J. Appl. Math. & Informatics Vol. **40**(2022), No. 3 - 4, pp. 795 - 805
https://doi.org/10.14317/jami.2022.795

CAYLEY SIGNED GRAPHS ASSOCIATED WITH ABELIAN GROUPS

PRANJALI, AMIT KUMAR* AND TANUJA YADAV

ABSTRACT. The aim of author's in this paper is to study the Cayley graph in the realm of signed graph. Moreover, we have characterized generating sets and finite abelian groups that corresponds to balanced Cayley signed graphs. The notion of Cayley signed graph has been demonstrated with the ample number of examples.

AMS Mathematics Subject Classification : 20D05, 05C75, 05C22.
Key words and phrases : Cayley Graph, finite group, generating set, signed graph.

1. Introduction

One principal step forward in the utility of graph-theoretic models in behavioral sciences the possibility of representing the qualitative nature of interpersonal relationships in a social group. For instance, in the 'acquaintance diagram' of the social group, which is essentially a graph (or, more generally, a graphical pattern), it cannot still be made out whether the two acquainted persons liked or disliked each other: Therefore, mere graphs cannot well model the 'structure of interpersonal relationships' in a social group. To represent the intrinsically dichotomous nature of most of the socio-psychological interpersonal relations, we have to assign weights $+1$ (or just the sign '$+$') or -1 (or just the sign '$-$') to each pair uv of 'acquaintances' in the social group according to whether the individuals u and v bear a qualitatively positive or negative 'attitudes' towards each other. Such considerations from the works of social psychologists (e.g. Heider [5]) led Harary [4] to the notion of a 'signed graph' and later on, Cartwright and Harary [2] to the notion of a 'signed digraph', which is defined formally as "A graph Γ equipped with a signature σ is called a *signed graph*, denoted by $\Sigma := (\Gamma, \sigma)$, where $\Gamma = (V, E)$ is an *underlying graph* and $\sigma : E \to \{+, -\}$ is the signature that labels each edge of Γ either by '$+$' or '$-$'. The edges which receive

Received December 13, 2021. Revised February 23, 2022. Accepted March 9, 2022.
*Corresponding author.

the '+'($-$) sign is called positive(negative) edge. A signed graph is an *all-positive* (*all-negative*) if all of its edges are positive (negative), further, it is said to be *homogeneous* if it is either an all-positive or an all-negative and *heterogeneous* otherwise. The *negative degree* $d^-(v)$ of a vertex v is the number of negative edges incident at v in Σ and the *positive degree* $d^+(v)$ is defined similarly."

One of the fundamental concept in the theory of signed graph is that of *balance*. Harary [4] introduced the fascinated concept of *balanced signed graphs* for the analysis of social networks, in which a positive edge stands for a positive relation and a negative edge is for a negative relation. A signed graph is *balanced* if every *cycle* has even numbers of negative edges. A *cycle* in a signed graph Σ is said to be *positive* if it contains an even number of negative edges. The following criteria for balance is well-known.

Lemma 1.1 (Zaslavsky [12]). *A signed graph in which every chordless cycle is positive is balanced.*

Due to numerous number of applications in various fields signed graphs are leading to vast variety of results and questions and number of papers with their applications have been published in the reputed international journals, for detailed bibliography of signed graphs the reader is referred to up-to-date creative survey article of Zaslavsky [11].

Throughout this article, all graphs are assumed to be *simple*, i.e., undirected graphs in which any two vertices are joined by at most one edge and without loops. For terminology and notation from group theory and graph theory not defined in this paper, we referred the reader to [6] and [3] respectively.

1.1. Preliminary Analysis. In this subsection, we briefly recall the notion of Cayley set and generating set and derived some observations needed in the sequel of this paper.

Let Γ be an abelian group. The group of integers modulo n, denoted by $\mathbb{Z}_n$ in which the sets $Z(\mathbb{Z}_n)$ and $U(\mathbb{Z}_n)$ are defined as; $Z(\mathbb{Z}_n) = \{x : gcd(x, n) \neq 1\}$ and $U(\mathbb{Z}_n) = \{y : gcd(y, n) = 1\}$. Also, $U(\mathbb{Z}_m \times \mathbb{Z}_n)$ is defined as, $U(\mathbb{Z}_m \times \mathbb{Z}_n) = \{(x, y) : gcd(x, m) = 1 \ \& \ gcd(y, n) = 1\}$.

