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ON LACUNARY STATISTICAL ϕ-CONVERGENCE FOR TRIPLE SEQUENCES OF SETS VIA IDEALS

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Abstract. In the present paper, we introduce some new notions of Wijsman $\mathcal{I}$-statistical convergence with the use of Orlicz function, lacunary sequence and triple sequences of sets, and obtain some analogous results from the new definitions point of views.

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1. Introduction

The concept of convergence of sequences of points has been extended by several authors to convergence of sequences of sets. The one of these extensions considered in this paper is the concept of Wijsman convergence in [42, 43]. In [22], Nuray and Rhoades presented definitions for statistical convergence of sequences of sets. Ulusu and Nuray [39, 40] studied the notion of Wijsman lacunary statistical convergence of sequence of sets. Nuray et al. [23] studied the concepts of Wijsman Cesàro summability and Wijsman lacunary convergence of double sequences of sets and investigated the relationship between them. Various applications of this concept can be found in [4, 14, 24, 27, 28, 36, 37, 38].

Fast [7] presented a generalization of the usual concept of sequential limit which is called statistical convergence. Šalát [26] gave some basic properties of statistical convergence. Various applications of this concept can be found in [9, 11, 17, 30, 31, 33]. The idea is based on the notion of natural density of subsets of $\mathbb{N}$, the set of all positive integers which is defined as follows: The natural density $\delta(A)$ of a subset A of $\mathbb{N}$ is defined by $\delta(A) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in A\}|$. Generalizing the concept of statistical convergence, Kostyrko et al. introduced the idea of $\mathcal{I}$-convergence in [15]. More investigations in this direction and more

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applications of ideals can be found in [5, 10, 12, 13, 21, 18, 19, 20, 34]. In another direction, a new type of convergence, called $\mathcal{I}$-statistical convergence, was introduced in [3]. Recently, various types of $\mathcal{I}$-statistical convergence for sequences have been studied by many authors (see for example [29, 41]).

In this paper, we study Wijsman $\mathcal{I}$-lacunary statistically ϕ-convergent, Wijsman $\mathcal{I}$-lacunary statistically ϕ-convergent and Wijsman strongly $\mathcal{I}$-lacunary ϕ-convergent concepts for triple sequence of sets and discuss the relationships between these new notions. We shall use lacunary triple sequence and Orlicz function ϕ to introduce these concepts. In addition to these definitions, natural inclusion theorems shall also be presented.

2. Definitions and Notations

First we recall some of the basic concepts which will be used in this paper. By $\mathbb{N}$ and $\mathbb{R}$, we mean the set of all natural and real numbers, respectively.

We say that a number sequence $x = (x_k)_{k\in\mathbb{N}}$ statistically converges to a point L if for each $\varepsilon > 0$ we have $\delta(K(\varepsilon)) = 0$, where $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ and in such situation we will write $L = st\text{-}\lim x_k$.

On the other hand in [2] a different direction was given to the study of statistical convergence where the notion of statistical convergence of order α, $0 < \alpha \leq 1$ was introduced by replacing n by n^α in the denominator in the definition of statistical convergence.

In several literary works, statistical convergence of any real sequence is identified relatively to absolute value. While we have known that the absolute value of real numbers is special of an Orlicz function [25], that is, a function $\phi : \mathbb{R} \to \mathbb{R}$ in such a way that it is even, non-decreasing on $\mathbb{R}^+$, continuous on $\mathbb{R}$, and satisfying

$$\phi(x) = 0 \text{ if and only if } x = 0 \text{ and } \phi(x) \to \infty \text{ as } x \to \infty.$$

Further, an Orlicz function $\phi : \mathbb{R} \to \mathbb{R}$ is said to satisfy the Δ_2 condition, if there exists an positive real number M such that $\phi(2x) \leq M.\phi(x)$ for every $x \in \mathbb{R}^+$.

