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# EXISTENCE OF NON-CONSTANT POSITIVE SOLUTION OF A DIFFUSIVE MODIFIED LESLIE-GOWER PREY-PREDATOR SYSTEM WITH PREY INFECTION AND BEDDINGTON DEANGELIS FUNCTIONAL RESPONSE

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Abstract. In this paper, a diffusive predator-prey system with Beddington DeAngelis functional response and the modified Leslie-Gower type predator dynamics when a prey population is infected is considered. The predator is assumed to predate both the susceptible prey and infected prey following the Beddington-DeAngelis functional response and Holling type II functional response, respectively. The predator follows the modified Leslie-Gower predator dynamics. Both the prey, susceptible and infected, and predator are assumed to be distributed in-homogeneous in space. A reaction-diffusion equation with Neumann boundary conditions is considered to capture the dynamics of the prey and predator population. The global attractor and persistence properties of the system are studied. The priori estimates of the non-constant positive steady state of the system are obtained. The existence of non-constant positive steady state of the system is investigated by the use of Leray-Schauder Theorem. The existence of non-constant positive steady state of the system, with large diffusivity, guarantees for the occurrence of interesting Turing patterns.

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## 1. Introduction

The mathematical modeling of epidemics has become a very important subject of research after the seminal model of Kermack and McKendric [1] on SIRS systems, in which the evolution of a disease which gets transmitted upon contact is described. Diseases have an effect on the health of any community and can regulate the human and animal population density. Thus, it is very important

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both from the ecological and mathematical points of view to study ecological systems under the influence of epidemiological factors. Anderson and May [2] were the pioneers for investigating the invasion, persistence and spread of diseases by formulating an eco-epidemiological predator prey model.

After the seminal work of Anderson and May [2], the researchers Jana and Kar [3], Chakraborty et al. [4], Sharma and Samanta [5], Meng et al. [6], Maji et al. [7], Hugo and Simanjilo [8] and Melese et al. [9] have studied a predator-prey system with infection in prey only. It was assumed that the disease spreads among the prey population only and the disease is not genetically inherited. The infected populations do not recover or become immune. Other researchers [10, 11, 12, 13, 14] consider the situation where disease is transmitted through the predator population. The authors Das et al., Gao et al. and Bera et al. [15, 16, 17] have investigated a predator-prey model with infection in both species.

In the real world, most species live in a habitat which is spatially heterogeneous. It is natural that predator and prey species moves from one place to another place in search of prey and to avoid being eaten by predators, respectively. Therefore the description of the spatial structure of the population becomes important. Such spatial structure cab be captured by reaction diffusion equations.

In recent times, different researchers have considered the spatio-temporal dynamics of a prey-predator system with infection [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. Chakraborty et al. [27] have considered a diffusive predator-prey model, where predator feeds on infected prey following type II response function and infection spreads among the prey species through horizontal transmission. They have studied the spatiotemporal complexity of the system. Li et al. [23] have considered an eco-epidemiological prey-predator system with infection in predator population and find the parameter ranges for the occurrence of Turing patterns. Wonlyul Ko and Inkyung [20], Ahn et al. [18] and Ryu [31] have investigated the existence and non-existence of non-constant positive steady states of a ratio-dependent prey-predator system with infection in prey. Chenglin Li [32] have considered a ratio-dependent invasion-diffusion predator-prey system with disease in the predator and found a sufficient conditions for the nonexistence and existence of non-constant positive solution of the system, which implies the existence of spatiotemporal pattern formation. Melese and Feyissa [28] have studied the stability and bifurcation analysis of a diffusive prey-predator system with disease in prey where predator predates the susceptible prey with Beddington-DeAngelis functional response and the infected prey following Holling type II functional response. However, Melese and Feyissa [28] did not study the existence and non-existence of non-constant positive steady state solutions of the system. Thus, in this paper the existence and non-existence of non-constant positive steady state solutions of a diffusive eco-epidemiological predator-prey system with prey infection, Beddington-DeAngelis type functional response and the modified Leslie-Gower type predator dynamics is studied.

The organization of this paper is as follows: in section 2.2, model formulation, the existence of a positively invariant attracting region for the spatio-temporal system (2), the boundedness and persistence properties of solutions to the system (2) are discussed. Section 3 is devoted to the existence of non-constant positive steady states of the system (2). Lastly, conclusions are given in section 4.

## 2. The Mathematical Model

**2.1. Model Equation.** Let $N(X,T)$ and $W(X,T)$ represent the total prey population densities and the predator population density, respectively at time $T$ and position $X$ in a habitat $\Omega \subset \mathcal{R}_+$ and the prey population is infected with a disease. In this paper, consider the following system due to [28]. (Please see the detail assumption in [28]).

$$\begin{cases} U_T - D_U \Delta U = r_1 \left(1 - \dfrac{U}{K}\right) U - \dfrac{aUV}{1+bV} - \dfrac{cUW}{B+U+\omega W},\ X \in \Omega,\ T > 0, \\ V_T - D_V \Delta V = \dfrac{aUV}{1+bV} - \dfrac{AWV}{1+AhV} - dV,\ X \in \Omega,\ T > 0, \\ W_T - D_W \Delta W = r_2 \left(1 - \dfrac{W}{s + s_2 U + s_3 V}\right) W,\ X \in \Omega,\ T > 0, \\ U_\nu = V_\nu = W_\nu = 0,\ X \in \partial\Omega,\ T > 0 \\ U(X,0) = U_0(X) \geq 0,\ V(X,0) = V_0(X) \geq 0,\ W(X,0) = W_0(X) \geq 0,\ X \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded region with smooth boundary $\partial\Omega$, and all the parameters in the model, which are given as in Table 1, are assumed to be positive. The initial functions $U_0(X)$, $V_0(X)$ and $W_0(X)$ are continuous functions on $\overline{\Omega}$. The variables $U$, $V$ and $W$ stands for the population densities of Susceptible prey, infected prey and predator, respectively.

