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QUASI HEMI-SLANT SUBMANIFOLDS OF KENMOTSU MANIFOLDS

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Abstract. The main purpose of the present paper is to introduce a brief analysis on some properties of quasi hemi-slant submanifolds of Kenmotsu manifolds. After discussing the introduction and some preliminaries about the Kenmotsu manifold, we worked out some important results in the direction of integrability of the distributions of quasi hemi-slant submanifolds of Kenmotsu manifolds. Afterward, we investigate the conditions for quasi hemi-slant submanifolds of a Kenmotsu manifold to be totally geodesic and later we provide some non-trivial examples to validate the existence of such submanifolds.

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1. Introduction

In 1972, the notion of Kenmotsu manifold was introduced by K. Kenmotsu [14]. After that several works on this manifold have been done by various authors such as [11, 12, 13, 20]. An interesting topic in the differential geometry is the theory of submanifolds in space endowed with additional structures [6, 7]. In [6], B. Y. Chen initiated the study of slant manifolds of an almost Hermitian manifold as a natural generalization of both holomorphic and totally real submanifolds. N. Papaghiuc have studied semi-invariant submanifolds in a Kenmotsu manifold [18, 19]. In [18] he studied the geometry of leaves on a semi-invariant $\xi^{\perp}$- submanifolds in a Kenmotsu manifolds. Afterwords, N. Papaghiuc introduced a class of submanifolds in an almost Hermitian manifold, called the semi-slant submanifolds, which includes the class of proper CR-submanifolds and slant submanifolds.

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A. Carriazo and others [4, 5] proposed the idea of bi-slant submanifold under the name anti-slant submanifold. Inspite of the fact that these bi-slant submanifolds are proposed as hemi-slant submanifolds by B. Sahin in [24](see also [17, 21, 22, 29]). Since then, this interesting subject has been studied broadly by several geometers during last two decades (for instance we refer [27, 28, 31]). A. Lotta [15] introduced the notion of slant immersion of a Riemannian manifold into an almost contact metric manifold. Further, the slant submanifolds were generalized as semi-slant submanifolds, pseudo-slant submanifolds, bi-slant submanifolds, and hemi-slant submanifolds etc. in different kinds of differentiable manifolds (see, [1, 2, 3, 8, 9, 10, 16, 21, 22, 23, 24, 25, 26]).

The paper is organized in 6 sections such that section 2 contains the basic definitions, formulas and some useful results on Kenmotsu manifolds. Section 3 consists the definition of quasi hemi-slant submanifolds of a Kenmotsu manifold and some important lemmas have been proved. Section 4 deals with the necessary and sufficient conditions for the distributions to be integrable. In section 5, we investigate the totally geodesic property of the distributions considered in the definition of quasi hemi slant submanifolds of a Kenmotsu manifold. Finally, in the last section we provide some examples of such submanifolds.

2. Preliminaries

Let $\mathcal{N}(\phi, \xi, \eta, g)$ be an almost contact manifold of dimension $n = 2m + 1$ admitting ϕ as a tensor field of (1,1) type, a vector field ξ and a 1-form η satisfying the following conditions:

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{1}$$

where I is an identity map defined on $\mathcal{TN}$. Also, on an almost contact manifold there exists a Riemannian metric g which satisfies the condition:

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V), \tag{2}$$

for $U, V \in \Gamma(\mathcal{TN})$, where $\Gamma(\mathcal{TN})$ represents the Lie algebra of vector fields on $\mathcal{N}$. A manifold $\mathcal{N}$ together with the structure (ϕ, ξ, η, g) is called an almost contact metric manifold [30].