Definition 2.1. ([30]) Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A sequence $x = (x_n)$ is said to be statistically ϕ-convergent to L if for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leq n : \phi(x_k - L) \geq \varepsilon\}| = 0.$$

The notion of statistical convergence was further generalized in the paper [15, 16] using the notion of an ideal of subsets of the set $\mathbb{N}$. We say that a non-empty family of sets $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is an ideal on $\mathbb{N}$ if $\mathcal{I}$ is hereditary (i.e. $B \subset A \in \mathcal{I} \Rightarrow B \in \mathcal{I}$) and additive (i.e. $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$). An ideal $\mathcal{I}$ on $\mathbb{N}$ for which $\mathcal{I} \neq \mathcal{P}(\mathbb{N})$ is called a non-trivial ideal. A non-trivial ideal $\mathcal{I}$ is said to be admissible if $\mathcal{I}$ contains every finite subset of $\mathbb{N}$. If not otherwise stated in the sequel $\mathcal{I}$ will denote an admissible ideal. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be any ideal. A class $\mathcal{F}(\mathcal{I}) = \{M \subseteq \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N}\backslash A\}$ called the filter associated with the ideal $\mathcal{I}$, is a filter on $\mathbb{N}$.

Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$ and $x = (x_k)$ be a real sequence. We say that the sequence x is $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in \mathcal{I}$. Take for $\mathcal{I}$ the class $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is a non-trivial admissible ideal and $\mathcal{I}_f$-convergence coincides with the usual convergence. For more information about $\mathcal{I}$-convergent, see the references in [18, 34].

We also recall that the concept of $\mathcal{I}$-statistically convergent is studied in [29]. A sequence (x_k) is said to be $\mathcal{I}$-statistically convergent to L if for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case, L is called $\mathcal{I}$-statistical limit of the sequence (x_k) and we write $\mathcal{I}$-st-$\lim_{k\to\infty} x_k = L$.

We now recall the following basic concepts from [27, 28, 32, 33, 35, 37, 39, 40] which will be needed throughout the paper.

A function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$) is called a real (or complex) triple sequence. A triple sequence (x_{jkl}) is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $|x_{jkl} - L| < \varepsilon$ whenever $j, k, l \geq n_0$.

Definition 2.2. A subset K of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is said to have natural density $\delta_3(K)$ if

$$\delta_3(K) = P - \lim_{n,k,l\to\infty} \frac{|K_{nkl}|}{nkl}$$

exists, where the vertical bars denote the number of (n, k, l) in K such that $p \leq n$, $q \leq k$, $r \leq l$. Then, a real triple sequence $x = (x_{nkl})$ is said to be statistically convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,

$$\delta_3(\{(n, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{nkl} - L| \geq \varepsilon\}) = 0.$$

Throughout the paper we consider the ideals of $\mathcal{P}(\mathbb{N})$ by $\mathcal{I}$; the ideals of $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ by $\mathcal{I}_2$ and the ideals of $\mathcal{P}(\mathbb{N} \times \mathbb{N} \times \mathbb{N})$ by $\mathcal{I}_3$.

Definition 2.3. A real triple sequence (x_{nkl}) is said to be $\mathcal{I}_3$-convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,

$$\{(n, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{nkl} - L| \geq \varepsilon\} \in \mathcal{I}_3.$$

In this case, one writes $\mathcal{I}_3$-$\lim x_{nkl} = L$.

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance $d(x, A)$ from x to A is defined by

$$d(x, A) = \inf_{a\in A} \rho(x, a).$$

Definition 2.4. ([1]) Let (X, ρ) be a metric space. For any non-empty closed subsets A; $A_k \subseteq X$; we say that the sequence (A_k) is Wijsman convergent to A if

$$\lim_{k\to\infty} d(x, A_k) = d(x, A)$$

for each $x \in X$.

Definition 2.5. Let (X, ρ) be a metric space and $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ be an admissible ideal. For any non-empty closed subsets A, $A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}$-statistical convergent to A or $S(\mathcal{I}_W)$-convergent to A if for each $\varepsilon > 0, \delta > 0$ and for each $x \in X$,

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}\right| \geq \delta\right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \to A\left(S\left(\mathcal{I}_W\right)\right)$.

By a lacunary $\theta = (k_r); r = 0, 1, 2, ...,$ where $k_0 = 0$, we shall mean an increasing sequence of nonnegative integers with $k_r - k_{r-1}$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.

Definition 2.6. Let (X, ρ) be a metric space, θ be a lacunary sequence and $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ be an admissible ideal. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}$-lacunary statistical convergent to A or $S_\theta(\mathcal{I}_W)$-convergent to A if for each $\varepsilon > 0, \delta > 0$ and for each $x \in X$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \left|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}\right| \geq \delta\right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \to A\left(S_\theta\left(\mathcal{I}_W\right)\right)$.