Definition 1.2. A nonempty subset S of Γ is called *Cayley set* or *symmetric Cayley set* if $e \notin S$ and for every $a \in S$, $a^{-1} \in S$. If Cayley set generates group Γ, then S is called *generating set* or *symmetric generating set.*

Consequently, for a given group Γ of order n

$$1 \leq |S| \leq n - 1. \tag{1}$$

However, if S generates Γ, then

$$2 \leq |S| \leq n - 1. \tag{2}$$

The following example illustrate the above concepts:

Example 1.3. Let $\Gamma \cong \mathbb{Z}_4$. Then possible Cayley sets are $S_1 = \{2\}$, $S_2 = \{1, 3\}$, $S_3 = \{1, 2, 3\}$ and out of them S_2 and S_3 are both generating sets.

If $|S|$ is either 1 or $(n-1)$, then such S is called an extreme Cayley set and if $|S|$ is either 2 or $(n-1)$, then such S is called an extreme generating set. Notice that if $|\Gamma|$ is odd, then $|S|$ can be even. However, if $|\Gamma|$ is even, then $|S|$ may be even or odd. Now we shall put our attention to algebraic graph. The notion of Cayley graph was introduced by A. Cayley [1] in 1978 as follows: The *Cayley graph* of Γ, denote by $Cay(\Gamma, S)$ is a simple graph with the vertex set Γ, and two vertices x and y are adjacent if and only if there exists $s \in S$ such that $x = sy$. For more details on the Cayley graphs we refer the reader to [8, 10].

The objective of this paper is to study the interplay between properties of Cayley graph together with parameters of signed graph. The main motivation behind this work is wide number of applications of signed graphs in allied areas and the importance of Cayley graph. Towards full-filling the objective here we study Cayley graph in the realm of signed graph. These graphs belongs to the family of algebraic signed graph for more details see [7]. In the course of investigation we found that if S is an extreme Cayley set of Γ, then Cayley signed graph is always balanced. Further, we determine the Cayley set whose associated signed graph is balanced. The advantage of examining these classes of graphs helps us to determine and extend certain graphical properties of associated algebraic structure and vice-versa.

2. Cayley Signed Graphs

At this stage one might be tempted to ask what is the appropriate way to extend the notion of signed graph in the realm of Cayley graph? Towards answering this, we have introduced the notion of *Cayley signed graph.* The formal definition of new notion is as follows:

Definition 2.1. Let S be a Cayley set of a finite group Γ. The *Cayley signed graph*, denoted by $Cay_\Sigma(\Gamma, S) := (Cay(\Gamma, S), \sigma)$ is a signed graph whose underlying graph is $Cay(\Gamma, S)$ with vertex set Γ and Cayley set S, and for an edge $(x, y) \in E(Cay(\Gamma, S))$, the signature σ is defined as

$$\sigma(x,\, y) = \begin{cases} +, & \text{if } x \in S \text{ or } y \in S; \\ -, & \text{otherwise.} \end{cases}$$

To illustrate the concept, we have the following example:

Example 2.2. Let $\Gamma \cong \mathbb{Z}_5$. Then there are three Cayley sets of Γ which are generating set also, namely, $S_1 = \{1, 4\}$, $S_2 = \{2, 3\}$, and $S_3 = \{1, 2, 3, 4\}$. The Cayely signed graph $Cay_\Sigma(\mathbb{Z}_5, S_1)$, $Cay_\Sigma(\mathbb{Z}_5, S_2)$ and $Cay_\Sigma(\mathbb{Z}_5, S_3)$ with respect to generating sets S_1, S_2 and S_3, respectively have shown in Figure 1.

Observation 2.1. From Figure 1, we observe that the Cayely signed graph with respect to generating sets S_1 and S_2 are both not balanced, as in both $Cay_\Sigma(\mathbb{Z}_5, S_1)$ and $Cay_\Sigma(\mathbb{Z}_5, S_2)$ there is only one(odd) negative edge in the cycle. However $Cay_\Sigma(\mathbb{Z}_5, S_3)$ is an all-positive signed graph and it is balanced trivially.

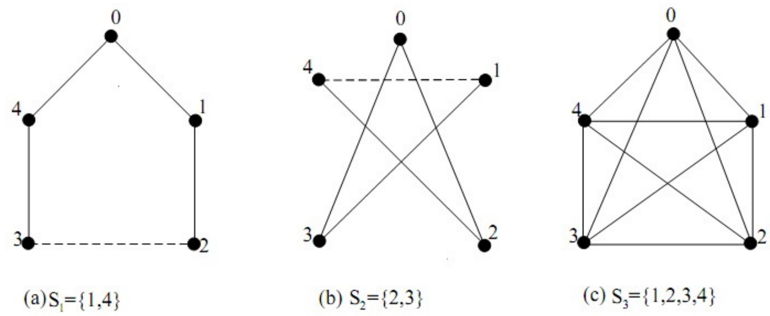

FIGURE 1. The Cayley signed graphs

Note that there do exist Cayley sets/generating sets with respect to which Cayley signed graph is not balanced. So it is noteworthy that before providing the main results, we need some preparatory observations for the advancement of the concept.