Consider the following non-dimensional variables and scaling parameters.

$$\begin{aligned} u &= \frac{U}{K},\ v = \frac{V}{K},\ w = \frac{W}{K},\ t = r_1 T,\ x = X\sqrt{\frac{r_1}{D_U}},\ D_2 = \frac{D_V}{D_U}, D_3 = \frac{D_W}{D_U}, \\ \alpha &= \frac{aK}{r_1},\ \kappa = bK,\ \beta = \frac{B}{K},\ \gamma = \frac{c}{r_1},\ \theta = \frac{AK}{r_1},\ \sigma = AhK,\ \eta = \frac{r_2}{r_1},\ s_1 = \frac{s}{K},\ \delta = \frac{d}{r_1}, \end{aligned}$$

Thus, the system (1) will take the following non-dimensional form as

$$\begin{cases} u_t - \Delta u = (1-u)u - \dfrac{\alpha uv}{1+\kappa v} - \dfrac{\gamma uw}{\beta + u + \omega w},\ x \in \Omega,\ t > 0, \\ v_t - D_2 \Delta v = \dfrac{\alpha uv}{1+\kappa v} - \dfrac{\theta vw}{1+\sigma v} - \delta v,\ x \in \Omega,\ t > 0, \\ w_t - D_3 \Delta w = \eta \left(1 - \dfrac{w}{s_1 + s_2 u + s_3 v}\right) w,\ x \in \Omega,\ t > 0, \\ u_\nu = v_\nu = w_\nu = 0,\ x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0,\ w(x,0) = w_0(x) \geq 0,\ x \in \Omega. \end{cases} \tag{2}$$

For simplicity, let us denote the reaction terms as

$$\begin{aligned} G_1(u,v,w) &:= (1-u)u - \frac{\alpha uv}{1+\kappa v} - \frac{\gamma uw}{\beta + u + \omega w}, \\ G_2(u,v,w) &:= \frac{\alpha uv}{1+\kappa v} - \frac{\theta vw}{1+\sigma v} - \delta v, \\ G_3(u,v,w) &:= \eta\left(1 - \frac{w}{s_1 + s_2 u + s_3 v}\right) w. \end{aligned}$$

Table 1. Biological Meaning of Parameters

| Parameters | Biological Meaning |
|---|---|
| $r_1$ | The intrinsic growth rate of susceptible prey, |
| $K$ | Environmental carrying capacity of prey , |
| $a$ | Infection rate of prey, |
| $b$ | Measure of Inhibition of prey, |
| $c$ | Predation rate of Predator on susceptible prey, |
| $B$ | Saturation constant, |
| $\omega$ | Predator interference, |
| $A$ | Half saturation constant, |
| $h$ | Handling time, |
| $d$ | Death rate of infected prey, |
| $r_2$ | Maximum per capita growth rate of the predator, |
| $s$ | Residual loss in predator population due to severe scarcity of its favorite food, |
| $s_2$ | Conversion factor of susceptible prey into predator, |
| $s_3$ | Conversion factor of infected prey into predator, |
| $D_U$ | Diffusion coefficient of susceptible prey, |
| $D_V$ | Diffusion coefficient of infected prey, |
| $D_W$ | Diffusion coefficient of predator. |

**2.2. Persistence and Boundedness.** The following lemma, due to Wang and Pang [33], is used to investigate the existence of a positively invariant attracting region, the boundedness and persistence of solutions of the system (2).

**Lemma 2.1.** *Let $f(s)$ be a positive $C^1$ function for $s \geq 0$, and let $d > 0$, $\tau \geq 0$ be constants. Further, let $T \in [0,\infty)$ and $\Phi \in C^{2,1}(\Omega \times (T,\infty)) \cap C^{1,0}(\overline{\Omega} \times [T,\infty))$ be a positive function.*

1 *If $\Phi$ satisfies*

$$\begin{cases} \Phi_t - d\Delta\Phi \leq \Phi^{1+\tau} f(\Phi)(\vartheta - \Phi), & (x,t) \in \Omega \times (T,\infty), \\ \Phi_\nu = 0, & (x,t) \in \partial\Omega \times [T,\infty), \end{cases}$$

*and the constant $\vartheta > 0$, then $\limsup_{t\to\infty} \max_{\overline{\Omega}} \Phi(.,t) \leq \vartheta$.*

2 *If* $\Phi$ *satisfies*

$$\begin{cases} \Phi_t - d\Delta\Phi \geq \Phi^{1+\tau} f(\Phi)(\vartheta - \Phi), (x,t) \in \Omega \times (T,\infty), \\ \quad \Phi_\nu = 0, \qquad\qquad\qquad\qquad (x,t) \in \partial\Omega \times [T,\infty), \end{cases}$$

*and the constant* $\vartheta > 0$*, then* $\liminf_{t\to\infty} \min_{\overline{\Omega}} \Phi(.,t) \geq \vartheta$*.*

3 *If* $\Phi$ *satisfies*

$$\begin{cases} \Phi_t - d\Delta\Phi \leq \Phi^{1+\tau} f(\Phi)(\vartheta - \Phi), (x,t) \in \Omega \times (T,\infty), \\ \quad \Phi_\nu = 0, \qquad\qquad\qquad\qquad (x,t) \in \partial\Omega \times [T,\infty), \end{cases}$$

*and the constant* $\vartheta \leq 0$*, then* $\limsup_{t\to\infty} \max_{\overline{\Omega}} \Phi(.,t) \leq 0$*.*

**Theorem 2.2.** *All solutions of (2) initiating in* $\mathbb{R}^3_+$ *are ultimately bounded and eventually enter into the positively invariant attracting region*

$$\Sigma = [0,1] \times \left[0, \frac{\alpha}{\delta\kappa}\right] \times \left[0, s_1 + s_2 + \frac{\alpha s_3}{\delta\kappa}\,\right]$$

*Proof.* See Theorem 4.2 in [28]. □

**Theorem 2.3.** *The system (2) is persistent if*

$$l_u = 1 - (\alpha/\kappa) - (\gamma/\omega) > 0, \;\; \alpha l_u - \left(\theta\left(s_1 + s_2 + \frac{\alpha s_3}{\delta\kappa}\right) + \delta\right) > 0.$$

*Proof.* See Theorem 4.3 in [28]. □

## 3. Non-constant Positive Steady States

This section is devoted to the investigation of the non-constant positive steady states, existence and non-existence of the system (2). The non-constant positive solutions of the system (2) are the solutions of the steady state problem

$$\begin{cases} -\Delta u = G_1(u,v,w), \qquad x \in \Omega, \\ -D_2\Delta v = G_2(u,v,w), \qquad x \in \Omega, \\ -D_3\Delta w = G_3(u,v,w), \qquad x \in \Omega, \\ \quad u_\nu = v_\nu = \; w_\nu = 0, \quad x \in \partial\Omega. \end{cases} \tag{3}$$

The classical solutions of the system (3) are assumed to be in $C^2(\Omega) \cap C^1(\overline{\Omega})$. For notational convenience, we shall write $\Lambda = \Lambda(\alpha, \beta, \gamma, \kappa, \theta, \sigma, \eta, \delta, s_1, s_2, s_3)$ in the sequel.