Definition 2.7. Let (X, ρ) be a metric space and $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ be an admissible ideal. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}$-statistical convergent of order α to A or $S(\mathcal{I}_W)^\alpha$-convergent to A, where $0 < \alpha \leq 1$, if for each $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha} \left|\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}\right| \geq \delta\right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \to A\left(S\left(\mathcal{I}_W\right)^\alpha\right)$.

3. Main results

Following the above definitions and results, we aim in this section to introduce some new notions of Wijsman $\mathcal{I}$-statistical convergence with the use of Orlicz function, lacunary sequence and triple sequences of sets and obtain some analogous results from the new definitions point of views.

Throughout the paper, we let (X, ρ) be a metric space and A, A_{jkl} be any non-empty closed subsets of X.

Definition 3.1. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. We say that the triple sequence $(A_{jkl})_{j,k,l \in \mathbb{N}}$ is Wijsman $\mathcal{I}_3$-statistically ϕ-convergent to A, if for each $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{(r, s, t) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{rst} \left|\{j \leq r, k \leq s, l \leq t :\right.\right.$$

$$\phi\left(d(x, A_{jkl}) - d\left(x, A\right)\right) \geq \varepsilon\}| \geq \delta\}$$

belongs to $\mathcal{I}_3$. In this case we write $\mathcal{I}_3$-st-$\lim_{W(S)} A_{jkl} = A$.

The set of Wijsman $\mathcal{I}_3$-statistically ϕ-convergent triple sequences will be denoted by

$$W_3S\left(\mathcal{I}_3 - \phi\right) := \left\{\{A_{jkl}\} : \mathcal{I}_3\text{-}st\text{-}\lim_{W(S)} A_{jkl} = A\right\}.$$

Furthermore, a new type of sequence called triple lacunary sequence was introduced in Esi and Savaş [6]. The triple sequence $\theta_3 = \theta_{r,s,t} = \{(j_r, k_s, l_t)\}$ is called triple lacunary sequence if there exist three increasing sequences of integers such that

$$j_0 = 0,\ h_r = j_r - j_{r-1} \to \infty \text{ as } r \to \infty,$$

$$k_0 = 0,\ h_s = k_s - k_{s-1} \to \infty \text{ as } s \to \infty,$$

and

$$l_0 = 0,\ h_t = l_t - l_{t-1} \to \infty \text{ as } t \to \infty.$$

Let $k_{r,s,t} = j_r k_s l_t$, $h_{r,s,t} = h_r h_s h_t$ and $\theta_{r,s,t}$ is determined by

$$I_{r,s,t} = \left\{(j,k,l) : j_{r-1} < j \leq j_r, k_{s-1} < k \leq k_s \text{ and } l_{t-1} < l \leq l_t\right\},$$

$$q_r = \frac{j_r}{j_{r-1}},\ q_s = \frac{k_s}{k_{s-1}},\ q_t = \frac{l_t}{l_{t-1}} \text{ and } q_{r,s,t} = q_r q_s q_t.$$

Let $D \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. The number

$$\delta_3^\theta(D) = \lim_{r,s,t} \frac{1}{h_{r,s,t}} \left|\{(j,k,l) \in I_{r,s,t} : (j,k,l) \in D\}\right|$$

is said to be as the $\theta_{r,s,t}$-density of D, provided the limit exists.

Definition 3.2. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $\theta_3 = \theta_{r,s,t} = \{(j_r, k_s, l_t)\}$ be a lacunary triple sequence. We say that the triple sequence $\{A_{jkl}\}$ is Wijsman $\mathcal{I}_3$-lacunary statistical ϕ-convergent to A, if for each $\varepsilon > 0, \delta > 0$ and for each $x \in X$,

$$\left\{(r,s,t) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s,t}} \left|\{(j,k,l) \in I_{r,s,t} : \phi\left(d(x, A_{jkl}) - d(x, A)\right) \geq \varepsilon\}\right| \geq \delta\right\}$$

belongs to $\mathcal{I}_3$. In this case, we write we write $\mathcal{I}_3$-st-$\lim_{W_\theta(S_\theta)} A_{jkl} = A$.