Observation 2.2. Consider $\Gamma \cong \mathbb{Z}_8$ and let $S_1, S_2 \subseteq \Gamma$, where $S_1 = \{2, 6\}$ and $S_2 = \{2, 4, 6\}$ be Cayley sets of Γ. Then $Cay_\Sigma(\Gamma, S_1)$ is balanced however $Cay_\Sigma(\Gamma, S_2)$ is not balanced. Further, if we choose $S_1 = \{3, 5\}$ and $S_2 = \{1, 3, 5, 7\}$, then both $Cay_\Sigma(\Gamma, S_1)$ and $Cay_\Sigma(\Gamma, S_2)$ are balanced.

Example 2.2 along with Observations 2.1, and Observation 2.2 prompts us to raise the following interesting problem:

Problem 2.1. Characterize the Cayley sets/Generating sets S with respect to which Cayley signed graph is balanced.

Towards attempting the Problem 2.1, several results have been established. First, we begin with a lemma which is advantageous to derive new results.

Lemma 2.3. *If a signed graph Σ is an all-positive, then it is balanced.*

Proof. If a signed graph Σ is an all-positive, then there is no negative edge. Therefore, each cycle in Σ consists of zero (even number) negative edges, and hence Σ is balanced trivially. $\square$

3. Balanced Cayley Signed Graphs

We shall move forward with some basic results about Cayley signed graphs. It appears more interesting that the balanced structure of Cayley signed graph depends upon the generating set 'S' as well as on group 'Γ'. Now with this consideration, we have two questions, What are those 'S' in $Cay_\Sigma(\Gamma, S)$ such that $Cay_\Sigma(\Gamma, S)$ is balanced? What are those group 'Γ' for which $Cay_\Sigma(\Gamma, S)$ is balanced? It is easy to notice that for any finite abelian group Γ and generating set S with $|S| = |\Gamma| - 1$, we will have an all-positive signed graph. Therefore, it is important to be noted here that the study of this particular case (i.e., $|S| = 1$ or $|S| = n - 1$) is scanty as clear from the existing literature and hence needs to be studied carefully.

From the foregoing analysis, we have the following result:

Theorem 3.1. *Let Γ be a finite abelian group and S be an extreme Cayley set. Then $Cay_\Sigma(\Gamma, S)$ is balanced.*

Proof. Let Γ be a finite abelian group and S be an extreme Cayley set. Since S is an extreme Cayley set, so either $|S| = 1$ or $|S| = n - 1$. If $|S| = 1$, then $Cay(\Gamma, S)$ is isomorphic to $\frac{n}{2}$-copies of K_2. Clearly, due to absence of cycle, $Cay_\Sigma(\Gamma, S)$ is trivially balanced. On the other hand if $|S| = n-1$, then all non-zero elements of Γ belongs to S. Therefore, in view of Definition 2.1, $Cay_\Sigma(\Gamma, S)$ is an all-positive, and hence balanced. $\square$

Now, we deeply pursuit of our basic objectives to characterize S. To do this, we shall impose some conditions on S and Γ.

Theorem 3.2. *Let Γ be a finite abelian group of order n, $(n > 3)$ and $S = \{a, a^{-1}\}$ be a Cayley set. Then $Cay_\Sigma(\Gamma, S)$ is balanced if and only if order of element a is even.*

Proof. Let Γ be a finite abelian group of order n, $(n > 3)$ and $S = \{a, a^{-1}\}$ be a Cayley set. Since $|S| = 2$, so $Cay(\Gamma, S)$ is a 2-regular graph. It means either $Cay(\Gamma, S)$ is isomorphic to a cycle or isomorphic to copies of cycles. Towards proving the necessity part, let us assume that $Cay_\Sigma(\Gamma, S)$ is balanced and our aim is to show that order of element a is even and this will be proved by contrapositive. Let us suppose that order of element a is odd. Now we shall tackle two cases depending upon the structure of $Cay(\Gamma, S)$.

Case-1: Let $O(a) = k = n$. Since the order of each element of a group divides the order of group, this gives $\frac{O(\Gamma)}{O(a)} = 1$. In this case $Cay(\Gamma, S)$ is isomorphic to a cycle C_k. Also note that an edge in $Cay_\Sigma(\Gamma, S)$ is negative if and only if none of its end vertices belongs to S. Therefore, its respective signed graph $Cay_\Sigma(\Gamma, S)$ is isomorphic to a cycle C_k with exactly four positive edges. Since k is odd, thus the negative edges in each cycle of $Cay_\Sigma(\Gamma, S)$ are also odd. Hence $Cay_\Sigma(\Gamma, S)$ is not balanced.

