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ROUGH $\Delta\mathcal{I}$-STATISTICAL CONVERGENCE

ÖMER KİŞİ*, ERDİNÇ DÜNDAR

Abstract. In this study, we examine rough $\Delta\mathcal{I}$-statistical convergence for difference sequences as an extension of rough convergence. We investigate the set of rough $\Delta\mathcal{I}$-statistical limit points of a difference sequence and analyze the results with proofs.

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1. Introduction and Background

Fast [1] examined the statistical convergence of a real number sequence. Some beneficial conclusions on this topic can be found in [2, 3, 4, 5, 6, 7, 8]. Kostyrko et al. [9] studied ideal convergence as a generalization of statistical convergence. Kostyrko et al. [10] researched some features of $\mathcal{I}$-convergence. Savaş and Das [11] investigated $\mathcal{I}$-statistical convergence. Later on it was studied by some researchers. For details, see [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25].

Rough convergence was firstly given by Phu [26] in finite-dimensional normed spaces. Phu [27] investigated rough continuity of linear operators and denoted that under the assumption of X and Y are normed spaces, every linear operator $f : X \to Y$ is rough continuous at every point X. Considering the results in [26], Phu [28] studied some properties of rough convergence in infinite-dimensional normed spaces. Aytar [29] defined rough statistical convergence. In another study [30], he worked rough limit set and the core of a real sequence. The generalization of rough statistical convergence which is known as rough $\mathcal{I}$-convergence was given by Pal et al. [31]. Recently, Dündar and Çakan [32, 33, 34] investigated the rough $\mathcal{I}$-convergence and examined the notions of rough convergence and rough $\mathcal{I}_2$-convergence of a double sequence. Rough $\mathcal{I}$-statistical convergence was firstly studied by Savaş et al [35]. In another study, Malik et al. [36] examined significant properties of this kind of convergence. Also, Arslan and

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Dündar [37, 38] introduced rough convergence in 2-normed spaces. Demir and Gümüş [39] studied rough statistical convergence of difference sequences. Rough convergence, rough statistical convergence and $\Delta\mathcal{I}$-convergence for difference sequences and for double difference sequences have been investigated. For details, see [40, 41, 42, 43, 44, 45].

In this study, our aim is to define the rough $\Delta\mathcal{I}$-statistical convergence for difference sequences and proved some significant theorems. As can be seen from the title of the article, there are four important notions that will form the basis of this article. These are; statistical convergence, $\mathcal{I}$-convergence, difference sequences and rough convergence.

2. Definitions and notations

In this section, some significant definitions and notations are given. (See [37, 38, 29, 30, 32, 33, 34, 45, 26, 27, 28]).

During the study, r denotes a nonnegative real number and $\mathbb{R}^n$ indicates the real n-dimensional space with the norm $\|.\|$. Think a sequence $x = (x_k) \subset X = \mathbb{R}^n$.

The sequence x is named to be r-convergent to x_*, showed by $x_k \xrightarrow{r} x_*$ on condition that

$$\forall \varepsilon > 0 \;\; \exists i_\varepsilon \in \mathbb{N} : \;\; k \geq i_\varepsilon \Rightarrow \|x_k - x_*\| < r + \varepsilon.$$

The set

$$\mathrm{LIM}^r x := \{x_* \in \mathbb{R}^n : x_k \xrightarrow{r} x_*\}$$

is given the r-limit set of the sequence x. If $\mathrm{LIM}^r x \neq \emptyset$ holds, then the $x = (x_i)$ is called r-convergent. Here, r indicates the convergence degree of the sequence x. For $r = 0$, we have the ordinary convergence.

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is named an ideal iff

(i) $\emptyset \in \mathcal{I}$,

(ii) for each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$,

(iii) for each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is named a filter in $\mathbb{N}$ iff

(i) $\emptyset \notin \mathcal{F}$,

(ii) for each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$,

(iii) for each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

If $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is a nontrivial ideal, then the family of sets

$$\mathcal{F}(\mathcal{I}) = \{P \subset \mathbb{N} : \exists K \in \mathcal{I} : P = \mathbb{N} \setminus K\}$$

is a filter of $\mathbb{N}$ and it is named as the filter connected with the ideal $\mathcal{I}$.

Mursaleen et al. [20] defined $\mathcal{I}$-statistical cluster point of real number sequence.