The set of Wijsman $\mathcal{I}_3$-lacunary statistically ϕ-convergent triple sequences will be denoted by

$$W_{\theta_3}S_\theta\left(\mathcal{I}_3 - \phi\right) := \left\{\{A_{jkl}\} : \mathcal{I}_3\text{-}st\text{-}\lim_{W_\theta(S_\theta)} A_{jkl} = A\right\}.$$

Definition 3.3. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. We say that the triple sequence $\{A_{jkl}\}$ is Wijsman $\mathcal{I}_3$-statistical ϕ-convergent of order α to A, where $0 < \alpha \leq 1$, if for each $\varepsilon > 0, \delta > 0$ and for each $x \in X$,

$$\left\{(r,s,t) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{r^\alpha s^\alpha t^\alpha} \left|\{j \leq r, k \leq s, l \leq t :\right.\right.$$

$$\left.\left.\phi\left(d(x, A_{jkl}) - d\left(x, A\right)\right) \geq \varepsilon\}\right| \geq \delta\right\}$$

belongs to $\mathcal{I}_3$. In this case we write $\mathcal{I}_3$-st-$\lim_{W^\alpha(S)} A_{jkl} = A$.

The set of Wijsman $\mathcal{I}_3$-statistically ϕ-convergent triple sequences of order α will be denoted by

$$W_3^{\alpha} S\left(\mathcal{I}_3-\phi\right):=\left\{\left\{A_{jkl}\right\}: \mathcal{I}_3\text{-}st\text{-}\lim_{W^{\alpha}(S)} A_{jkl}=A\right\}.$$

Definition 3.4. Let $\phi: \mathbb{R} \rightarrow \mathbb{R}$ be an Orlicz function and $\theta_3=\theta_{r,s,t}=\left\{\left(j_r, k_s, l_t\right)\right\}$ be a lacunary triple sequence. We say that the triple sequence $\left\{A_{jkl}\right\}$ is Wijsman $\mathcal{I}_3$-lacunary statistically ϕ-convergent of order α to A, where $0<\alpha \leq 1$, if for each $\varepsilon>0, \delta>0$ and for each $x \in X$,

$$\left\{(r, s, t) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \frac{1}{h_{r,s,t}^{\alpha}}\left|\left\{(j, k, l) \in I_{r,s,t}: \phi\left(d(x, A_{jkl})-d(x, A)\right) \geq \varepsilon\right\}\right| \geq \delta\right\}$$

belongs to $\mathcal{I}_3$. In this case, we write we write $\mathcal{I}_3$-st-$\lim_{W_{\theta}^{\alpha}(S_{\theta})} A_{jkl}=A$.

The set of Wijsman $\mathcal{I}_3$-lacunary statistically ϕ-convergent triple sequences of order α will be denoted by

$$W_{\theta_3}^{\alpha} S_{\theta}\left(\mathcal{I}_3-\phi\right):=\left\{\left\{A_{jkl}\right\}: \mathcal{I}_3\text{-}st\text{-}\lim_{W_{\theta}^{\alpha}(S_{\theta})} A_{jkl}=A\right\}.$$

Theorem 3.5. *Let $0<\alpha \leq \beta \leq 1$. Then $W_3^{\alpha} S\left(\mathcal{I}_3-\phi\right) \subset W_3^{\beta} S\left(\mathcal{I}_3-\phi\right)$.*

Proof. Let $0<\alpha \leq \beta \leq 1$. Then

$$\begin{aligned}
& \frac{1}{r^{\beta} s^{\beta} t^{\beta}}\left|\left\{j \leq r, k \leq s, l \leq t: \phi\left(d(x, A_{jkl})-d(x, A) \geq \varepsilon\right\}\right| \\
& \leq \frac{1}{r^{\alpha} s^{\alpha} t^{\alpha}}\left|\left\{j \leq r, k \leq s, l \leq t: \phi\left(d(x, A_{jkl})-d(x, A) \geq \varepsilon\right\}\right|
\end{aligned}$$

and so for any $\delta>0$,

$$\begin{aligned}
& \left\{(r, s, t) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \frac{1}{r^{\beta} s^{\beta} t^{\beta}}\left|\left\{j \leq r, k \leq s, l \leq t:\right.\right.\right. \\
& \qquad\qquad \left.\left.\left.\phi\left(d(x, A_{jkl})-d\left(x, A\right)\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \\
& \subset\left\{(r, s, t) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \frac{1}{r^{\alpha} s^{\alpha} t^{\alpha}}\left|\left\{j \leq r, k \leq s, l \leq t:\right.\right.\right. \\
& \qquad\qquad \left.\left.\left.\phi\left(d(x, A_{jkl})-d\left(x, A\right)\right) \geq \varepsilon\right\}\right| \geq \delta\right\}.
\end{aligned}$$