Case-2: Let $O(a) = k < n$. Since the order of each element of a group divides the order of group, this gives us $\frac{O(\Gamma)}{O(a)} = \frac{n}{k}$. In this case $Cay(\Gamma, S)$ is isomorphic to $\underbrace{C_k \cup C_k \cup \cdots \cup C_k}_{(n/k)-times}$ and its respective signed graph $Cay_\Sigma(\Gamma, S)$ is isomorphic to $\underbrace{C_k \cup C_k \cup \cdots \cup C_k}_{(n/k)-times}$ in which one component has exactly four positive edges out of k edges and remaining other components(cycles) are an all-negative. Since k is odd, therefore, each cycle consists of an odd number of negative edges in $Cay_\Sigma(\Gamma, S)$ this indicates that $Cay_\Sigma(\Gamma, S)$ is not balanced.

Thus we have contradiction in each of the possible cases, and hence the necessity follows by contraposition.

Conversely, let order of element a is k, which is even and our objective is to show that $Cay_{\Sigma}(\Gamma, S)$ is balanced. Towards proving this we shall deal with two cases listed below:

Case-1: Let $O(a) = k = n$. Now to tackle this case similar arguments as reported above in Case-1 of necessity can be given to show that $Cay_{\Sigma}(\Gamma, S)$ is isomorphic to a cycle C_k with exactly four positive edges. Since k is even, so the negative edges in each cycle of $Cay_{\Sigma}(\Gamma, S)$ are also even in number. Therefore, $Cay_{\Sigma}(\Gamma, S)$ is balanced.

Case-2: Let $O(a) = k < n$. Now to tackle this case similar arguments as reported above in Case-2 of necessity, we found that $Cay_{\Sigma}(\Gamma, S)$ is isomorphic to $\underbrace{C_k \cup C_k \cup \cdots \cup C_k}_{(n/k)-times}$ in which one component has exactly four positive edges out of k edges and remaining other components(cycles) are an all-negative. Since k is even, therefore, each cycle consists of an even number of negative edges in $Cay_{\Sigma}(\Gamma, S)$ and which ensures that $Cay_{\Sigma}(\Gamma, S)$ is balanced. □

Theorem 3.3. *Let Γ be an abelian group of order n and let S be generating set with $|S| = n - 2$. Then $Cay_{\Sigma}(\Gamma, S)$ is balanced.*

Proof. Let S be generating set of Γ with $|S| = n - 2$. In view of definition of generating set it can be found that Γ must possesses non-trivial self inverse element. Also note that $|S| = n - 2$ ensure that only self inverse element except identity does not belong to S and this guarantee that in $Cay_{\Sigma}(\Gamma, S)$, there is no negative edge. Thus $Cay_{\Sigma}(\Gamma, S)$ is an all-positive, and hence balanced. □

From the above results, it can easily be observed that only the condition of cardinality can not suffices our main goal as there are several other generating sets deserving serious attention to develop the concept. At this place, it becomes important to explore specific classes of generating set that provide the balanced structure of Cayley signed graph.

Theorem 3.4. *Let Γ be a finite cyclic group and p be a prime number. Then $Cay_{\Sigma}(\Gamma, S)$ is balanced if one of the following holds:*

(a) $S \subseteq U(\Gamma)$, *when* $|\Gamma|$ *is even,*
(b) $S = S' \cup \{p\}$, *when* $|\Gamma| = 2p$, *where* $S' \subseteq U(\Gamma)$,
(c) $S = \{\frac{|\Gamma|}{4}, \frac{3|\Gamma|}{4}\}$, *when* $|\Gamma|$ *is multiple of* 4,
(d) $S = U(\Gamma)$, *when* $|\Gamma| = p^k$, $k \geq 1$,
(e) $S = \{a : \ a \neq a^{-1}, \ \forall \ a \in \Gamma\}$, *when* $|\Gamma|$ *is even.*

Proof. (a) Let Γ be a finite cyclic group of even order (say n) and $S \subseteq U(\Gamma)$. To prove this case we shall consider the following possibilities:
If $n = 2$, then $Cay(\Gamma, S)$ is 1-regular. Therefore, due to absence of a cycle $Cay_{\Sigma}(\Gamma, S)$ is trivially balanced.