### 3.1. A priori estimates of non-constant positive steady state.

**Theorem 3.1.** *For any classical solution* $\boldsymbol{u} = (u,v,w)^T$ *of the system (3),*

$$\max_{\overline{\Omega}} u \leq 1, \;\; \max_{\overline{\Omega}} v \leq \frac{\alpha}{\delta\kappa}, \;\; \max_{\overline{\Omega}} w \leq s_1 + s_2 + \frac{\alpha s_3}{\delta\kappa}. \tag{4}$$

*Proof.* It directly follows from equations (25), (26) and (27) of [28] and the fact that

$$\max_{\overline{\Omega}} u \leq \limsup_{t\to\infty} \max_{\overline{\Omega}} u(.,t),$$
$$\max_{\overline{\Omega}} v \leq \limsup_{t\to\infty} \max_{\overline{\Omega}} v(.,t) \text{ and } \max_{\overline{\Omega}} w \leq \limsup_{t\to\infty} \max_{\overline{\Omega}} w(.,t).$$

□

**Theorem 3.2.** *For any classical solution* $\boldsymbol{u} = (u, v, w)^T$ *of the system (3), if*

$$l_u = 1 - (\alpha/\kappa) - (\gamma/\omega) > 0, \ l_v = \alpha l_u - \left(\theta\left(s_1 + s_2 + \frac{\alpha s_3}{\delta\kappa}\right) + \delta\right) > 0,$$

*then*

$$\begin{aligned} \min_{\overline{\Omega}} u &\geq 1 - (\alpha/\kappa) - (\gamma/\omega) = l_u, \\ \min_{\overline{\Omega}} v &\geq \frac{\alpha l_u - \left(\theta\left(s_1 + s_2 + \frac{\alpha s_3}{\delta\kappa}\right) + \delta\right)}{\left(\theta\left(s_1 + s_2 + \frac{\alpha s_3}{\delta\kappa}\right) + \delta\right)\kappa} = l_v, \\ \min_{\overline{\Omega}} w &\geq s_1 + s_2 l_u + s_3 l_v = l_w. \end{aligned} \tag{5}$$

*Proof.* The proof directly follows from equations (28), (29) and (30) of [28] and the fact that

$$\min_{\overline{\Omega}} u \geq \liminf_{t\to\infty} \min_{\overline{\Omega}} u(.,t),$$
$$\min_{\overline{\Omega}} v \geq \liminf_{t\to\infty} \min_{\overline{\Omega}} v(.,t) \text{ and } \min_{\overline{\Omega}} w \geq \liminf_{t\to\infty} \min_{\overline{\Omega}} w(.,t).$$

□

**3.2. Non-existence of non-constant positive steady state.** In this subsection, the non-existence of a non-constant positive steady state of the system (2) is proved. But before this, let us have the following notations.

**Notations:**

i) $0 = \mu_0 < \mu_1 < \mu_2 < \mu_3 < ...$ are the eigenvalues of the operator $-\Delta$ on $\Omega$ under the homogeneous Neumann boundary condition.

ii) $E(\mu_i)$ is the eigenspace corresponding to the eigenvalue $\mu_i$.

iii) $\mathbf{X}_{ij} := \{\mathbf{c}.\psi_{ij} : \mathbf{c} \in \mathbf{R}^3\}$, where $\{\psi_{ij}\}$ are orthonormal basis of $\mathbf{X}_i$ for $j = 1, 2, 3, ..., dim[E_i]$.

iv) $\mathbf{X} := \{\mathbf{u} = (u, v, w) \in [C^1(\overline{\Omega})]^3 | \frac{\partial \mathbf{u}}{\partial \nu} = 0 \text{ on } \partial\Omega\}$, and so $\mathbf{X} = \bigoplus_{i=0}^{\infty} \mathbf{X}_i$, where

$$\mathbf{X}_i = \bigoplus_{j=1}^{dim[E(\mu_i)]} \mathbf{X}_{ij}.$$

We can observe that the system (2) has a unique constant positive steady state $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v}, \tilde{w})$, where

$$\tilde{w} = s_1 + s_2\tilde{u} + s_3\tilde{v}, \ \tilde{u} = \frac{(1+\kappa\tilde{v})(\delta + s_1\theta + (s_3\theta + \sigma\delta)\tilde{v})}{\alpha - s_2\theta + (\alpha\sigma - s_2\theta\kappa)\tilde{v}}$$

and $\tilde{v}$ is the unique positive root of the quartic equation

$$A_5 v^5 + A_4 v^4 + A_3 v^3 + A_2 v^2 + A_1 v + A_0 = 0, \tag{6}$$

where the coefficients $A_i, (i = 1, 2, 3, 4, 5)$ are given in the Appendix 4.

**Theorem 3.3.** *Let $\mu_1$ be the smallest positive eigenvalue of the operator $-\Delta$ on $\Omega$ with homogeneous Neumann boundary condition and $D_3^*$ be a fixed positive constant satisfying $D_3^* \geq \frac{\eta}{\mu_1}$. Then there exists a positive constant $D^* = D^*(D_2, \Lambda)$ such that the system (3) has no non-constant positive solution provided* $\min\{D_2, 1\} \geq D^*$ *and $D_3 \geq D_3^*$.*

*Proof.* Let $\mathbf{u} = (u, v, w)^T$ be a positive solution of the system (3). Let $\overline{u} = |\Omega|^{-1} \int_\Omega u\,dx$, $\overline{v} = |\Omega|^{-1} \int_\Omega v\,dx$ and $\overline{w} = |\Omega|^{-1} \int_\Omega w\,dx$. Now, we multiply the first, second and third equations in (3) by $(u-\overline{u})$, $(v-\overline{v})$ and $(w-\overline{w})$, respectively, and integrate the system (3) over $\Omega$ by parts and then apply Green's first identity. Thus, we have