Hence if the set on the right hand side belongs to the ideal $\mathcal{I}_3$ then obviously the set on the left hand side also belongs to $\mathcal{I}_3$. We obtain the desired result. □

Corollary 3.6. *If a triple sequence is Wijsman $\mathcal{I}_3$-statistically ϕ-convergent of order α to A for some $0<\alpha \leq 1$ then it is Wijsman $\mathcal{I}_3$-statistically ϕ-convergent.*

Similarly we can show that

Theorem 3.7. *Let $0<\alpha \leq \beta \leq 1$. Then $W_{\theta_3}^{\alpha} S_{\theta}\left(\mathcal{I}_3-\phi\right) \subset W_{\theta_3}^{\beta} S_{\theta}\left(\mathcal{I}_3-\phi\right)$ and in particular $W_{\theta_3}^{\alpha} S_{\theta}\left(\mathcal{I}_3-\phi\right) \subset W_{\theta_3} S_{\theta}\left(\mathcal{I}_3-\phi\right)$.*

Definition 3.8. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $\theta_3 = \theta_{r,s,t} = \{(j_r, k_s, l_t)\}$ be a lacunary triple sequence. We say that the triple sequence $\{A_{jkl}\}$ is Wijsman strongly $\mathcal{I}_3$-lacunary ϕ-convergent to A, if for each $\varepsilon > 0$ and for each $x \in X$,

$$\left\{(r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s,t}} \sum_{(j,k,l)\in I_{r,s,t}} \phi\left(d(x, A_{jkl}) - d(x,A)\right) \geq \varepsilon\right\}$$

belongs to $\mathcal{I}_3$. In this case, we write we write $\mathcal{I}_3\text{-}\lim_{W_\theta(N_\theta)} A_{jkl} = A$.

The set of Wijsman strongly $\mathcal{I}_3$-lacunary ϕ-convergent triple sequences will be denoted by

$$W_{\theta_3} N_\theta\left(\mathcal{I}_3 - \phi\right) := \left\{\{A_{jkl}\} : \mathcal{I}_3\text{-}\lim_{W_\theta(N_\theta)} A_{jkl} = A\right\}.$$

Theorem 3.9. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $\theta_3 = \theta_{r,s,t} = \{(j_r, k_s, l_t)\}$ be a lacunary triple sequence. Then,*

$$\mathcal{I}_3\text{-}\lim_{W_\theta(N_\theta)} A_{jkl} = A \textit{ implies } \mathcal{I}_3\text{-}st\text{-}\lim_{W_\theta(S_\theta)} A_{jkl} = A.$$

Proof. If $\varepsilon > 0$ and $\mathcal{I}_3\text{-}\lim_{W_\theta(N_\theta)} A_{jkl} = A$, we can write, for each $x \in X$

$$\begin{aligned}
&\sum_{(j,k,l)\in I_{r,s,t}} \phi\left(d(x, A_{jkl}) - d(x,A)\right) \\
&\geq \sum_{\substack{(j,k,l)\in I_{r,s,t} \\ \phi(d(x,A_{jkl})-d(x,A))\geq\varepsilon}} \phi\left(d(x, A_{jkl}) - d(x,A)\right) \\
&\geq \varepsilon \left|\{(j,k,l) \in I_{r,s,t} : \phi\left(d(x, A_{jkl}) - d(x,A)\right) \geq \varepsilon\}\right|
\end{aligned}$$

and so

$$\begin{aligned}
&\frac{1}{\varepsilon h_{r,s,t}} \sum_{(j,k,l)\in I_{r,s,t}} \phi\left(d(x, A_{jkl}) - d(x,A)\right) \\
&\geq \frac{1}{h_{r,s,t}} \left|\{(j,k,l) \in I_{r,s,t} : \phi\left(d(x, A_{jkl}) - d(x,A)\right) \geq \varepsilon\}\right|.
\end{aligned}$$

