)}(x,y) = \sum_{j=0}^{l}\sum_{n=0}^{j}\binom{l}{j}\binom{n+k-1}{n}n!x^nS_2(j,n)y^{l-j}. \tag{1.13}
$$

The following elementary properties of the $\omega_n(x)$, $\omega_n(x,y)$, $\omega_n^{(k)}(x)$, and $\omega_n^{(k)}(x,y)$ are readily derived from (1.3), (1.11), and (1.13). We, therefore, choose to omit the details involved.

Theorem 1.1. *For any positive integer n, we have*

(1) $\omega_l(x) = \sum_{n=0}^{l} n!x^nS_2(l,n)$.

(2) $\omega_l(x,y) = \sum_{j=0}^{l}\sum_{n=0}^{j}\binom{l}{j}n!x^nS_2(j,n)y^{l-j}$.

(3) $\sum_{n=0}^{\infty}\omega_n^{(k)}(x)\frac{t^n}{n!} = \int_0^{\infty} e^{-\lambda(1-x(e^t-1))^k}d\lambda$.

(4) $\sum_{n=0}^{\infty}\omega_n^{(k)}(x,y)\frac{t^n}{n!} = \int_0^{\infty} e^{-\lambda(1-x(e^t-1))^k}e^{yt}d\lambda$.

Recently, many mathematicians have studied the differential equations arising from the generating functions of special polynomials(see [6, 9, 13, 14, 15, 16, 17]). In this paper, we study differential equations arising from the generating functions of geometric polynomials. We use the coefficients of this differential equation to obtain explicit identities of geometric polynomials. In addition, we investigate the zeros of the geometric polynomials with numerical methods. Finally, we observe an interesting phenomenon of 'scattering' of the zeros of geometric polynomials.

2. Explicit identities for the geometric polynomials

Differential equations arising from the generating functions of special polynomials are studied by many authors in order to give explicit identities for special polynomials(see [6, 9, 13, 14, 15, 16, 17]). In this section, we study differential equations arising from the generating functions of geometric polynomials. We give explicit identities for the geometric polynomials.

Let

$$\begin{aligned} F = F(t,x) &= \frac{1}{1-x(e^t-1)} = \sum_{n=0}^{\infty} \omega_n(x)\frac{t^n}{n!}, \\ F^k = F^k(t,x) &= \left(\frac{1}{1-x(e^t-1)}\right)^k = \sum_{n=0}^{\infty} \omega_n^{(k)}(x)\frac{t^n}{n!}. \end{aligned} \tag{2.1}$$

Then, by (2.1), we have

$$\begin{aligned} F^{(1)} = \frac{d}{dt}F(t,x) &= \frac{d}{dt}\left(\frac{1}{1-x(e^t-1)}\right) = \frac{xe^t}{(1-x(e^t-1))^2} \\ &= xe^tF^2, \end{aligned} \tag{2.2}$$

$$\begin{aligned} F^{(2)} = \frac{d}{dt}F^{(1)} &= xe^tF^2 + 2xe^tFF^{(1)} \\ &= xe^tF^2 + 2xe^tFxe^tF^2 = xe^tF^2 + 2x^2e^{2t}F^3 \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} F^{(3)} = \frac{d}{dt}F^{(2)} &= xe^tF^2 + 2xe^tFF^{(1)} + 4x^2e^{2t}F^3 + 6x^2e^{2t}F^2F^{(1)} \\ &= xe^tF^2 + 2xe^tF(xe^tF^2) + 4x^2e^{2t}F^3 + 6x^2e^{2t}F^2(xe^tF^2) \\ &= xe^tF^2 + 6x^2e^{2t}F^3 + 6x^3e^{3t}F^4. \end{aligned}$$

Continuing this process, we can guess that

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t,x) \\ &= \sum_{i=0}^{N} a_i(N,x)e^{it}F^{i+1} \\ &= a_0(N,x)F + a_1(N,x)e^tF^2 + \cdots + a_N(N,x)e^{Nt}F^{N+1}, (N=0,1,2,\ldots). \end{aligned} \tag{2.4}$$

Therefore, by (2.4), we can construct the following differential equation.

$$F^{(N)} - a_0(N,x)F - a_1(N,x)e^tF^2 - \cdots - a_N(N,x)e^{Nt}F^{N+1} = 0.$$

Using the coefficients of this differential equation, we give explicit identities for the geometric polynomials.