On the other hand, if $n > 2$, then there are two possibilities, either S contains all odd positive integers upto n or S contains some odd positive integers upto n. If S contains all odd positive integers, then there is no negative edge in $Cay_\Sigma(\Gamma, S)$. If S contains some odd positive integers then $(\Gamma \setminus S)$ contains even as well as odd positive integers. The one end of negative edges incident only at odd positive integers belong to $(\Gamma \setminus S)$ and at multiples of these odd positive integers $(\Gamma \setminus S)$. These odd vertices incident with even number of negative edges and also, every cycle in $Cay_\Sigma(\Gamma, S)$ is of even length because $S \subseteq U(\Gamma)$ and $|\Gamma|$ is even. Thus every cycle contains odd and even positive integers alternatively with even number of negative edges. Therefore, $Cay_\Sigma(\Gamma, S)$ is balanced.

(b) Consider $|\Gamma| = 2p$ and $S = U(\Gamma) \cup \{p\}$, where p is prime number. Since the elements in $(\Gamma \setminus S)$ are multiple of 2, then for arbitrary $x, y \in (\Gamma \setminus S)$, the difference $x - y$ is multiple of 2 and this indicates the absence of negative edge in $Cay_\Sigma(\Gamma, S)$. Therefore, $Cay_\Sigma(\Gamma, S)$ is an-all positive signed graph, and hence balanced.

Next, if $S = S' \cup \{p\}$, where $S' \subset U(\Gamma)$, then $U(\Gamma)$ have only odd positive integers. Although, $(\Gamma \setminus S)$ consists of even as well as odd positive integers, which are member of $U(\Gamma)$. In this case $Cay(\Gamma, S)$ is isomorphic to $|S|$-regular bipartite graph. In order to see the respective signed graph let $x, y \in (\Gamma \setminus S)$, then x and y are adjacent with negative edge if and only if x is odd and y is even (or vice-versa). Choose any arbitrary cycle in $Cay_\Sigma(\Gamma, S)$, then its adjacent vertices are labeled by odd positive integer, even positive integer, odd positive integer, alternatively. Now using the arguments analogues to those used in the previous case it can be shown that there does not exist a cycle consisting of odd number of negative edges, which would ensure that $Cay_\Sigma(\Gamma, S)$ is balanced.

(c) If $S = \{\frac{n}{4}, \frac{3n}{4}\}$, where n is multiple of 4 (say $n = 4k$), then $Cay(\Gamma, S)$ is isomorphic to $\underbrace{C_4 \cup C_4 \cup \cdots \cup C_4}_{k-times}$ and its respective signed graph $Cay_\Sigma(\Gamma, S)$ is isomorphic to $\underbrace{C_4 \cup C_4 \cup \cdots \cup C_4}_{k-times}$ in which exactly one component formed by the vertices, namely $0, \frac{n}{4}, \frac{3n}{4}$ and $\frac{n}{2}$ is an all-positive, and remaining $(k-1)$ components are an all-negative. Therefore, each cycle consists of an even number of negative edges. Hence $Cay_\Sigma(\Gamma, S)$ is balanced.

(d) Let $|\Gamma| = p^k$, where p is prime and k is a positive integer and $S = U(\Gamma)$. Note that an edge in $Cay_\Sigma(\Gamma, S)$ is negative if and only if none of end vertices belongs to S. Consider two arbitrary elements x and y from $\Gamma \setminus S$, then clearly the difference $x - y$ is a multiple of p, which does not belongs to S. This indicates that there does not exist negative edges in $Cay_\Sigma(\Gamma, S)$. Therefore, $Cay_\Sigma(\Gamma, S)$ is an all-positive signed graph and hence it is balanced.

(e) Let $|\Gamma|$ be even and $S = \{a : \ a \neq a^{-1}, \ \forall \, a \in \Gamma\}$. Since Γ is a cyclic group of even order, so Γ necessarily has one non-zero self inverse element. This implies that $|S| = |\Gamma| - 2$ and in view of Theorem 3.3, $Cay_\Sigma(\Gamma, S)$ is balanced. $\square$

Theorem 3.5. *Let Γ be a finite abelian group with $|\Gamma| < 9$ and $S = U(\Gamma)$. Then $Cay_\Sigma(\Gamma, S)$ is balanced.*