Then, for each $x \in X$ and for any $\delta > 0$,

$$\begin{aligned}
&\Big\{(r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s,t}} \left|\{(j,k,l) \in I_{r,s,t} :\right. \\
&\qquad \left.\phi\left(d(x, A_{jkl}) - d\left(x,A\right)\right) \geq \varepsilon\}\right| \geq \delta\Big\}
\end{aligned}$$

$$\subseteq \left\{(r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s,t}} \sum_{(j,k,l)\in I_{r,s,t}} \phi\left(d(x, A_{jkl}) - d(x,A)\right) \geq \varepsilon.\delta\right\} \in \mathcal{I}_3.$$

This proof is completed. □

Definition 3.10. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A triple sequence (A_{jkl}) is said to be bounded if there exists $M > 0$ such that $\phi\left(A_{jkl}\right) \leq M$ for all $j,k,l \in \mathbb{N}$. We denote the space of all bounded triple sequences by ℓ^3_∞.

Theorem 3.11. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $\theta_3 = \theta_{r,s,t} = \{(j_r, k_s, l_t)\}$ be a lacunary triple sequence. If $(A_{jkl}) \in \ell_\infty^3$ and (A_{jkl}) is Wijsman $\mathcal{I}_3$-lacunary statistical ϕ-convergent to A, then (A_{jkl}) is Wijsman strongly $\mathcal{I}_3$-lacunary ϕ-convergent to A.*

Proof. Suppose that (A_{jkl}) belongs to the space ℓ_∞^3 and $\mathcal{I}_3$-st-$\lim_{W_\theta(S_\theta)} A_{jkl} = A$. Then, we can assume that

$$\phi\left(d(x, A_{jkl}) - d(x, A)\right) \leq M, \text{ for each } x \in X \text{ and all } j, k \text{ and } l.$$

Given $\varepsilon > 0$, for each $x \in X$ we have

$$\begin{aligned}
&\frac{1}{h_{r,s,t}} \sum_{(j,k,l) \in I_{r,s,t}} \phi\left(d(x, A_{jkl}) - d(x, A)\right) \\
&= \frac{1}{h_{r,s,t}} \sum_{\substack{(j,k,l) \in I_{r,s,t} \\ \phi(d(x,A_{jkl}) - d(x,A)) \geq \frac{\varepsilon}{2}}} \phi\left(d(x, A_{jkl}) - d(x, A)\right) \\
&+ \frac{1}{h_{r,s,t}} \sum_{\substack{(j,k,l) \in I_{r,s,t} \\ \phi(d(x,A_{jkl}) - d(x,A)) < \frac{\varepsilon}{2}}} \phi\left(d(x, A_{jkl}) - d(x, A)\right) \\
&\leq \frac{M}{h_{r,s,t}} \left| \left\{ (j,k,l) \in I_{r,s,t} : \phi\left(d(x, A_{jkl}) - d(x, A)\right) \geq \frac{\varepsilon}{2} \right\} \right| + \frac{\varepsilon}{2}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\left\{ (r,s,t) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s,t}} \sum_{(j,k,l) \in I_{r,s,t}} \phi\left(d(x, A_{jkl}) - d(x, A)\right) \geq \varepsilon \right\} \\
&\subseteq \Bigg\{ (r,s,t) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s,t}} \big| \{ (j,k,l) \in I_{r,s,t} : \\
&\qquad\qquad \phi\left(d(x, A_{jkl}) - d\left(x, A\right)\right) \geq \frac{\varepsilon}{2} \} \big| \geq \frac{\varepsilon}{2M} \Bigg\} \in \mathcal{I}_3.
\end{aligned}$$

Therefore, $\mathcal{I}_3$-$\lim_{W_\theta(N_\theta)} A_{jkl} = A$. □

From Theorem 3.9 and Theorem 3.11, we have following Corollary.

Corollary 3.12. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $\theta_3 = \theta_{r,s,t} = \{(j_r, k_s, l_t)\}$ be a lacunary triple sequence. Then, the following statements hold:*

$$W_{\theta_3} S_\theta\left(\mathcal{I}_3 - \phi\right) \cap \ell_\infty^3 = W_{\theta_3} N_\theta\left(\mathcal{I}_3 - \phi\right) \cap \ell_\infty^3.$$

We will now investigate the relationship between Wijsman $\mathcal{I}_3$-statistical ϕ-convergence and Wijsman $\mathcal{I}_3$-lacunary statistical ϕ-convergence for triple sequence.