$$
\begin{aligned}
& I \equiv \int_\Omega \left\{ |\nabla u|^2 + D_2 |\nabla v|^2 + D_3 |\nabla w|^2 \right\} dx, \\
= \quad & \int_\Omega \Big\{ (u-\overline{u})(G_1(u,v,w) - G_1(\overline{u},\overline{v},\overline{w})) + (v-\overline{v})(G_2(u,v,w) - G_2(\overline{u},\overline{v},\overline{w})) \\
+ \quad & (w-\overline{w})(G_3(u,v,w) - G_3(\overline{u},\overline{v},\overline{w})) \Big\} dx, \\
= \quad & \int_\Omega \Big\{ (u-\overline{u})^2 \left( 1 - (u+\overline{u}) - \frac{\alpha v}{1+\kappa v} - \frac{\gamma w(\beta + \omega \overline{w})}{\chi_2} \right) + (v-\overline{v})^2 \left( \frac{\alpha \overline{u}}{\chi_1} - \delta - \frac{\theta \overline{w}}{\chi_3} \right) \\
+ \quad & (u-\overline{u})(v-\overline{v}) \left( \frac{\alpha v}{1+\kappa v} - \frac{\alpha \overline{u}}{\chi_1} \right) + (u-\overline{u})(w-\overline{w}) \left( \frac{\eta s_2 \overline{w}^2}{\chi_4} - \frac{\gamma \overline{u}(\beta + u)}{\chi_2} \right) \\
+ \quad & (w-\overline{w})(v-\overline{v}) \left( \frac{\eta s_3 \overline{w}^2}{\chi_4} - \frac{\theta v}{1+\sigma v} \right) + (w-\overline{w})^2 \left( \eta - \frac{\eta(\overline{w}+w)}{s_1 + s_2 u + s_3 v} \right), \\
\leq \quad & \int_\Omega \Big\{ (u-\overline{u})^2 + \alpha \overline{u}(v-\overline{v})^2 + |u-\overline{u}||v-\overline{v}| \left( \frac{\alpha v}{1+\kappa v} + \frac{\alpha \overline{u}}{\chi_1} \right) + (w-\overline{w})^2 \eta \\
+ \quad & |u-\overline{u}||w-\overline{w}| \left( \frac{\eta s_2 \overline{w}^2}{\chi_4} + \frac{\gamma \overline{u}(\beta+u)}{\chi_2} \right) + |w-\overline{w}||v-\overline{v}| \left( \frac{\eta s_3 \overline{w}^2}{\chi_4} + \frac{\theta v}{1+\sigma v} \right), \\
\leq \quad & \int_\Omega \Big\{ (u-\overline{u})^2 + \alpha \overline{u}(v-\overline{v})^2 + |u-\overline{u}||v-\overline{v}| \left( \overline{u} + \frac{1}{\kappa} \right) \alpha + (w-\overline{w})^2 \eta \\
+ \quad & |u-\overline{u}||w-\overline{w}| \left( \frac{\eta s_2 \overline{w}^2}{s_1^2} + \gamma \right) + |w-\overline{w}||v-\overline{v}| \left( \frac{\eta s_3 \overline{w}^2}{s_1^2} + \frac{\theta}{\sigma} \right), \\
\leq \quad & \int_\Omega \Big\{ (u-\overline{u})^2 + \alpha (v-\overline{v})^2 + |u-\overline{u}||v-\overline{v}| \left( 1 + \frac{1}{\kappa} \right) \alpha + (w-\overline{w})^2 \eta \\
+ \quad & |u-\overline{u}||w-\overline{w}| \left( \frac{\eta s_2 M_w^2}{s_1^2} + \gamma \right) + |w-\overline{w}||v-\overline{v}| \left( \frac{\eta s_3 M_w^2}{s_1^2} + \frac{\theta}{\sigma} \right) \Big\} dx \equiv I_1,
\end{aligned}
$$

where

$$
\begin{aligned}
\chi_1 &= (1+\kappa v)(1+\kappa \overline{v}), \ \chi_2 = (\beta + \overline{u} + \omega \overline{w})(\beta + u + \omega w), \ \chi_3 = (1+\sigma v)(1+\sigma \overline{v}), \\
\chi_4 &= (s_1 + s_2 u + s_3 v)(s_1 + s_2 \overline{u} + s_3 \overline{v}), \ M_w = \max_{\overline{\Omega}} w.
\end{aligned}
$$

For the positive constants $\xi_1 = \left(1+\frac{1}{\kappa}\right)\alpha$, $\xi_2 = \left(\frac{\eta s_3 M_w^2}{s_1^2} + \frac{\theta}{\sigma}\right)$, $\xi_3 = \left(\frac{\eta s_2 M_w^2}{s_1^2} + \gamma\right)$ and arbitrary positive constants $\epsilon_1, \epsilon_2, \epsilon_3$, the Young's inequality yields

$$|u-\overline{u}||v-\overline{v}| \leq \frac{\xi_1}{2\epsilon_1}(u-\overline{u})^2 + \frac{\alpha_1\epsilon_1}{2}(v-\overline{v})^2,$$
$$|v-\overline{v}||w-\overline{w}| \leq \frac{\xi_2}{2\epsilon_2}(v-\overline{v})^2 + \frac{\xi_2\epsilon_2}{2}(w-\overline{w})^2,$$
$$|u-\overline{u}||w-\overline{w}| \leq \frac{\xi_3}{\epsilon_3}(u-\overline{u})^2 + \frac{\xi_3\epsilon_3}{2}(w-\overline{w})^2.$$

Thus, we have

$$\begin{aligned} I \leq I_1 \quad &\leq \int_\Omega \left\{ \left(1+\frac{\xi_1}{2\epsilon_1}+\frac{\xi_3}{\epsilon_3}\right)(u-\overline{u})^2 + \left(\alpha + \frac{\xi_1\epsilon_1}{2} + \frac{\xi_2}{2\epsilon_2}\right)(v-\overline{v})^2 \right. \\ &+ \left. \left(\eta + \frac{\xi_2\epsilon_2}{2} + \frac{\xi_3\epsilon_3}{2}\right)(w-\overline{w})^2 \right\} dx \equiv I_2. \end{aligned} \tag{7}$$

Further, due to the Poincare inequality, we get

$$I \geq \int_\Omega \left\{ \mu_1(u-\overline{u})^2 + \mu_1 D_2(v-\overline{v})^2 + \mu_1 D_3(w-\overline{w})^2 \right\} dx \equiv I_3. \tag{8}$$

From (7) and (8), it follows that

$$I_2 \geq I_3. \tag{9}$$

Since $\mu_1 D_3^* > \eta$ by the assumption, we can find a sufficiently small $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ such that
$\mu_1 D_3^* \geq (\eta + \frac{\xi_2\epsilon_2}{2} + \frac{\xi_3\epsilon_3}{2})$. Let $D_{11}^* := \mu_1^{-1}\left(1+\frac{\xi_1}{2\epsilon_1}+\frac{\xi_3}{\epsilon_3}\right)$, $D_{21}^* := \mu_1^{-1}\left(\alpha + \frac{\xi_1\epsilon_1}{2} + \frac{\xi_2}{2\epsilon_2}\right)$ and $D^* = \max\{D_{11}^*, D_{21}^*\}$. Therefore, we conclude that $u = \overline{u} =$ constant, $v = \overline{v} =$ constant and $w = \overline{w} =$ constant provided $\min\{D_2, 1\} \geq D^*$ and $D_3 \geq D_3^*$. □

**3.3. Existence of non-constant positive steady state.** The main aim of this section is to discuss the existence of non-constant positive solutions to the system (3) by using Leray-Schauder Theorem. Theorem (3.3) implies that when the assumptions of the theorem holds then the system (3) will not have non-constant positive solution.