Theorem 2.1. *If an $\mathcal{I}$-statistically bounded sequence has one cluster point then it is $\mathcal{I}$-statistically convergent.*

A sequence $x = (x_k)$ is called to be r-$\mathcal{I}$-convergent to x_* with the roughness degree r, demonstrated by $x_k \xrightarrow{r-\mathcal{I}} x_*$ provided that

$$\{k \in \mathbb{N} : \|x_k - x_*\| \geq r + \varepsilon\} \in \mathcal{I}$$

for every $\varepsilon > 0$. Additionally, we write $x_k \xrightarrow{r-\mathcal{I}} x_*$ iff the $\|x_k - x_*\| < r + \varepsilon$ holds for every $\varepsilon > 0$ and almost all k.

(Δx_k) is called to be rough $\mathcal{I}$-convergent to x_* or r-$\mathcal{I}$-convergent to x_* if for any $\varepsilon > 0$

$$\{k \in \mathbb{N} : \|\Delta x_k - x_*\| \geq r + \varepsilon\} \in \mathcal{I}.$$

In this case x_* is named the r-$\mathcal{I}$-limit of (Δx_k) and we indicate it by $\Delta x \xrightarrow{r-\mathcal{I}} x_*$.

3. MAIN RESULTS

Definition 3.1. A sequence (Δx_k) in X is said to be rough $\mathcal{I}$-statistically convergent to x_* or r-$\mathcal{I}$-statistically convergent to x_*, demonstrated by $\Delta x \xrightarrow{r-\mathcal{I}-st} x_*$, provided that

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \leq n : \|\Delta x_k - x_*\| \geq r + \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}$$

for any $\varepsilon > 0$ and $\delta > 0$, or correspondingly we can say

$$\mathcal{I} - st \limsup \|\Delta x_k - x_*\| \leq r.$$

For $r = 0$, we have $\Delta\mathcal{I}$-statistical convergence. So, our main attention is when $r > 0$.

If $\mathcal{I}$ is an admissible ideal, then usual rough statistical convergence for a difference sequence (Δx_k) implies rough $\mathcal{I}$-statistical convergence.

The idea of rough $\mathcal{I}$-statistical convergence for a difference sequence can be explained with the following example.

Example 3.2. As an example presume that the sequence (Δy_k) is $\mathcal{I}$-statistically convergent which can not be measured absolutely. We can select an approximated sequence (Δx_k) satisfying $\{k \in \mathbb{N} : \|\Delta x_k - \Delta y_k\| > r\} \in \mathcal{I}$. Then, $\mathcal{I}$-statistically convergence of sequence (Δx_k) is not assured, but the inclusion,

$$\left\{n \in \mathbb{N} : \tfrac{1}{n} \left|\{k \leq n : \|\Delta x_k - x_*\| \geq r + \varepsilon\}\right| \geq \delta\right\}$$

$$\subseteq \left\{n \in \mathbb{N} : \tfrac{1}{n} \left|\{k \leq n : \|\Delta y_k - x_*\| \geq \varepsilon\}\right| \geq \delta\right\}$$

holds, so the sequence (Δx_k) is r-$\mathcal{I}$-statistically convergent.

Generally the r-$\mathcal{I}$-statistical limit of a sequence may not be unique for $r > 0$. We identify the set of all r-$\mathcal{I}$-statistical limit of (Δx_k) with

$$\mathcal{I} - st - \text{LIM}^r_{(\Delta x_k)} = \left\{ x_* \in X : \Delta x_k \overset{r-\mathcal{I}-st}{\longrightarrow} x_* \right\}.$$

If $\mathcal{I} - st - LIM^r_{(\Delta x_k)} \neq \emptyset$ holds, then the sequence (Δx_k) is called r-$\mathcal{I}$-statistically convergent. It is obvious that if $\mathcal{I} - st - LIM^r_{(\Delta x_k)} \neq \emptyset$ for a sequence (Δx_k), then we get

$$\mathcal{I} - st - \text{LIM}^r_{(\Delta x_k)} = \left[\mathcal{I} - st - \limsup (\Delta x_k) - r,\ \mathcal{I} - st - \liminf (\Delta x_k) - r\right].$$

As seen in the example below, there is an unbounded difference sequence which is not rough convergence but it can be r-$\mathcal{I}$-statistically convergent.