Taking the derivative with respect to t in (2.4), we get

$$\begin{aligned}F^{(N+1)} &= \frac{dF^{(N)}}{dt}\\ &= \sum_{i=0}^{N} a_i(N,x) i e^{it} F^{i+1} + \sum_{i=0}^{N} a_i(N,x)(i+1)e^{it}F^{i}F^{(1)}\\ &= \sum_{i=0}^{N} a_i(N,x) i e^{it} F^{i+1} + \sum_{i=0}^{N} a_i(N,x)(i+1)e^{it}F^{i}(xe^{t}F^{2})\\ &= \sum_{i=0}^{N} a_i(N,x) i e^{it} F^{i+1} + \sum_{i=0}^{N} a_i(N,x)(i+1)xe^{(i+1)t}F^{i+2}\\ &= \sum_{i=0}^{N} a_i(N,x) i e^{it} F^{i+1} + \sum_{i=1}^{N+1} a_{i-1}(N,x) i x e^{it}F^{i+1}.\end{aligned} \tag{2.5}$$

On the other hand, by replacing N by $N+1$ in (2.4), we have

$$F^{(N+1)} = \sum_{i=0}^{N+1} a_i(N+1,x)e^{it}F^{i+1}. \tag{2.6}$$

Comparing the coefficients on both sides of (2.5) and (2.6), we obtain

$$a_0(N+1,x) = 0, \quad a_{N+1}(N+1,x) = (N+1)xa_N(N,x), \tag{2.7}$$

and

$$a_i(N+1,x) = ixa_{i-1}(N,x) + ia_i(N,x), (1 \le i \le N). \tag{2.8}$$

In addition, by (2.4), we get

$$F(t,x) = F^{(0)}(t,x) = a_0(0,x)F(t,x). \tag{2.9}$$

By (2.9), we get

$$a_0(0,x) = 1. \tag{2.10}$$

It is not difficult to show that

$$\begin{aligned}xe^{t}F^{2}(t,x) &= F^{(1)}(t,x)\\ &= \sum_{i=0}^{1} a_i(1,x)e^{it}F^{i+1}(t,x)\\ &= a_0(1,x)F(t,x) + a_1(1,x)e^{t}F^{2}(t,x).\end{aligned} \tag{2.11}$$

Thus, by (2.11), we also get

$$a_0(1,x) = 0, \quad a_1(1,x) = x. \tag{2.12}$$

From (2.7), we note that

$$a_0(N+1,x) = a_0(N,x) = \cdots = a_0(1,x) = 0, a_0(0,x) = 1, \tag{2.13}$$

and

$$a_{N+1}(N+1,x,\lambda) = (N+1)xa_N(N,x) = \cdots = (N+1)!x^{N+1}. \tag{2.14}$$

For $i = 1, 2, 3$ in (2.8), we have

$$a_1(N+1,x) = x \sum_{k=0}^{N} a_0(N-k,x), \tag{2.15}$$

$$a_2(N+1,x) = 2x \sum_{k=0}^{N-1} 2^k a_1(N-k,x), \tag{2.16}$$

and

$$a_3(N+1,x) = 3x \sum_{k=0}^{N-2} 3^k a_2(N-k,x). \tag{2.17}$$

Continuing this process, we can deduce that, for $1 \le i \le N$,

$$a_i(N+1,x) = ix \sum_{k=0}^{N-i+1} i^k a_{i-1}(N-k,x). \tag{2.18}$$

Here, we note that the matrix $a_i(j,x)_{0\le i,j\le N+1}$ is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & x & x & x & \cdots & x \\ 0 & 0 & 2!x^2 & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 3!x^3 & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (N+1)!x^{N+1} \end{pmatrix}$$