Now, define $\mathbf{X}^+ = \{\mathbf{u} \in \mathbf{X} | u > 0, v > 0, w > 0\}$ on $\overline{\Omega}$, $B(C) = \{\mathbf{u} \in \mathbf{X} | C^{-1} < u, v, w < C\}$ on $\overline{\Omega}$, where $C$ is a positive constant in which its existence is ensured by theorems (3.1) and (3.2). Let $\mathbf{D} = diag(1, D_2, D_3)$. Thus, the system (3) is equivalent with

$$\begin{cases} -\mathbf{D}\Delta\mathbf{u} = \mathbf{G}(\mathbf{u}), & x \in \Omega, \\ \mathbf{u}_\nu = 0, & x \in \partial\Omega. \end{cases} \tag{10}$$

$\mathbf{u}$ is a positive solution of (10) if and only if

$$\varphi(\mathbf{u}) \triangleq \mathbf{u} - (\mathbf{I} - \Delta)^{-1}\{\mathbf{D}^{-1}\mathbf{G}(\mathbf{u}) + \mathbf{u}\} = 0 \text{ in } \mathbf{X}^+,$$

where $\mathbf{I}$ is the identity map from $C^1(\Omega)$ to itself and $(\mathbf{I} - \Delta)^{-1}$ is the inverse of $\mathbf{I} - \Delta$ in $\mathbf{X}$ subject to Neumann boundary condition. It can be noticed that the Leray-Schauder degree $deg(\varphi(.), 0, B)$ is well defined if $\varphi(\mathbf{u}) \neq 0$ for any $\mathbf{u} \in \partial B(C)$. Direct computation gives

$$D_{\mathbf{u}}\varphi(\widetilde{\mathbf{u}}) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\{\mathbf{D}^{-1}\mathbf{G}_{\mathbf{u}}(\widetilde{\mathbf{u}}) + \mathbf{I}\}.$$

Note that, $\lambda$ is an eigenvalue of the matrix $D_{\mathbf{u}}\varphi(\widetilde{\mathbf{u}})$ on $\mathbf{X}_i$ if and only if it is an eigenvalue of the matrix $\frac{1}{1+\mu_i}\left\{\mu_i\mathbf{I} - \mathbf{D}^{-1}\mathbf{G}_{\mathbf{u}}(\widetilde{\mathbf{u}})\right\}$. Thus, the matrix $D_{\mathbf{u}}\varphi(\widetilde{\mathbf{u}})$ is invertible if and only if $\frac{1}{1+\mu_i}\left\{\mu_i\mathbf{I} - \mathbf{D}^{-1}\mathbf{G}_{\mathbf{u}}(\widetilde{\mathbf{u}})\right\}$ is non-singular for any $i \geq 1$.

Let

$$\Psi(\mu) \triangleq det\left(\mu\mathbf{I} - \mathbf{D}^{-1}\mathbf{G}_{\mathbf{u}}(\widetilde{\mathbf{u}})\right), \tag{11}$$

$$\psi(\mu) \triangleq det\left(\mu\mathbf{D} - \mathbf{G}_{\mathbf{u}}(\widetilde{\mathbf{u}})\right). \tag{12}$$

Then,

$$\Psi(\mu) = \frac{1}{D_2D_3}\psi(\mu). \tag{13}$$

Note that the number of negative eigenvalues $\mu$ of $D_{\mathbf{u}}\varphi(\widetilde{\mathbf{u}})$ on $\mathbf{X}_i$ is odd if and only if $\Psi(\mu_i, 0) < 0$.

**Proposition 3.4.** *Suppose $\Psi(\mu_i) \neq 0$; $i \geq 1$. Let $m(\mu_i)$ be the multiplicity of the eigenvalue $\mu_i$ and $\rho = \sum_{i\geq 1, \Psi(\mu_i)<0} m(\mu_i)$. Then, $index(\varphi(.), \widetilde{\mathbf{u}}) = (-1)^{\rho}$.*

The above proposition indicates that the sign of $\Psi(\mu_i)$ has a paramount importance for calculating the value of $index(\varphi(.), \widetilde{\mathbf{u}})$.

Now,

$$\psi(\mu) = \psi_1(D_3)\mu^3 + \psi_2(D_3)\mu^2 + \psi_3(D_3)\mu - det\left(\mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}})\right), \tag{14}$$

where

$$\begin{aligned}\psi_1(D_3) &= D_2D_3,\ \psi_2(D_3) = \eta D_2 - (a_{22} + a_{11}D_2)D_3,\\ \psi_3(D_3) &= -\eta(s_3a_{23} + a_{22}) - \eta(s_2a_{13} + a_{11})D_2 + (-a_{12}a_{21} + a_{11}a_{22})D_3\end{aligned}$$

and

$$a_{11} = \tilde{u}\left(\frac{\gamma\tilde{w}}{(\beta + \tilde{u} + \omega\tilde{w})^2} - 1\right),\ a_{12} = -\frac{\alpha\tilde{u}}{(\kappa\tilde{v} + 1)^2},\ a_{13} = -\frac{\gamma\tilde{u}(\beta + \tilde{u})}{(\beta + \tilde{u} + \omega\tilde{w})^2},$$
$$a_{21} = \frac{\alpha\tilde{v}}{\kappa\tilde{v} + 1},\ a_{22} = -\frac{\alpha\kappa\tilde{u}}{(\kappa\tilde{v} + 1)^2} + \frac{\theta\sigma\tilde{w}}{(\sigma\tilde{v} + 1)^2},\ a_{23} = -\frac{\theta\tilde{v}}{\sigma\tilde{v} + 1}.$$

Let $\overline{\mu}_1, \overline{\mu}_2, \overline{\mu}_3$ be the three roots of $\psi(\mu) = 0$. Then, $\overline{\mu}_1\overline{\mu}_2\overline{\mu}_3 = det\left(\mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}})\right)$, where

$$det\left(\mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}})\right) = -\eta(a_{11}a_{22} - a_{12}a_{21} + s_3(a_{11}a_{23} - a_{13}a_{21}) + s_2(a_{13}a_{22} - a_{12}a_{23})).$$

One can see that $det\left(\mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}})\right) < 0$ if

$$a_{11}a_{22} - a_{12}a_{21} > 0,\ a_{11}a_{23} - a_{13}a_{21} > 0,\ a_{13}a_{22} - a_{12}a_{23} > 0. \tag{15}$$

Hence, since $\psi_1 > 0$, one of $\overline{\mu}_1, \overline{\mu}_2, \overline{\mu}_3$ is real and negative, and the product of the other two is positive.