Example 3.3. Let $\mathcal{I}$ be an admissible ideal and A be an infinite set such that $A \in \mathcal{I}$. Take the difference sequence

$$(\Delta x_k) = \begin{cases} (-1)^k, & \text{if } k \notin A, \\ k, & \text{if } k \in A. \end{cases}$$

It is clear that (Δx_k) is unbounded and r-$\mathcal{I}$-statistically convergent. Because,

$$\mathcal{I} - st - \text{LIM}^r_{(\Delta x_k)} = \begin{cases} \emptyset, & \text{if } r < 1, \\ [1 - r, r - 1], & \text{otherwise.} \end{cases}$$

Definition 3.4. A sequence (Δx_k) is called to be $\mathcal{I}$-statistically bounded if there consists a number K such that

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ k \leq n : \|\Delta x_k\| > K \} \right| > \delta \right\} \in \mathcal{I}.$$

Theorem 3.5. *For a sequence* (Δx_k)*,*

$$diam\left(\mathcal{I} - st - \text{LIM}^r_{(\Delta x_k)}\right) \leq 2r.$$

Generally $diam\left(\mathcal{I} - st - \text{LIM}^r_{(\Delta x_k)}\right)$ *has no smaller bound.*

Proof. Presume that $diam\left(\mathcal{I} - st - LIM^r_{(\Delta x_k)}\right) > 2r$. Then, there are $y, z \in \mathcal{I}$-st-$\text{LIM}^r_{(\Delta x_k)}$ such that $\|y - z\| > 2r$. Now select $\varepsilon > 0$ so that $\varepsilon \in \left(0, \frac{\|y-z\|}{2} - r\right)$. Let

$$A_1 = \{ k \in \mathbb{N} : \|\Delta x_k - y\| \geq r + \varepsilon \} \in \mathcal{I}$$

and

$$A_2 = \{ k \in \mathbb{N} : \|\Delta x_k - z\| \geq r + \varepsilon \} \in \mathcal{I}.$$

Then, we have

$$\frac{1}{n} \left| \{ k \leq n : k \in A_1 \cup A_2 \} \right| \leq \frac{1}{n} \left| \{ k \leq n : k \in A_1 \} \right| + \frac{1}{n} \left| \{ k \leq n : k \in A_2 \} \right|$$

and by the feature of $\mathcal{I}$-convergence we get

$$\mathcal{I}-\lim_{n\to\infty}\tfrac{1}{n}\left|\{k\leq n:k\in A_1\cup A_2\}\right|\leq\mathcal{I}-\lim_{n\to\infty}\tfrac{1}{n}\left|\{k\leq n:k\in A_1\}\right|$$

$$+\mathcal{I}-\lim_{n\to\infty}\tfrac{1}{n}\left|\{k\leq n:k\in A_2\}\right|=0.$$

Thus, we get

$$\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\leq n:k\in A_1\cup A_2\}\right|\geq\delta\right\}\in\mathcal{I}$$

for all $\delta>0$. Let

$$M=\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\leq n:k\in A_1\cup A_2\}\right|\geq\frac{1}{2}\right\}.$$

Obviously, $M\in\mathcal{I}$. Now, select $n_0\in\mathbb{N}\setminus M$. Then

$$\frac{1}{n_0}\left|\{k\leq n_0:k\in A_1\cup A_2\}\right|<\frac{1}{2}.$$

So

$$\frac{1}{n_0}\left|\{k\leq n_0:k\notin A_1\cup A_2\}\right|\geq 1-\frac{1}{2}=\frac{1}{2}$$

i.e., $\{k:k\notin A_1\cup A_2\}$ is a nonempty set. Take $k_0\in\mathbb{N}$ such that $k_0\notin A_1\cup A_2$. Then $k_0\in A_1^c\cap A_2^c$ and hence $\|\Delta x_{k_0}-y\|<r+\varepsilon$ and $\|\Delta x_{k_0}-z\|<r+\varepsilon$. So

$$\|y-z\|\leq\|\Delta x_{k_0}-y\|+\|\Delta x_{k_0}-z\|<2\left(r+\varepsilon\right)<\|y-z\|,$$

As we can see this is a absurd. Thus,

$$diam\left(\mathcal{I}-st-\mathrm{LIM}^r_{(\Delta x_k)}\right)\leq 2r.$$

For evidence of the second part, think a sequence (Δx_k) such that $\mathcal{I}-st-\lim\Delta x_k=x_*$. Let $\varepsilon>0$ and $\delta>0$ then, we get

$$A=\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\leq n:\|\Delta x_k-x_*\|\geq\varepsilon\}\right|\geq\delta\right\}\in\mathcal{I}.$$

Then, for $n\notin A$ we obtain

$$\frac{1}{n}\left|\{k\leq n:\|\Delta x_k-x_*\|\geq\varepsilon\}\right|<\delta,$$

i.e.,

$$\frac{1}{n}\left|\{k\leq n:\|\Delta x_k-x_*\|<\varepsilon\}\right|\geq 1-\delta.$$