Proof. Let Γ be a finite abelian group with $|\Gamma| < 9$ and $S = U(\Gamma)$. If the order of Γ belongs to $\{2, 3, 5, 7\}$, then Γ is cyclic and hence by Theorem 3.4 (d), $Cay_\Sigma(\Gamma, S)$ is balanced. Now we shall consider remaining exhaustive cases for the cardinality of Γ, i.e, $|\Gamma| = 4$ or $|\Gamma| = 6$ or $|\Gamma| = 8$. Let $|\Gamma| = 4$. Then upto isomorphism the possible abelian groups are $\mathbb{Z}_4$, and $\mathbb{Z}_2 \times \mathbb{Z}_2$. Then for $\mathbb{Z}_4$, $Cay_\Sigma(\Gamma, S)$ is balanced due to Theorem 3.4 (d) and for $\mathbb{Z}_2 \times \mathbb{Z}_2$ we have $|S| = 1$, so in view of Theorem 3.1, $Cay_\Sigma(\Gamma, S)$ is balanced. Next, let $|\Gamma| = 6$. Then, $Cay_\Sigma(\Gamma, S)$ is isomorphic to a cycle graph C_6 having exactly two negative edges forming a section. Therefore by definition $Cay_\Sigma(\Gamma, S)$ is balanced. Finally, let $|\Gamma| = 8$. Then upto isomorphism, precisely the abelian groups are $\mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. For $\Gamma \cong \mathbb{Z}_8$, $Cay_\Sigma(\Gamma, S)$ is balanced due to Theorem 3.4 (d). For group $\mathbb{Z}_2 \times \mathbb{Z}_4$, $Cay_\Sigma(\Gamma, S)$ is isomorphic to $C_4 \cup C_4$, in which one component is an all-positive and other is an all-negative, therefore, each cycle consists of even number of negative edges in $Cay_\Sigma(\Gamma, S)$, and hence balanced. For the group $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $Cay(\Gamma, S) \cong K_2 \cup K_2 \cup K_2 \cup K_2$, which indicates that $Cay_\Sigma(\Gamma, S)$ is trivially balanced due to absence of a cycle. Thus in each of the above mentioned cases, it follows that for each Γ ($|\Gamma| < 9$) $Cay_\Sigma(\Gamma, S)$ is balanced. $\square$

In view of the Theorem 3.5 an obvious but important point that needs to be noted here, is the following remark.

Remark 3.1. From the foregoing analysis for generating set $S = U(\Gamma)$, one can easily verify that the smallest order of an abelian group Γ for which $Cay_\Sigma(\Gamma, S)$ is not balanced is 9, and precisely the group is $\mathbb{Z}_3 \times \mathbb{Z}_3$.

In fact, It would thus be interesting to see the impact of Cayley set $S = Z^0(\Gamma)$ on the balanced structure of $Cay_\Sigma(\Gamma, S)$, which is stated in the following remark:

Remark 3.2. The smallest order of an abelian group Γ with Cayley set $S = Z^0(\Gamma)$ for which $Cay_\Sigma(\Gamma, S)$ is not balanced is 6 and precisely the group is $\mathbb{Z}_6$.

3.1. Balanced Cayley Signed Graphs with Specific Generating Sets. In this section, we shall choose some specific generating sets to establish necessary and sufficient conditions for balanced Cayley signed graph.

Theorem 3.6. *Let $\Gamma \cong \mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \times \cdots \times \mathbb{Z}_{p_t^{k_t}}$ ($k_i \geq 1$), where p_t's are prime and $S = U(\Gamma)$ be generating set. Then $Cay_\Sigma(\Gamma, S)$ is balanced if and only if either*

(i) *$t = 1$ or*
(ii) *at least one of $\mathbb{Z}_{p_i^{k_i}}$ ($1 \leq i \leq t$) has $\mathbb{Z}_2$ as a quotient.*

Proof. Necessity: Let $\Gamma \cong \mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \times \cdots \times \mathbb{Z}_{p_t^{k_t}}$ ($k_i \geq 1$) and $S = U(\Gamma)$ be generating set. Suppose $Cay_\Sigma(\Gamma, S)$ is balanced and our aim is to show