For a sufficiently large $D_3$, i.e $D_3 \to \infty$, we have

$$\lim_{D_3\to\infty}\left\{\frac{\Psi(\mu)}{D_3}\right\} = \mu(D_2\mu^2 - (a_{22} + a_{11}D_2)\mu + a_{11}a_{22} - a_{12}a_{21}).$$

Thus, we have the following result.

**Proposition 3.5.** *Assume that* $a_{11} > 0,\ a_{22} + a_{11}D_2 > 0$, *the constant positive equilibrium point* $\tilde{\mathbf{u}}$ *exists and (15) hold. Then, there exists a positive constant* $\tilde{D}$ *such that when* $D_3 \geq \tilde{D}$, *the three roots* $\overline{\mu}_1(D_3), \overline{\mu}_2(D_3), \overline{\mu}_3(D_3)$ *of* $\psi(\mu) = 0$ *are all real and satisfy*

$$\begin{cases} \lim\limits_{D_3\to\infty} \overline{\mu}_1(D_3) = \dfrac{(a_{22} + a_{11}D_2) - \sqrt{(a_{22} + a_{11}D_2)^2 - 4D_2(a_{11}a_{22} - a_{12}a_{21})}}{2D_2} \\ \qquad\qquad\quad \equiv \widetilde{\mu}_1 > 0, \\ \lim\limits_{D_3\to\infty} \overline{\mu}_2(D_3) = \dfrac{(a_{22} + a_{11}D_2) + \sqrt{(a_{22} + a_{11}D_2)^2 - 4D_2(a_{11}a_{22} - a_{12}a_{21})}}{2D_2} \\ \qquad\qquad\quad \equiv \widetilde{\mu}_2 > 0, \\ \lim\limits_{D_3\to\infty} \overline{\mu}_3(D_3) = 0. \end{cases} \tag{16}$$

*Moreover, we have*

$$\begin{cases} -\infty < \overline{\mu}_3(D_3) < 0 < \overline{\mu}_1(D_3) < \overline{\mu}_2(D_3), \\ \psi(\mu) < 0, \ \text{when } \mu \in (-\infty, \overline{\mu}_3(D_3)) \cup (\overline{\mu}_1(D_3), \overline{\mu}_2(D_3)), \\ \psi(\mu) > 0, \ \text{when } \mu \in (\overline{\mu}_3(D_3), \overline{\mu}_1(D_3)) \cup (\overline{\mu}_2(D_3), \infty). \end{cases} \tag{17}$$

The following theorem proves the global existence of non-constant positive solution to the system (3) for sufficiently large $D_3$ while other parameters are fixed.

**Theorem 3.6.** *Assume that the parameters* $\Lambda$ *and* $D_2$ *are fixed, the constant positive equilibrium point* $\tilde{\mathbf{u}}$ *exists,* $a_{11} > 0,\ a_{22} + a_{11}D_2 > 0$ *and (15) hold. Let* $\widetilde{\mu}_1$ *and* $\widetilde{\mu}_2$ *be given by the limit (16). If* $\widetilde{\mu}_1 \in (\mu_n, \mu_{n+1})$, $\widetilde{\mu}_2 \in (\mu_p, \mu_{p+1})$ *for some* $p \geq n \geq 1$ *and the sum* $\rho_p = \sum_{i=n+1}^{p} m(\mu_i)$ *is odd then there exists a positive constant* $\tilde{D}_{21}$ *such that, if* $D_3 \geq \tilde{D}_{21}$, *then the system (3) admits at least one non-constant positive solution.*

*Proof.* From proposition (3.5), it follows that there exists a positive constant $\tilde{D}$ such that, when $D_3 \geq \tilde{D}$, (17) holds and

$$\overline{\mu}_1 \in (\mu_n, \mu_{n+1}), \ \overline{\mu}_2 \in (\mu_p, \mu_{p+1}). \tag{18}$$

Now we prove that, for any $D_3 \geq \tilde{D}_{21}$, the system (3) admits at least one non-constant positive solution. Assume that the assertion is not true for some

$D_3 = \tilde{D}_3 \geq \tilde{D}_{21}$. By using the homotopy invariance of the topological degree, we can derive a contradiction in the sequel.

Fix $D_3 = \tilde{D}_3$, $D_3^* = \frac{\eta}{\mu_1}$. Thus, by Theorem (3.3), we get a positive constant $D^* = D^*(D_2, \Lambda)$. Fix $\hat{D}_3 \geq D_3^*$, $\hat{D}_2 \geq \max\{D^*, D_2\}$. For $t \in [0,1]$, define $\mathbf{D}(t) = diag(1, D_2(t), D_3(t))$ with $D_i(t) = tD_i + (1-t)\hat{D}_i, i = 2, 3$ and consider the problem

$$\begin{cases} -\mathbf{D}(t)\Delta \mathbf{u} = \mathbf{G}(\mathbf{u}), & x \in \Omega, \\ \quad \mathbf{u}_\nu = 0, & x \in \partial\Omega. \end{cases} \tag{19}$$

Thus, $\mathbf{u}$ is a non-constant positive solution of the system (3) if and only if it is a positive solution of (19) for $t = 1$. Clearly, $\tilde{\mathbf{u}}$ is the unique constant positive solution of (19) for any $0 \leq t \leq 1$. For any $0 \leq t \leq 1$, $\mathbf{u}$ is a positive solution of (3) if and only if it is a solution of the following problemma

$$\varphi(t; \mathbf{u}) \triangleq \mathbf{u} - (\mathbf{I} - \Delta)^{-1} \{\mathbf{D}^{-1}(t)\mathbf{G}(\mathbf{u}) + \mathbf{u}\} = 0 \text{ in } \mathbf{X}^+.$$

It is clear that $\varphi(1; \mathbf{u}) = \varphi(0; \mathbf{u})$. From theorem (3.3) it follows that $\varphi(0; \mathbf{u}) = 0$ has only the positive solution $\tilde{\mathbf{u}}$ in $\mathbf{X}^+$. It is easy to see that

$$D_{\mathbf{u}}\varphi(t; \tilde{\mathbf{u}}) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1} \{\mathbf{D}^{-1}(t)\mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}}) + \mathbf{I}\}.$$

In particular,

$$\begin{aligned} D_{\mathbf{u}}\varphi(0; \tilde{\mathbf{u}}) &= \mathbf{I} - (\mathbf{I} - \Delta)^{-1} \{\widehat{\mathbf{D}}^{-1}\mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}}) + \mathbf{I}\}, \\ D_{\mathbf{u}}\varphi(1; \tilde{\mathbf{u}}) &= \mathbf{I} - (\mathbf{I} - \Delta)^{-1} \{\mathbf{D}^{-1}\mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}}) + \mathbf{I}\} = D_{\mathbf{u}}\varphi(\tilde{\mathbf{u}}). \end{aligned}$$

where $\widehat{\mathbf{D}} = diag(1, \hat{D}_2, \hat{D}_3)$.