Now, for each

$$y\in\overline{B}_r\left(x_*\right)=\{y\in X:\|y-x_*\|\leq r\}$$

we have

$$\|\Delta x_k-y\|\leq\|\Delta x_k-x_*\|+\|x_*-y\|\leq\|\Delta x_k-x_*\|+r.$$

Let

$$P_n=\left\{k\leq n:\|\Delta x_k-x_*\|<\varepsilon\right\}.$$

Then, for $k \in P_n$ we get $\|\Delta x_k - y\| < r + \varepsilon$. Hence

$$P_n \subset \{k \leq n : \|\Delta x_k - y\| < r + \varepsilon\}.$$

This means,

$$\frac{|P_n|}{n} \leq \frac{1}{n} |\{k \leq n : \|\Delta x_k - y\| < r + \varepsilon\}|$$

i.e.,

$$\frac{1}{n} |\{k \leq n : \|\Delta x_k - y\| < r + \varepsilon\}| \geq 1 - \delta.$$

Thus, for all $n \notin A$,

$$\frac{1}{n} |\{k \leq n : \|\Delta x_k - y\| \geq r + \varepsilon\}| < 1 - (1 - \delta).$$

Hence, we get

$$\left\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : \|\Delta x_k - y\| \geq r + \varepsilon\}| \geq \delta\right\} \subset A.$$

Since $A \in \mathcal{I}$, so

$$\left\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : \|\Delta x_k - y\| \geq r + \varepsilon\}| \geq \delta\right\} \in \mathcal{I}$$

which gives that $y \in \mathcal{I} - st - LIM^r_{(\Delta x_k)}$ and as a consequence, we obtain $\mathcal{I}$-st-$\mathrm{LIM}^r_{(\Delta x_k)} \supset \overline{B}_r(x_*)$. Therefore, $diam\left(\mathcal{I} - st - LIM^r_{(\Delta x_k)}\right) \geq 2r$, so $diam\left(\mathcal{I} - st - LIM^r_{(\Delta x_k)}\right) = 2r$. □

Theorem 3.6. *A sequence (Δx_k) is $\mathcal{I}$-st-bounded iff there is $r > 0$ such that $\mathcal{I} - st - \mathrm{LIM}^r_{(\Delta x_k)} \neq \emptyset$.*

Proof. Take $\mathcal{I}$-st-bounded sequence (Δx_k). Then, there is $K > 0$ such that

$$P = \left\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : \|\Delta x_k\| > K\}| > \delta\right\} \in \mathcal{I}.$$

Let $\overline{r} := \sup\{\|\Delta x_k\| : k \in P^c\}$. The set $\mathcal{I} - st - LIM^{\overline{r}}_{(\Delta x_k)}$ includes the origin of $\mathbb{R}^n$ and so $\mathcal{I} - st - LIM^{\overline{r}}_{(\Delta x_k)} \neq \emptyset$.

On the contrary, presume that $\mathcal{I} - st - LIM^{\overline{r}}_{(\Delta x_k)} \neq \emptyset$ for some $r \geq 0$. Then, there is $x_* \in \mathcal{I} - st - LIM^{\overline{r}}_{(\Delta x_k)}$ i.e.,

$$\left\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : \|\Delta x_k - x_*\| \geq r + \varepsilon\}| \geq \delta\right\} \in \mathcal{I},$$

for each $\varepsilon > 0$ and $\delta > 0$. So we can say that almost all Δx_k's are contained in some ball with any radius greater than r and (Δx_k) is $\mathcal{I}$-st-bounded. □

Theorem 3.7. *The set $\mathcal{I} - st - \mathrm{LIM}^r_{(\Delta x_k)}$ of a sequence (Δx_k) is a closed set.*

Proof. If $\mathcal{I}-st-LIM^r_{(\Delta x_k)} = \emptyset$, then it is obvious. Think that $\mathcal{I}-st-LIM^r_{(\Delta x_k)} \neq \emptyset$. Now, take a sequence (Δy_k) in $\mathcal{I}-st-LIM^r_{(\Delta x_k)}$ with $\lim_{k\to\infty} \Delta y_k = y_*$. Choose $\varepsilon > 0$. Then, there is $i_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that

$$\|\Delta y_k - y_*\| < \frac{\varepsilon}{2}$$

for all $k > i_{\frac{\varepsilon}{2}}$. Let $k_0 > i_{\frac{\varepsilon}{2}}$ such that $y_{k_0} \in \mathcal{I} - st - LIM^r_{(\Delta x_k)}$. So,

$$A = \left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : \|\Delta x_k - y_{k_0}\| \geq r + \frac{\varepsilon}{2} \right\} \right| \geq \delta \right\} \in \mathcal{I}.$$