that any one of the conditions hold. We shall prove this by contrapositive, to do this, let us suppose $Cay_{\Sigma}(\Gamma, S)$ is balanced and each of listed condition (i) and (ii) is false. Assume that $t > 1$ and none of $\mathbb{Z}_{p_i^{k_i}}$ has $\mathbb{Z}_2$ as a quotient and this indicates that $p_i > 2\ \forall\, i$. Let $x, y, z \in \Gamma$, where $x = (0, 0, \ldots, 0, 2)$, $y = (2, 2, \ldots, 2, 0)$ and $z = (1, 1, \ldots, 1)$. Then x, y and z are mutually adjacent in $Cay(\Gamma, S)$ whence in their respective signed graph $Cay_{\Sigma}(\Gamma, S)$, x is adjacent to y with negative edge and remaining pair of vertices are adjacent with positive edge as $z \in U(\Gamma)$. Thus one can see the presence of a triangle with exactly one negative edge in $Cay_{\Sigma}(\Gamma, S)$. This implies that $Cay_{\Sigma}(\Gamma, S)$ is not balanced, a contradiction. Hence by contrapositive necessity holds.
Sufficiency: Let us suppose one of conditions is true and our aim is to show that $Cay_{\Sigma}(\Gamma, S)$ is balanced. First, if condition (i) is true, then by Theorem 3.4, $Cay_{\Sigma}(\Gamma, S)$ is balanced. Next, let at least one of $\mathbb{Z}_{p_i^{k_i}}$ has $\mathbb{Z}_2$ as a quotient for some i in Γ. Without loss of generality assume that $\mathbb{Z}_{p_j^{k_j}}$ has $\mathbb{Z}_2$ as a quotient. If we choose two elements from S, then clearly they can not be adjacent in $Cay(\Gamma, S)$ because $U(\Gamma)$ contains the elements of the form $(u_1, u_2, \ldots, u_t)$, where u_j is odd. In this case $Cay(\Gamma, S)$ is $|U(\Gamma)|$-regular bipartite graph in which one partite set consists of $U(\Gamma)$ along with all those elements of $Z^0(\Gamma)$ in which j^{th}-coordinate belongs to $U(\mathbb{Z}_{p_j^{k_j}})$ and the remaining elements of Γ will be in other partite set. In order to see the respective signed graph, let $x, y \in \Gamma \setminus S$, where $x = (x_1, x_2, \ldots, x_t)$ and $y = (y_1, y_2, \ldots, y_t)$. Then x is adjacent to y with negative edge if and only if $x - y \in S$. Since $Cay(\Gamma, S)$ is a bipartite graph, so x and y must belongs to different partite sets with $x_i \in U(\mathbb{Z}_{p_i^{k_i}})$ and $y_i \in Z(\mathbb{Z}_{p_i^{k_i}})$ (or vice-versa). Since this holds for all i $(1 \le i \le t)$ and each cycle in $Cay_{\Sigma}(\Gamma, S)$ is of even length. Therefore, each cycle in $Cay_{\Sigma}(\Gamma, S)$ consists of even number of negative edges in $Cay_{\Sigma}(\Gamma, S)$ and hence balanced. □

Theorem 3.6 suggests the validity of the following remark:

Remark 3.3. The above derived result is for arbitrary finite abelian group Γ and generating set $S = U(\Gamma)$, thus the result established in [9, Theorem 4] becomes a corollary to Theorem 3.6.

Theorem 3.7. *Let $\Gamma \cong \mathbb{Z}_{{p_1}^{k_1}} \times \mathbb{Z}_{{p_2}^{k_2}} \times \cdots \times \mathbb{Z}_{{p_t}^{k_t}}$ be a finite abelian group, $p_i's$ are prime, $k_i's$, i and t are positive integers. Assume that $S = Z^0(\Gamma)$ be a Cayley set of Γ. If at least one $\mathbb{Z}_{{p_i}^{k_i}}$ is isomorphic to $\mathbb{Z}_2$ and $|U(\Gamma)| \ge 3$, then $Cay_{\Sigma}(\Gamma, S)$ is not balanced.*

Proof. Let $\Gamma \cong \mathbb{Z}_{{p_1}^{k_1}} \times \mathbb{Z}_{{p_2}^{k_2}} \times \cdots \times \mathbb{Z}_{{p_t}^{k_t}}$ be a finite abelian group and $S = Z^0(\Gamma)$ be a Cayley set of Γ. In order to prove the desire result it suffices to show the existence of a negative cycle in $Cay_{\Sigma}(\Gamma, S)$. If atleast one of $\mathbb{Z}_{{p_i}^{k_i}}$ is isomorphic to $\mathbb{Z}_2$, then we can choose $u = (1, x_2, \ldots, x_t)$, $v = (1, y_2, \ldots, y_t)$ and $w = (1, z_2, \ldots, z_t)$ three distinct elements from $U(\Gamma)$ as $|U(\Gamma)| \ge 3$, where $x_i, y_i, z_i \in U(\mathbb{Z}_{{p_t}^{k_t}})$ $(2 \le i \le t)$. Note that in $Cay_{\Sigma}(\Gamma, S)$ an edge is negative if and only if none of its end vertices belongs to S. Since the vertices u, v and w are mutually adjacent to each other in $Cay_\Sigma(\Gamma, S)$ through negative edge, so there exist an all-negative triangle, which would ensures that $Cay_\Sigma(\Gamma, S)$ is not balanced. □