For $t = 1$, by (17), (18) and (13), we have

$$\begin{cases} \Psi(\mu_0) = \Psi(0) > 0, \\ \Psi(\mu_i) > 0, \ 1 \leq i \leq n, \\ \Psi(\mu_i) < 0, \ n+1 \geq i \leq p, \\ \Psi(\mu_{i+1}) > 0, \ i \geq p+1. \end{cases}$$

Hence, zero is not the eigenvalue of the matrix $\mu_i\mathbf{I} - \mathbf{D}^{-1}\mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}})$ for all $i \geq 0$ and $\displaystyle\sum_{i \geq 0, \Psi(\mu_i) < 0} m(\mu_i) = \sum_{i}^{p} m(\mu_i) = \rho_p$, which is odd. Then proposition (3.4) yields

$$index(\varphi(1; .), \tilde{\mathbf{u}}) = (-1)^\rho = (-1)^{\rho_p} = -1. \tag{20}$$

Similarly, it is possible to prove

$$index(\varphi(0; .), \tilde{\mathbf{u}}) = (-1)^\rho = (-1)^0 = 1. \tag{21}$$

In view of theorems (3.1) and (3.2), there exists a positive constant $C$ such that, for all $0 \leq t \leq 1$, the positive solutions of (19) satisfy $C^{-1} < u, v, w < C$ and hence $\varphi(t;\mathbf{u}) \neq 0$ on $\partial B(C)$. By the homotopy invariance of the topological degree, we have

$$deg(\varphi(1;.), 0, B(C)) = deg(\varphi(0;.), 0, B(C)). \tag{22}$$

Since both equations $\varphi(1;\mathbf{u}) = 0$ and $\varphi(0;\mathbf{u}) = 0$ have the unique positive solution $\tilde{\mathbf{u}}$ in $B(C)$, by (20) and (21), we have

$$\begin{aligned} deg(\varphi(0;.), 0, B(C)) &= index(\varphi(0;.), \tilde{\mathbf{u}}) = 1, \\ deg(\varphi(1;.), 0, B(C)) &= index(\varphi(1;.), \tilde{\mathbf{u}}) = -1. \end{aligned}$$

This contradicts (22). Hence the proof is complete. □

## 4. Conclusion

In this paper, a diffusive predator-prey system with disease in prey, Beddington-DeAngelis functional response and the modified Leslie-Gower type predator dynamics under homogeneous Neumann boundary condition was investigated. In the context, we have shown the global attractor and persistence nature of the system. In addition, under a certain condition the non-constant positive steady state of the system (2) does not exist and hence pattern formation is not possible (c.f. Theorem (3.3)). On the other hand, under a suitable condition and a sufficiently large diffusion constant ratio $D_3$, non-constant positive steady state of the system (2) exists as stated and proved in Theorem (3.6). As a result interesting Turing patterns, which are induced by large diffusion coefficient ratio $D_3$, can occur.

From the qualitative analysis of the model, one can observe that our model can be used to describe any dynamical interaction between prey and predator populations with a communicable/infectious disease in the prey population. For example, the model can be used to describe a phytoplankton-zooplankton system where the phytoplankton population is affected with an infection disease. The model can also be used to describe the macro and micro parasitic infections with constant predator for the hyper-trophic plankton fish system.