Since $\mathcal{I}$ is admissible ideal, hence $M = \mathbb{N} \setminus A$ is nonempty. Take $n \in M$. Then

$$\tfrac{1}{n} \left| \left\{ k \leq n : \|\Delta x_k - y_{k_0}\| \geq r + \tfrac{\varepsilon}{2} \right\} \right| < \delta$$

$$\Rightarrow \tfrac{1}{n} \left| \left\{ k \leq n : \|\Delta x_k - y_{k_0}\| < r + \tfrac{\varepsilon}{2} \right\} \right| \geq 1 - \delta.$$

Select $P_n = \left\{ k \leq n : \|\Delta x_k - y_{k_0}\| < r + \frac{\varepsilon}{2} \right\}$. Then, for $k \in P_n$

$$\|\Delta x_k - y_*\| \leq \|\Delta x_k - y_{k_0}\| + \|y_{k_0} - y_*\| < r + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = r + \varepsilon.$$

Therefore, we obtain $P_n \subset \{k \leq n : \|\Delta x_k - y_*\| < r + \varepsilon\}$, which means that

$$1 - \delta \leq \frac{|P_n|}{n} \leq \frac{1}{n} \left| \{k \leq n : \|\Delta x_k - y_*\| < r + \varepsilon\} \right|.$$

So, we get

$$\frac{1}{n} \left| \{k \leq n : \|\Delta x_k - y_*\| \geq r + \varepsilon\} \right| < 1 - (1 - \delta) = \delta.$$

Then, we have

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{k \leq n : \|\Delta x_k - y_*\| \geq r + \varepsilon\} \right| \geq \delta \right\} \subset A \in \mathcal{I}.$$

Hence, $y_* \in \mathcal{I} - st - LIM^r_{(\Delta x_k)}$ and so, $\mathcal{I} - st - LIM^r_{(\Delta x_k)}$ is a closed set. □

Theorem 3.8. *The set $\mathcal{I} - st - \mathrm{LIM}^r_{(\Delta x_k)}$ of a sequence (Δx_k) is convex.*

Proof. Let $y_0, y_1 \in \mathcal{I} - st - LIM^r_{(\Delta x_k)}$ and $\varepsilon > 0$ be given. Let

$$K_0 = \{k \in \mathbb{N} : \|\Delta x_k - y_0\| \geq r + \varepsilon\}$$

and

$$K_1 = \{k \in \mathbb{N} : \|\Delta x_k - y_1\| \geq r + \varepsilon\}.$$

Then, for $\delta > 0$ we get

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{k \leq n : k \in K_0 \cup K_1\} \right| \geq \delta \right\} \in \mathcal{I}.$$

Select $0 < \delta_1 < 1$ such that $0 < 1 - \delta_1 < \delta$. Take

$$K = \left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{k \leq n : k \in K_0 \cup K_1\} \right| \geq 1 - \delta_1 \right\}.$$

Then, $K \in \mathcal{I}$. Now for each $n \notin K$ we get

$$\frac{1}{n}\left|\{k \leq n : k \in K_0 \cup K_1\}\right| < 1 - \delta_1$$

$$\Rightarrow \frac{1}{n}\left|\{k \leq n : k \notin K_0 \cup K_1\}\right| \geq \{1 - (1 - \delta_1)\} = \delta_1.$$

Therefore, $\{k \in \mathbb{N} : k \notin K_0 \cup K_1\}$ is a nonempty set. Take $k_0 \in K_1^c \cap K_2^c$ and $\lambda \in [0, 1]$. Then

$$\|\Delta x_{k_0} - (1-\lambda)\, y_0 - \lambda y_1\| = \|(1-\lambda)\, \Delta x_{k_0} + \lambda \Delta x_{k_0} - [(1-\lambda)\, y_0 + \lambda y_1]\|$$

$$\leq (1-\lambda)\, \|\Delta x_{k_0} - y_0\| + \lambda\, \|\Delta x_{k_0} - y_1\| < (1-\lambda)\,(r+\varepsilon) + \lambda\,(r+\varepsilon) = r + \varepsilon.$$

Let

$$T = \{k \in \mathbb{N} : \|\Delta x_k - (1-\lambda)\, y_0 - \lambda y_1\| \geq r + \varepsilon\}.$$

Then obviously, $K_1^c \cap K_2^c \subset T^c$. So for $n \notin K$,

$$\delta_1 \leq \frac{1}{n}\left|\{k \leq n : k \notin K_0 \cup K_1\}\right| \leq \frac{1}{n}\left|\{k \leq n : k \notin T\}\right|$$

$$\Rightarrow \frac{1}{n}\left|\{k \leq n : k \in T\}\right| < 1 - \delta_1 < \delta.$$