Now, we give explicit expressions for $a_i(N+1,x)$. By (2.15), (2.16), and (2.17), we get

$$a_1(N+1,x) = x \sum_{k_1=0}^{N} a_0(N-k_1,x) = x,$$

$$\begin{aligned} a_2(N+1,x) &= 2!x \sum_{k_2=0}^{N-1} 2^{k_2} a_1(N-k_2,x) \\ &= 2!x^2 \sum_{k_2=0}^{N-1} 2^{k_2}, \end{aligned}$$

and

$$\begin{aligned} a_3(N+1,x) &= 3x \sum_{k_3=0}^{N-2} 3^{k_3} a_2(N-k_3,x) \\ &= 3!x^3 \sum_{k_3=0}^{N-2} \sum_{k_2=0}^{N-2-k_3} 3^{k_3} 2^{k_2}. \end{aligned}$$

Continuing this process, we have

$$a_i(N+1,x) = i!x^i \sum_{k_i=0}^{N-i+1} \sum_{k_{i-1}=0}^{N-i+1-k_i} \cdots \sum_{k_2=0}^{N-i+1-k_i-\cdots-k_3} i^{k_i} \cdots 3^{k_3} 2^{k_2}. \tag{2.19}$$

Therefore, by (2.19), we obtain the following theorem.

Theorem 2.1. *For $N = 0, 1, 2, \ldots$, the differential equation*

$$F^{(N)} = \sum_{i=0}^{N} a_i(N,x) e^{it} F^{i+1}(t,x) = \sum_{i=0}^{N} a_i(N,x) e^{it} F^{i+1}(t,x)$$

has a solution

$$F = F(t,x) = \frac{1}{1 - x(e^t - 1)},$$

where

$$a_0(0,x) = 1, \quad a_0(i,x) = 0, (1 \leq i \leq N),$$
$$a_1(i,x) = x, (1 \leq i \leq N),$$
$$a_N(N,x) = N!x^N,$$
$$a_i(N,x) = i!x^i \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-i-k_i} \cdots \sum_{k_2=0}^{N-i-k_i-\cdots-k_3} i^{k_1} \cdots 3^{k_3} 2^{k_2}, (2 \leq i \leq N).$$

From (2.1), we note that

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,x) = \sum_{k=0}^{\infty} \omega_{k+N}(x) \frac{t^k}{k!}. \tag{2.20}$$

From Theorem 1 and (2.20), we can derive the following equation:

$$\begin{aligned} \sum_{k=0}^{\infty} \omega_{k+N}(x) \frac{t^k}{k!} = F^{(N)} &= \sum_{i=0}^{N} a_i(N,x) e^{it} \left(\frac{1}{1 - x(e^t - 1)} \right)^{i+1} \\ &= \sum_{i=0}^{N} a_i(N,x) \left(\sum_{k=0}^{\infty} \omega_k^{(i+1)}(x,i) \frac{t^k}{k!} \right) \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{N} a_i(N,x) \omega_k^{(i+1)}(x,i) \right) \frac{t^k}{k!}. \end{aligned} \tag{2.21}$$

By comparing the coefficients on both sides of (2.21), we obtain the following theorem.

Theorem 2.2. *For $k, N = 0, 1, 2, \ldots$, we have*

$$\omega_{k+N}(x) = \sum_{i=0}^{N} a_i(N,x) \omega_k^{(i+1)}(x,i), \tag{2.22}$$

where

$$a_0(0,x) = 1, \quad a_0(i,x) = 0, (1 \leq i \leq N),$$
$$a_1(i,x) = x, (1 \leq i \leq N),$$
$$a_N(N,x) = N!x^N,$$
$$a_i(N,x) = i!x^i \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-i-k_i} \cdots \sum_{k_2=0}^{N-i-k_i-\cdots-k_3} i^{k_1} \cdots 3^{k_3} 2^{k_2}, (2 \leq i \leq N).$$

Now, by (2.22), we obtain explicit identities for the geometric polynomials. Let us take $k = 0$ in (2.22). Then, we have the following corollary.