## Appendix

$$
\begin{aligned}
A_5 &= \kappa^2(\delta\sigma+\theta s_3)(\delta\kappa\sigma(s_2\omega+1)+s_3(\alpha\sigma\omega+\theta\kappa))\\
A_4 &= \kappa^3(\delta\sigma(\theta(s_2\omega(s_1+s_2)+2s_1-s_2(\beta+\gamma s_2-1))+2\delta(s_2\omega+1))+\theta s_3\delta(s_2\omega+2))\\
&+ \kappa^2\left(s_3(\sigma(\alpha(\beta-1)\theta+6\delta\theta+2\alpha\omega(\delta+\theta s_1))+\theta s_2(\sigma\omega(\alpha+3\delta)-\alpha(\gamma\sigma+\theta))))\right)\\
&+ \kappa^2\left(\delta\sigma\left(\sigma(3\delta+\alpha(\beta+s_1\omega-1))-\alpha\theta s_2^2\omega+s_2(3\delta\sigma\omega-\alpha(-\gamma\sigma+\theta+\sigma\omega))\right)\right)\\
&+ \kappa\alpha\sigma\left(\alpha\delta\sigma(s_2\omega+1)+s_3(2\delta\sigma\omega+\alpha(\gamma\sigma+\theta-\omega(\sigma+\theta s_2)))+2\theta s_3^2\omega\right)\\
&+ \kappa^3\theta s_3\theta(2s_1-\beta s_2+s_2)+\kappa^2\theta s_3^2(\alpha\omega+3\theta)+\alpha^3\sigma^2 s_3\omega\\
A_3 &= \kappa^3\left(\delta\theta(s_2\omega(s_1+s_2)+2s_1-s_2(\beta+\gamma s_2-1))+\theta^2(s_1+s_2)(s_1-\beta s_2)+\delta^2(s_2\omega+1)\right)\\
&+ \kappa^2(\sigma(\alpha(\beta+s_1\omega-1)(2\delta+\theta s_1)+6\delta(\delta+\theta s_1))+\theta s_2^2(\alpha(\beta\theta-\delta\omega)+3\delta\sigma(\omega-\gamma)))\\
&+ \kappa^2(s_3(\alpha\omega(\delta+\theta(2s_1+s_2))+3\theta(\theta(2s_1-\beta s_2+s_2)+\delta(s_2\omega+2))+\alpha\theta(\beta-\gamma s_2-1)))\\
&+ \kappa^2(s_2(3\delta\sigma(-\beta\theta+2\delta\omega+\theta+\theta s_1\omega)-\alpha(\delta(-2\gamma\sigma+\theta+2\sigma\omega)+\theta(s_1(\gamma\sigma+\theta-\sigma\omega)-2\beta\sigma))))\\
&+ \kappa(4\alpha\sigma s_3\omega(\delta+\theta s_1)-2\alpha\delta\theta\sigma s_2^2\omega+\beta\left(2\alpha\delta\sigma^2+\alpha^2(-\sigma)(\sigma+2\theta s_2)+2\alpha\theta\sigma s_3\right)+2\alpha\delta\sigma^2 s_1\omega)\\
&+ \kappa(2\alpha^2\delta\sigma(s_2\omega+1)+3\delta^2\sigma^2(s_2\omega+1)-2\alpha\delta\sigma^2-2\alpha\theta\sigma s_3+3\delta\theta\sigma s_3(s_2\omega+2)+\theta s_3^2(2\alpha\omega+3\theta))\\
&+ \kappa(\alpha^2\sigma s_1(\gamma\sigma+\theta-\omega(\sigma+\theta s_2))-2\alpha\delta\sigma s_2(\sigma(\omega-\gamma)+\theta)+\alpha^2 s_3(2\sigma(\gamma-\omega)+\theta+\theta(-s_2)\omega))\\
&+ \kappa(-2\alpha\theta s_2 s_3(\gamma\sigma+\theta-\sigma\omega))+s_3\left(2\alpha^3\sigma\omega+\alpha^2\sigma^2(\gamma-\omega)+\alpha\delta\sigma^2\omega+\alpha^2\theta\sigma(1-s_2\omega)\right)\\
&+ \alpha^3\beta\sigma^2+\alpha^2\delta\sigma^2+\alpha^3\sigma^2 s_1\omega+\alpha^2\delta\sigma^2 s_2\omega+\alpha\theta\sigma s_3^2\omega\\
A_2 &= \kappa^2(\alpha(\delta(\beta+s_1\omega+\gamma s_2-s_2\omega-1)+\theta(\beta s_1-\gamma s_1 s_2+s_1\omega(s_1+s_2)-s_1+2\beta s_2)))\\
&+ \kappa(3\left(\delta\theta(s_2\omega(s_1+s_2)+2s_1-s_2(\beta+\gamma s_2-1))+\theta^2(s_1+s_2)(s_1-\beta s_2)+\delta^2(s_2\omega+1)\right))\\
&+ 2\kappa\alpha(\sigma(\beta+s_1\omega-1)(2\delta+\theta s_1)-s_2(\delta(-2\gamma\sigma+\theta+2\sigma\omega)+\theta(s_1(\gamma\sigma+\theta-\sigma\omega)-2\beta\sigma)))\\
&+ \kappa(2\alpha\theta s_2^2(\beta\theta-\delta\omega)+\alpha^2(\delta+s_1(2\gamma\sigma+\theta-2\sigma\omega+\theta(-s_2)\omega)-2\beta(\sigma+\theta s_2)+\delta s_2\omega))\\
&+ \kappa(3\delta\sigma(\theta(s_2\omega(s_1+s_2)+2s_1-s_2(\beta+\gamma s_2-1))+2\delta(s_2\omega+1))+s_3\alpha^2(\gamma-\omega))\\
&+ \kappa s_3(2\alpha(\delta\omega+\theta(\beta+2s_1\omega-\gamma s_2+s_2\omega-1))+3\theta(\theta(2s_1-\beta s_2+s_2)+\delta(s_2\omega+2)))\\
&+ s_3\left(\alpha\sigma((\beta-1)\theta+2\omega(\delta+\theta s_1))+\alpha^2(2\sigma(\gamma-\omega)+\theta+\theta(-s_2)\omega)-\alpha\theta s_2(\gamma\sigma+\theta-\sigma\omega)\right)\\
&+ s_3(\alpha^3\omega+\delta\theta\sigma(s_2\omega+2))-\sigma\alpha\delta\left(-\sigma(\beta+s_1\omega-1)+\theta s_2^2\omega+s_2(\sigma(\omega-\gamma)+\theta)\right)\\
&+ \theta s_3^2(\alpha\omega+\theta)+\sigma\alpha^2(s_1(\gamma\sigma+\theta-\omega(\sigma+\theta s_2))-\beta(\sigma+2\theta s_2)+2\delta(s_2\omega+1))\\
&+ \sigma(2\alpha^3(\beta+s_1\omega)+\delta^2\sigma(s_2\omega+1))\\
A_1 &= \kappa(2\alpha(\delta(\beta+s_1\omega+\gamma s_2-s_2\omega-1)+\theta(\beta s_1-\gamma s_1 s_2+s_1\omega(s_1+s_2)-s_1+2\beta s_2)))\\
&+ \kappa(\delta\theta(s_2\omega(3(s_1+s_2)+s_3)+6s_1-3s_2(\beta+\gamma s_2-1)+2s_3)+3\delta^2(s_2\omega+1))\\
&+ \kappa(\alpha^2(\beta+s_1(\omega-\gamma))+\theta^2\left(3s_1^2-\beta s_2(3(s_1+s_2)+s_3)+3s_1 s_2+2s_1 s_3+s_2 s_3\right))\\
&+ \alpha(\sigma(\beta+s_1\omega-1)(2\delta+\theta s_1)+s_3\omega(\delta+\theta(2s_1+s_2))+\theta s_2^2(\beta\theta-\delta\omega))\\
&+ \alpha(-s_2(\delta(-2\gamma\sigma+\theta+2\sigma\omega)+\theta(s_1(\gamma\sigma+\theta-\sigma\omega)-2\beta\sigma)))+\alpha^3(\beta+s_1\omega)\\
&+ \alpha^2(\delta+\theta s_1-\omega(2\sigma s_1+\theta s_1 s_2-\delta s_2+s_3)+\gamma(2\sigma s_1+s_3)-2\beta(\sigma+\theta s_2))\\
&+ \delta\sigma(\theta(s_2\omega(s_1+s_2)+2s_1-s_2(\beta+\gamma s_2-1))+2\delta(s_2\omega+1))+\alpha\theta s_3(\beta-\gamma s_2-1)\\
A_0 &= \alpha^2(-\beta-s_1(\omega-\gamma))+\alpha(\delta(\beta+s_1\omega+\gamma s_2-s_2\omega-1)+\theta(\beta s_1-\gamma s_1 s_2)\\
&+ \delta\theta(s_2\omega(s_1+s_2)+2s_1-s_2(\beta+\gamma s_2-1))+\theta^2(s_1+s_2)(s_1-\beta s_2)\\
&+ \delta^2(s_2\omega+1)+\theta\alpha(s_1\omega(s_1+s_2)-s_1+2\beta s_2)
\end{aligned}
$$

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