Therefore, $K^c \subset \left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : k \in T\}\right| < \delta\right\}$. Since $K^c \in \mathcal{F}(\mathcal{I})$, so

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : k \in T\}\right| < \delta\right\} \in \mathcal{F}(\mathcal{I})$$

and so

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : k \in T\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Hence, the set $\mathcal{I} - st - LIM^r_{(\Delta x_k)}$ is convex. □

Theorem 3.9. *Take $r > 0$. Then, a sequence $(\Delta x_k) \in X$ is r-$\mathcal{I}$-statistically convergent to x_* iff there is a difference sequence $(\Delta y_k) \in X$ such that $\mathcal{I} - st - \lim \Delta y = x_*$ and $\|\Delta x_k - \Delta y_k\| \leq r$ for all $k \in \mathbb{N}$.*

Proof. Take $(\Delta y_k) \in X$, which is $\mathcal{I} - st - \lim \Delta y = x_*$ and $\|\Delta x_k - \Delta y_k\| \leq r$ for all $k \in \mathbb{N}$. Then for any $\varepsilon > 0$ and $\delta > 0$, the set

$$K = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : \|\Delta y_k - x_*\| \geq \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Take $n \notin K$. Then

$$\frac{1}{n}\left|\{k \leq n : \|\Delta y_k - x_*\| \geq \varepsilon\}\right| < \delta$$

$$\Rightarrow \frac{1}{n}\left|\{k \leq n : \|\Delta y_k - x_*\| < \varepsilon\}\right| \geq 1 - \delta.$$

Take

$$T_n = \{k \leq n : \|\Delta y_k - x_*\| < \varepsilon\}$$

for $n \in \mathbb{N}$. Then for $k \in T_n$, we get

$$\|\Delta x_k - x_*\| \leq \|\Delta x_k - \Delta y_k\| + \|\Delta y_k - x_*\| < r + \varepsilon.$$

Therefore,

$$T_n \subset \{k \leq n : \|\Delta x_k - x_*\| < r + \varepsilon\}$$

$$\Rightarrow \tfrac{|T_n|}{n} \leq \tfrac{1}{n}\left|\{k \leq n : \|\Delta x_k - x_*\| < r + \varepsilon\}\right|$$

$$\Rightarrow \tfrac{1}{n}\left|\{k \leq n : \|\Delta x_k - x_*\| < r + \varepsilon\}\right| \geq 1 - \delta$$

$$\Rightarrow \tfrac{1}{n}\left|\{k \leq n : \|\Delta x_k - x_*\| \geq r + \varepsilon\}\right| < 1 - (1 - \delta) = \delta.$$

Hence,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : \|\Delta x_k - x_*\| \geq r + \varepsilon\}\right| \geq \delta\right\} \subset K$$

and since $K \in \mathcal{I}$, we get

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : \|\Delta x_k - x_*\| \geq r + \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Therefore, $\Delta x \overset{r-\mathcal{I}-st}{\longrightarrow} x_*$.

Conversely, presume that $\Delta x \overset{r-\mathcal{I}-st}{\longrightarrow} x_*$. Then, for $\varepsilon > 0$ and $\delta > 0$,

$$K = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : \|\Delta x_k - x_*\| \geq r + \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Take $n \notin K$. Then

$$\tfrac{1}{n}\left|\{k \leq n : \|\Delta x_k - x_*\| \geq r + \varepsilon\}\right| < \delta$$

$$\Rightarrow \tfrac{1}{n}\left|\{k \leq n : \|\Delta x_k - x_*\| < r + \varepsilon\}\right| \geq 1 - \delta.$$

Take

$$T_n = \{k \leq n : \|\Delta x_k - x_*\| < r + \varepsilon\}.$$

Now, we consider a difference sequence (Δy_k) as follows:

$$\Delta y_k := \begin{cases} x_*, & \text{if } \|\Delta x_k - x_*\| \leq r, \\ \Delta x_k + r\frac{x_* - \Delta x_k}{\|\Delta x_k - x_*\|}, & \text{otherwise.} \end{cases}$$

Then,

$$\|\Delta y_k - \Delta x_k\| = \begin{cases} \|x_* - \Delta x_k\| \leq r, & \text{if } \|\Delta x_k - x_*\| \leq r, \\ \\ r, & \text{otherwise.} \end{cases}$$