Corollary 2.3. *For $N = 0, 1, 2, \ldots,$ we have*

$$\omega_N(x) = \sum_{i=0}^{N} a_i(N,x)\omega_0^{(i+1)}(x,i) = \sum_{i=0}^{N} a_i(N,x).$$

3. Zeros of the geometric polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the geometric polynomials $\omega_n(x)$. By using computer, the geometric polynomials $\omega_n(x)$ can be determined explicitly. We display the shapes of the geometric polynomials $\omega_n(x)$ and investigate the zeros of the geometric polynomials $\omega_n(x)$. Several conjectures are presented through numerical experiments. We expect these conjectures to be theoretically resolved in the future. The first few examples of geometric polynomials are

$$\omega_0(x) = 1,$$
$$\omega_1(x) = x,$$
$$\omega_2(x) = x + 2x^2,$$
$$\omega_3(x) = x + 6x^2 + 6x^3,$$
$$\omega_4(x) = x + 14x^2 + 36x^3 + 24x^4,$$
$$\omega_5(x) = x + 30x^2 + 150x^3 + 240x^4 + 120x^5,$$
$$\omega_6(x) = x + 62x^2 + 540x^3 + 1560x^4 + 1800x^5 + 720x^6.$$

Now, we investigate the beautiful zeros of the geometric polynomials $\omega_n(x)$ by using a computer. We plot the zeros of the $\omega_n(x)$ for $n = 5, 10, 15, 20$, and $x \in \mathbb{C}$(Figure 1). In Figure 1(top-left), we choose $n = 10$. In Figure 1(top-right), we choose $n = 20$. In Figure 1(bottom-left), we choose $n = 25$. In Figure 1(bottom-right), we choose $n = 30$.

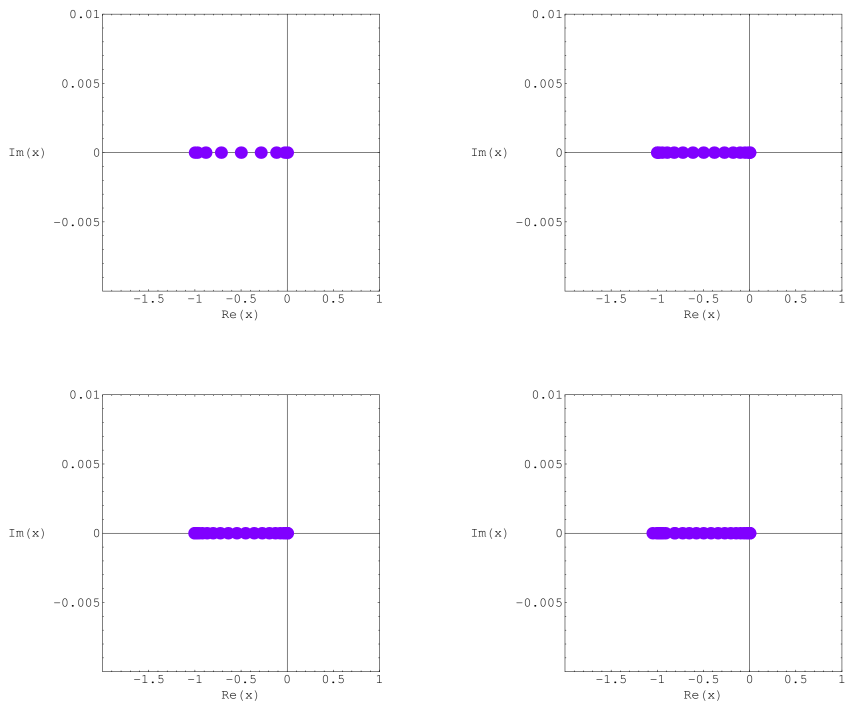

FIGURE 1. Zeros of $\omega_n(x)$

Stacks of zeros of the geometric polynomials $\omega_n(x)$ for $1 \leq n \leq 30$ from a 3-D structure are presented(Figure 2).