Theorem 3.13. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $\theta_3 = \theta_{r,s,t} = \{(j_r, k_s, l_t)\}$ be a lacunary triple sequence with $\liminf q_{r,s,t} > 1$. Then, $\mathcal{I}_3$-st-$\lim_{W(S)} A_{jkl} = A$ implies $\mathcal{I}_3$-st-$\lim_{W_\theta(S_\theta)} A_{jkl} = A$.*

Proof. Suppose that $\liminf q_{r,s,t} > 1$. Then, there exists a $\gamma > 0$ such that $q_{r,s,t} \geq 1+\gamma$ for sufficiently large r, s, t, which implies

$$\frac{h_{r,s,t}}{k_{r,s,t}} \geq \frac{\gamma}{1+\gamma}.$$

If $\mathcal{I}_3$-*st*-$\lim_{W(S)} A_{jkl} = A$, then for every $\varepsilon > 0$, for each $x \in X$ and for sufficiently large r, s, t, we have

$$\begin{aligned}
&\frac{1}{k_{r,s,t}} \left|\{j \leq j_r, k \leq k_s, l \leq l_t : \phi\left(d(x, A_{jkl}) - d(x,A)\right) \geq \varepsilon\}\right| \\
&\geq \frac{1}{k_{r,s,t}} \left|\{(j,k,l) \in I_{r,s,t} : \phi\left(d(x, A_{jkl}) - d(x,A)\right) \geq \varepsilon\}\right| \\
&\geq \frac{\gamma}{1+\gamma}\frac{1}{h_{r,s,t}} \left|\{(j,k,l) \in I_{r,s,t} : \phi\left(d(x, A_{jkl}) - d(x,A)\right) \geq \varepsilon\}\right|.
\end{aligned}$$

Then, for each $x \in X$ and for any $\delta > 0$, we get

$$\begin{aligned}
&\left\{(r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s,t}} \left|\{(j,k,l) \in I_{r,s,t} : \phi\left(d(x, A_{jkl}) - d(x,A)\right) \geq \varepsilon\}\right| \geq \delta\right\} \\
&\subseteq \left\{(r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{k_{r,s,t}} \left|\{j \leq j_r, k \leq k_s, l \leq l_t :\right.\right. \\
&\qquad\qquad \left.\left.\phi\left(d(x, A_{jkl}) - d\left(x,A\right)\right) \geq \varepsilon\}\right| \geq \frac{\gamma\delta}{1+\gamma}\right\} \in \mathcal{I}_3.
\end{aligned}$$

This completes the proof. $\square$

Theorem 3.14. *Let $\mathcal{I}_3 = \mathcal{I}_3^{\text{fin}} = \{J : J \text{ is a finite set}\}$ be a non-trivial ideal, and let $\theta_3 = \{(j_r, k_s, l_t)\}$ be a lacunary sequence with $\limsup q_{r,s,t} < \infty$. Then, the following statements hold:*

$$\mathcal{I}_3\text{-}st\text{-}\lim_{W_\theta(S_\theta)} A_{jkl} = A \textit{ implies } \mathcal{I}_3\text{-}st\text{-}\lim_{W(S)} A_{jkl} = A\ .$$

Proof. If $\limsup_{r,s,t} q_{r,s,t} < \infty$, then without loss of generality, we can assume that there exists a $D > 0$ such that $q_{r,s,t} < D$ for all $r, s, t \geq 1$. Suppose that $\mathcal{I}_3$-*st*-$\lim_{W_\theta(S_\theta)} A_{jkl} = A$ and for $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$ define the sets

$$F_{r,s,t} = \left|\{(j,k,l) \in I_{r,s,t} : \phi\left(d(x, A_{jkl}) - d(x,A)\right) \geq \varepsilon\}\right|$$

and

$$\begin{aligned}
&\left\{(r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s,t}} \left|\{(j,k,l) \in I_{r,s,t} : \phi\left(d(x, A_{jkl}) - d(x,A)\right) \geq \varepsilon\}\right| \geq \delta\right\} \\
&= \left\{(r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{F_{r,s,t}}{h_{r,s,t}} \geq \delta\right\} \in \mathcal{I}_3,
\end{aligned}$$

and, therefore, it is a finite set. We choose integers $r_0, s_0, t_0 \in \mathbb{N}$ such that

$$\frac{F_{r,s,t}}{h_{r,s,t}} < \delta \text{ for all } r > r_0,\ s > s_0,\ t > t_0.$$