Also,

$$\|\Delta y_k - x_*\| = \begin{cases} 0, & \text{if } \|\Delta x_k - x_*\| \leq r, \\ \left\|\Delta x_k - x_* + r\frac{x_* - \Delta x_k}{\|\Delta x_k - x_*\|}\right\|, & \text{otherwise.} \end{cases}$$

$$\|\Delta y_k - x_*\| = \begin{cases} 0, & \text{if } \|\Delta x_k - x_*\| \leq r, \\ \|\Delta x_k - x_*\| - r, & \text{otherwise.} \end{cases}$$

Let $k \in T_n$. Then

$$\|\Delta y_k - x_*\| = \begin{cases} 0, & \text{if } \|\Delta x_k - x_*\| \leq r, \\ < \varepsilon, & \text{if } r < \|\Delta x_k - x_*\| < r + \varepsilon. \end{cases}$$

Therefore, we obtain

$$T_n \subset \{k \leq n : \|\Delta y_k - x_*\| < \varepsilon\}$$

$$\Rightarrow \tfrac{|T_n|}{n} \leq \tfrac{1}{n} \left|\{k \leq n : \|\Delta y_k - x_*\| < \varepsilon\}\right|$$

$$\Rightarrow \tfrac{1}{n} \left|\{k \leq n : \|\Delta y_k - x_*\| < \varepsilon\}\right| \geq 1 - \delta$$

$$\Rightarrow \tfrac{1}{n} \left|\{k \leq n : \|\Delta y_k - x_*\| \geq \varepsilon\}\right| < 1 - (1 - \delta) = \delta.$$

Thus

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \leq n : \|\Delta y_k - x_*\| \geq \varepsilon\}\right| \geq \delta\right\} \subset K.$$

Since $K \in \mathcal{I}$, $\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \leq n : \|\Delta y_k - x_*\| \geq \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}$ and so $\mathcal{I} - st - \lim \Delta y = x_*$. □

Definition 3.10. An element $c \in X$ is named as $\mathcal{I}$-statistical cluster point of a sequence (Δx_k) if for any $\varepsilon > 0$

$$d_{\mathcal{I}}\left(\{k : \|\Delta x_k - c\| < \varepsilon\}\right) \neq 0$$

where

$$d_{\mathcal{I}}(A) = \mathcal{I}\text{-}\lim_{n \to \infty} \frac{1}{n} \left|\{k \leq n : k \in A\}\right|,$$

if exists. The set of all $\mathcal{I}$-statistical cluster point of (Δx_k) can be indicated by $\mathcal{I}\text{-S}\big(\Gamma_{(\Delta x_k)}\big)$.

Theorem 3.11. *Let (Δx_k) be a sequence and $c \in \mathcal{I}\text{-}S\big(\Gamma_{(\Delta x_k)}\big)$. Then, $\|x_* - c\| \leq r$ for all $x_* \in \mathcal{I} - st - \mathrm{LIM}^r_{(\Delta x_k)}$.*

Proof. If it is possible assume that there is $x_* \in \mathcal{I} - st - LIM^r_{(\Delta x_k)}$ such that $\|x_* - c\| > r$. Let $\varepsilon = \frac{\|x_* - c\| - r}{2}$. Then, we get

$$\{k \in \mathbb{N} : \|\Delta x_k - x_*\| \geq r + \varepsilon\} \supseteq \{k \in \mathbb{N} : \|\Delta x_k - c\| < \varepsilon\}. \tag{1}$$

Since $c \in \mathcal{I}\text{-S}\big(\Gamma_{(\Delta x_k)}\big)$, so $d_{\mathcal{I}}\left(\{k : \|\Delta x_k - c\| < \varepsilon\}\right) \neq 0$. Hence, by (1) we get

$$d_{\mathcal{I}}\left(\{k : \|\Delta x_k - x_*\| \geq r + \varepsilon\}\right) \neq 0,$$

which contradicts that $x_* \in \mathcal{I} - st - LIM^r_{(\Delta x_k)}$. Hence, $\|x_* - c\| \leq r$. □

Theorem 3.12. *A sequence (Δx_k) is r-$\mathcal{I}$-statistically convergent to x_* iff $\mathcal{I} - st - \mathrm{LIM}^r_{(\Delta x_k)} = \overline{B}_r(x_*)$.*

Proof. The necessary part of the theorem is already proved in the 2nd part of *Theorem 3.5*. For the sufficiency, let $\mathcal{I}-st-LIM^r_{(\Delta x_k)} = \overline{B}_r(x_*) \neq \emptyset$. Thus, the sequence (Δx_k) is $\mathcal{I}$-statistically bounded. Assume that (Δx_k) has $\Delta\mathcal{I}$-statistical cluster point x'_* different from x_*. The point

$$\overline{x}_* := x_* + \frac{r}{\|x_* - x'_*\|}(x_* - x'_*)$$

satisfies,

$$\|\overline{x}_* - x'_*\| = \left(\frac{r}{\|x_* - x'_*\|} + 1\right)\|x_* - x'_*\| = r + \|x_* - x'_*\| > r.$$