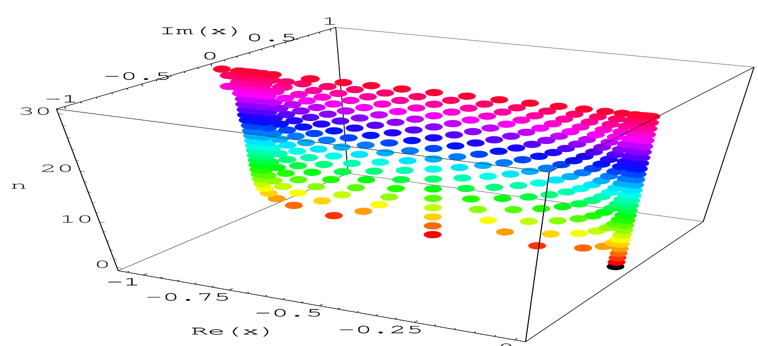

FIGURE 2. Stacks of zeros of $\omega_n(x), 1 \leq n \leq 30$

Our numerical results for approximate solutions of real zeros of the geometric polynomials $\omega_n(x)$ are displayed(Tables 1, 2).

Table 1. Numbers of real and complex zeros of $\omega_n(x)$

degree n	real zeros	complex zeros
1	1	0
2	2	0
3	3	0
4	4	0
5	5	0
6	6	0
7	7	0
8	8	0
9	9	0
10	10	0
11	11	0
12	12	0
13	13	0
14	14	0

Plot of real zeros of $\omega_n(x)$ for $1 \leq n \leq 30$ structure are presented(Figure 3). We observe a remarkably regular structure of the complex roots of the geomet-

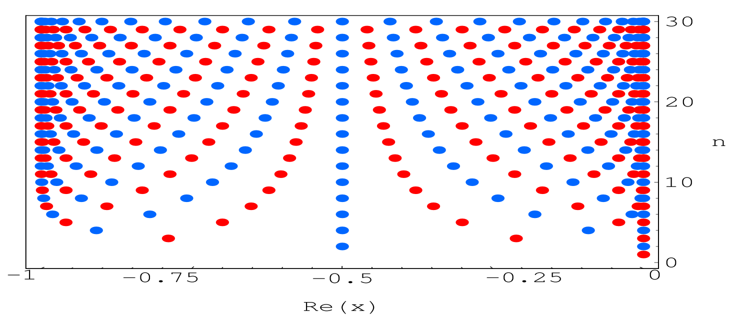

FIGURE 3. Real zeros of $\omega_n(x)$ for $1 \leq n \leq 30$

ric polynomials $\omega_n(x)$. We hope to verify a remarkably regular structure of the complex roots of the geometric polynomials $\omega_n(x)$(Table 1). By means of numerical experiments, we make a series of the following conjectures:

Conjecture 1. The numbers of complex zeros $C_{\omega_n(x)}$ of $\omega_n(x)$ is 0(see Table 1).

Next, we calculated an approximate solution satisfying $\omega_n(x) = 0, x \in \mathbb{R}$. The results are given in Table 2.

Table 2. Approximate solutions of $\omega_n(x) = 0, x \in \mathbb{R}$

degree n	x
1	0
2	−0.5, 0
3	−0.788675, −0.211325, 0
4	−0.908248, −0.5, −0.0917517, 0
5	−0.958684, −0.699019, −0.300981, −0.0413157, 0
6	−0.98085, −0.819557, −0.5, −0.180443 −0.0191503, 0
7	−0.990934, −0.890825, −0.651347, −0.348653, −0.109175, −0.00906575, 0
8	−0.995643, −0.93314, −0.758317, −0.5, −0.241683, −0.0668599, −0.00435709, 0

From Table 2 we can make the following conjecture.

Conjecture 2. Prove that $\omega_n(x) = 0$ has n distinct solutions.

Conjecture 3. If $n \equiv 0 \pmod 2$, then

$$\omega_n\left(-\frac{1}{2}\right) = 0.$$

Conjecture 4. For any positive integer n, one has

$$\omega_n(0) = 0.$$

The authors have no doubt that investigations along this line will lead to a new approach employing numerical method in the research field of the geometric polynomials $\omega_n(x)$ to appear in mathematics and physics. The reader may refer to [1, 12, 17] for the details.

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