Let $F = \max\{F_{r,s,t} : 1 \leq r \leq r_0, 1 \leq s \leq s_0, 1 \leq t \leq t_0\}$ and p, q, r be any three integers with $j_{r-1} < p \leq j_r$, $k_{s-1} < q \leq k_s$ and $l_{t-1} < r \leq l_t$, then we have

$$\begin{aligned}
&\frac{1}{pqr}\left|\{j \leq p, k \leq q, l \leq r : \phi\left(d(x, A_{jkl}) - d(x, A)\right) \geq \varepsilon\}\right| \\
&\leq \frac{1}{j_{r-1}k_{s-1}l_{t-1}}\left|\{j \leq j_r, k \leq k_s, l \leq l_t : \phi\left(d(x, A_{jkl}) - d(x, A)\right) \geq \varepsilon\}\right| \\
&= \frac{1}{j_{r-1}k_{s-1}l_{t-1}}\left\{\left|\{(j,k,l) \in I_{1,1,1} : \phi\left(d(x, A_{jkl}) - d(x, A)\right) \geq \varepsilon\}\right|\right. \\
&\left. + \ldots + \frac{1}{j_{r-1}k_{s-1}l_{t-1}}\left|\{(j,k,l) \in I_{r,s,t} : \phi\left(d(x, A_{jkl}) - d(x, A)\right) \geq \varepsilon\}\right|\right\} \\
&= \frac{1}{j_{r-1}k_{s-1}l_{t-1}}\left\{F_{1,1,1} + F_{2,2,2} + \ldots + F_{r_0,s_0,t_0} + F_{r_0+1,s_0+1,t_0+1} + \ldots + F_{r,s,t}\right\} \\
&= \frac{F}{j_{r-1}k_{s-1}l_{t-1}}r_0s_0t_0 + \frac{1}{j_{r-1}k_{s-1}l_{t-1}}\left\{h_{r_0+1,s_0+1,t_0+1}\left(\frac{F_{r_0+1,s_0+1,t_0+1}}{h_{r_0+1,s_0+1,t_0+1}}\right)\right. \\
&\qquad \left. + \ldots + h_{r,s,t}\frac{F_{r,s,t}}{h_{r,s,t}}\right\} \\
&= \frac{F}{j_{r-1}k_{s-1}l_{t-1}}r_0s_0t_0 \\
&+ \frac{1}{j_{r-1}k_{s-1}l_{t-1}}\left(\sup_{r>r_0,s>s_0,t>t_0}\frac{F_{r,s,t}}{h_{r,s,t}}\right)\left\{h_{r_0+1,s_0+1,t_0+1} + \ldots + h_{r,s,t}\right\} \\
&\leq \frac{F}{j_{r-1}k_{s-1}l_{t-1}}r_0s_0t_0 + \delta\left(\frac{j_rk_sl_t - j_{r_0}k_{s_0}l_{t_0}}{j_{r-1}k_{s-1}l_{t-1}}\right) \\
&\leq \frac{F}{j_{r-1}k_{s-1}l_{t-1}}r_0s_0t_0 + \delta q_{r,s,t} \\
&\leq \frac{F}{j_{r-1}k_{s-1}l_{t-1}}r_0s_0t_0 + \delta D.
\end{aligned}$$

Since $j_{r-1} \to \infty$, $k_{s-1} \to \infty$, $l_{t-1} \to \infty$ as $p \to \infty$, $q \to \infty$, $r \to \infty$, respectively, it follows that $\mathcal{I}_3 - st - \lim_{W(S)} A_{jkl} = A$. This completes the proof of the theorem. □

From Theorem 3.13 and Theorem 3.14, we have following Corollary :

Corollary 3.15. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $\theta_3 = \theta_{r,s,t} = \{(j_r, k_s, l_t)\}$ be a lacunary triple sequence with $1 < \inf q_{r,s,t} \leq \sup q_{r,s,t} < \infty$. Then, the following statements hold:*

$$\mathcal{I}_3\text{-}st\text{-}\lim_{W_\theta(S_\theta)} A_{jkl} = \mathcal{I}_3\text{-}st\text{-}\lim_{W(S)} A_{jkl}.$$

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