Since, $x'_* \in \mathcal{I} - S(\Gamma_x)$, by *Theorem 3.11*, $\overline{x}_* \notin \mathcal{I} - st - LIM^r_{(\Delta x_k)}$. But this is not possible as

$$\|\overline{x}_* - x'_*\| = r \text{ and } \mathcal{I} - st - LIM^r_{(\Delta x_k)} = \overline{B}_r(x_*).$$

Therefore x_* is the unique $\mathcal{I}$-statistical cluster point of (Δx_k). So, (Δx_k) is r-$\mathcal{I}$-statistically convergent to x_*. □

Theorem 3.13. *(i) If $c \in \Gamma_{(\Delta x_k)}(\mathcal{I})$, then $\mathcal{I} - st - \mathrm{LIM}^r_{(\Delta x_k)} \subseteq \overline{B}_r(c)$.*

(ii) $\mathcal{I}-st-\mathrm{LIM}^r_{(\Delta x_k)} = \bigcap_{c\in\Gamma_{(\Delta x_k)}(\mathcal{I})} \overline{B}_r(c) = \left\{x_* \in \mathbb{R}^n : \Gamma_{(\Delta x_k)}(\mathcal{I}) \subseteq \overline{B}_r(x_*)\right\}.$

Proof. (i) If $x_* \in \mathcal{I} - st - LIM^r_{(\Delta x_k)}$ and $c \in \Gamma_{(\Delta x_k)}(\mathcal{I})$, then $\|x_* - c\| \leq r$. So, the consequence follows.

(ii) By (i) we can note

$$\mathcal{I} - st - \mathrm{LIM}^r_{(\Delta x_k)} \subseteq \bigcap_{c\in\Gamma_{(\Delta x_k)}(\mathcal{I})} \overline{B}_r(c).$$

Assume that $y \in \bigcap_{c\in\Gamma_{(\Delta x_k)}(\mathcal{I})} \overline{B}_r(c)$. We have $\|y - c\| \leq r$ for all $c \in \Gamma_{(\Delta x_k)}(\mathcal{I})$ and so

$$\Gamma_{(\Delta x_k)}(\mathcal{I}) \subseteq \overline{B}_r(x_*).$$

Then, clearly

$$\bigcap_{c\in\Gamma_{(\Delta x_k)}(\mathcal{I})} \overline{B}_r(c) = \left\{x_* \in \mathbb{R}^n : \Gamma_{(\Delta x_k)}(\mathcal{I}) \subseteq \overline{B}_r(x_*)\right\}.$$

If it is possible, let $y \notin \mathcal{I} - st - LIM^r_{(\Delta x_k)}$. Then, there is an $\varepsilon > 0$ such that

$$K = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : \|\Delta x_k - y\| \geq r + \varepsilon\}\right| < \delta\right\} \notin \mathcal{I},$$

which means the existence of an $\mathcal{I}$-cluster point c of the sequence (Δx_k) with $\|y - c\| \geq r + \varepsilon$. Hence

$$\Gamma_{(\Delta x_k)}(\mathcal{I}) \subseteq \overline{B}_r(y) \text{ and } y \notin \left\{x_* \in \mathbb{R}^n : \Gamma_{(\Delta x_k)}(\mathcal{I}) \subseteq \overline{B}_r(x_*)\right\}.$$

Ultimately the fact that $y \in \mathcal{I} - st - LIM^{r}_{(\Delta x_k)}$ follows from the examination that

$$y \in \left\{ x_* \in \mathbb{R}^n : \Gamma_{(\Delta x_k)}(\mathcal{I}) \subseteq \overline{B}_r(x_*) \right\}.$$

□

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Ömer KİŞİ received M.Sc. from Cumhuriyet University and Ph.D. from Afyon Kocatepe University. Since 2014 he has been at Bartın University. His research interests include summability theory, sequences spaces, and fuzzy sequence spaces through functional analysis.

Department of Mathematics, Faculty of Science, Bartın University, Bartın, Turkey.
e-mail: okisi@bartin.edu.tr

Erdinç Dündar is a Associate Professor in Department of Mathematics, Afyon Kocatepe University. He received the Ph.D. degree in Mathematics for İnönü University, Malatya, Turkey. His area of expertise includes: Summability theory, sequences spaces and fuzzy sequence spaces through functional analysis. He is the author of many international published papers.

Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar, Turkey.
e-mail: edundar@aku.edu.tr