Theorem 3.8. *Let be a finite abelian group. Assume that $S = Z^0(\Gamma)$ be a Cayley set of Γ. Then Cayley signed graph $Cay_\Sigma(\Gamma, S)$ is balanced if and only if either Γ is isomorphic to $\mathbb{Z}_4$ or ${\mathbb{Z}_2}^t$.*

Proof. Necessity: Let $\Gamma \cong \mathbb{Z}_{{p_1}^{k_1}} \times \mathbb{Z}_{{p_2}^{k_2}} \times \cdots \times \mathbb{Z}_{{p_t}^{k_t}}$ be a finite abelian group, where $p_i's$ are prime, $k_i's$, i and t are positive integers and $S = Z^0(\Gamma)$ be a Cayley set of Γ. Suppose if possible that $Cay_\Sigma(\Gamma, S)$ is balanced and Γ is neither isomorphic to $\mathbb{Z}_4$ nor ${\mathbb{Z}_2}^t$ $(t > 1)$. As $U(\Gamma)$ can have at least two elements, so consider two distinct elements u_1, and u_2 from the set $\Gamma \setminus S$ and $z \in S$, where $u_1 = (1, 1, \ldots, 1)$, $u_2 = (1, 1, \ldots, a)$ and $z = (1, 0, \ldots, 0)$. The vertices u_1, u_2 and z are mutually adjacent in $Cay(\Gamma, S)$ and in their respective signed graph $Cay_\Sigma(\Gamma, S)$, u_1 is adjacent to u_2 with negative edge and other two pair of vertices are adjacent with positive edge. Therefore, there exist a triangle, viz., $u_1 - u_2 - z - u_1$ in $Cay_\Sigma(\Gamma, S)$ with exactly one negative edge. This ensure the existence of a cycle with an odd numbers of negative edge. Thus $Cay_\Sigma(\Gamma, S)$ is not balanced, a contradiction, hence by contrapositive necessity hold.
Sufficiency: Let $\Gamma \cong \mathbb{Z}_{{p_1}^{k_1}} \times \mathbb{Z}_{{p_2}^{k_2}} \times \cdots \times \mathbb{Z}_{{p_t}^{k_t}}$ be a finite abelian group, $p_i's$ are prime numbers, $k_i's$, i and t are positive integers. Let $S = Z^0(\Gamma)$ be a Cayley set of Γ. Here our aim is to show that $Cay_\Sigma(\Gamma, S)$ is balanced in each of the above listed group. We will prove each case separately as follows:
(i) If $\Gamma \cong \mathbb{Z}_4$, then $|S| = 1$. Therefore, in view of Theorem 3.1, $Cay_\Sigma(\Gamma, S)$ is balanced.
(ii) Next, if $\Gamma \cong {\mathbb{Z}_2}^t$, then S contains all non-zero elements except the one element, namely, $\underbrace{(1, 1, 1, \ldots, 1)}_{t-times}$. Since there is only one element which does not belong to S, so there is no negative edge in $Cay_\Sigma(\Gamma, S)$, and hence it is an all-positive signed graph and therefore, $Cay_\Sigma(\Gamma, S)$ is trivially balanced. □

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Dr. Pranjali has completed her doctorate degree from University of Delhi, India in the field of Algebraic Graph Theory. She is serving the field of academics over last 11 years and is currently working in the department of Mathematics, University of Rajasthan, Jaipur, India. Dr. Pranjali has authored of more than 30 research papers. Her area of interest includes Algebra and Graph theory. Dr. Pranjali is Editorial Board member of National and International Journals. She is Life Member several Mathematical Society of India.

Department of Mathematics, University of Rajasthan, JLN Marg, Jaipur-302004, India.
`e-mail: pranjali48@gmail.com`

Dr. Amit Kumar has received his doctorate degree from National Institute of Technology, Hamirpur (HP), India in the field of Functional Analysis. He has qualified several examinations including GATE, NET and NBHM. He is having teaching and research experience of more than 12 years and is presently serving in the department of Mathematics and Statistics, Banasthali Vidyapith, Banasthali, Rajasthan, India. Dr. Kumar has published more than 28 research articles including one patent. He is also referee of several National and International journals.

Department of Mathematics and Statistics, Banasthali Vidyapith, Banasthali-304022, India.
`e-mail: amitsu48@gmail.com`

Tanuja Yadav received her M.Sc. degree from the Department of Mathematics and Statistics, Banasthali Vidyapith, Banasthali, Rajasthan. She is pursuing her doctorate degree from Banasthali Vidyapith. Her research interests includes Algebra and Graph theory.

Department of Mathematics and Statistics, Banasthali Vidyapith, Banasthali-304022, India.
`e-mail: yadav.tanuja17@gmail.com`