Due to the above equations (1) and (2), we obtain following consequences:

$$g(U, \xi) = \eta(U), \tag{3}$$

and

$$g(\phi U, V) = -g(U, \phi V), \tag{4}$$

for all vector fields $U, V \in \Gamma(\mathcal{TN})$.
Now if

$$(\overline{\nabla}_U \phi)(V) = g(\phi U, V)\xi - \eta(V)\phi U, \tag{5}$$

and

$$\overline{\nabla}_U \xi = U - \eta(U)\xi \tag{6}$$

for any U, V tangent to $\mathcal{N}$, where $\overline{\nabla}$ is the Levi-civita connection, then $(\mathcal{N}, \phi, \xi, \eta, g)$ is called a Kenmotsu manifold.
In an $n-$dimensional Kenmotsu manifold, the following relations hold [14]:

$$
\begin{aligned}
(\overline{\nabla}_U \eta)V &= g(U,V) - \eta(U)\eta(V),\\
S(U,\xi) &= -(n-1)\eta(U),\\
R(U,V)\xi &= \eta(U)V - \eta(V)U,\\
\eta(R(U,V),Z) &= -g(V,Z)\eta(U) + g(U,Z)\eta(V),\\
Q\xi &= -(n-1)\xi,\\
R(\xi,U)V &= -R(U,\xi)V = \eta(V)U - g(U,V)\xi,
\end{aligned}
$$

where S is the Ricci tensor and R is the Riemannian tensor of the manifold.
Let $\mathcal{M}$ be a submanifold of a Kenmotsu manifold $\mathcal{N}$ with structure (ϕ, ξ, η, g) and assume that ∇ represents the induced connection on the tangent bundle $\mathcal{TM}$, $\nabla^{\perp}$ represents the induced connection on the normal bundle $\mathcal{T}^{\perp}\mathcal{M}$ of $\mathcal{M}$ and we also denote g as induced Riemannian metric on $\mathcal{M}$ throughout this paper.
The Gauss and Weingarten equations are given, respectively by [6]

$$\overline{\nabla}_U V = \nabla_U V + h(U,V), \tag{7}$$

and

$$\overline{\nabla}_U N = -A_N U + \nabla^{\perp}_U N \tag{8}$$

for any vector fields $U, V \in \Gamma(\mathcal{TM})$ and $N \in \Gamma(\mathcal{T}^{\perp}\mathcal{M})$. Also, the relation between second fundamental form h and shape operator A_N is given by

$$g(h(U,V),N) = g(A_N U, V) \tag{9}$$

for any vector fields $U, V \in \Gamma(\mathcal{TM})$ and $N \in \Gamma(\mathcal{T}^{\perp}\mathcal{M})$.
The mean curvature vector is denoted and defined by the following equation

$$\mathcal{H} = \frac{1}{m}\sum_{j=1}^{m} h(e_j, e_j),$$

where m is the dimension of submanifold $\mathcal{M}$ and $\{e_j\}_{j=1}^{m}$ is the local orthonormal frame defined on $\mathcal{M}$.

For any $U \in \Gamma(\mathcal{TM})$ and $N \in \Gamma(\mathcal{T}^{\perp}\mathcal{M})$, we have the following conditions:

$$\phi U = \nu U + \omega U, \tag{10}$$

and

$$\phi N = FN + GN, \tag{11}$$

where νU and FN represents the tangential components of ϕU and ϕN, respectively; and ωU and GN are normal components of ϕU and ϕN, respectively.

The covariant derivatives of the tensor fields ν, ω, F and G are defined as follows [21]:

$$(\overline{\nabla}_U \nu)V = \nabla_U \nu V - \nu \nabla_U V, \tag{12}$$

$$(\overline{\nabla}_U \omega)V = \nabla^{\perp}_U \omega V - \omega \nabla_U V, \tag{13}$$

$$(\overline{\nabla}_U F)N = \nabla_U FN - F\nabla^{\perp}_U N, \tag{14}$$

$$(\overline{\nabla}_U G)N = \nabla^{\perp}_U GN - G\nabla^{\perp}_U N \tag{15}$$

for any $U, V \in \Gamma(\mathcal{TM})$ and $N \in \Gamma(\mathcal{T}^{\perp}\mathcal{M})$. If ω is identically zero, i.e., $\phi U \in \Gamma(\mathcal{TM})$ for any $U \in \Gamma(\mathcal{TM})$, then submanifold $\mathcal{M}$ is called an *invariant submanifold*. Moreover, if ν is identically zero, i.e., $\phi U \in \Gamma(\mathcal{T}^{\perp}\mathcal{M})$ for any $U \in \Gamma(\mathcal{TM})$, then submanifold $\mathcal{M}$ is known as *anti-invariant submanifold.*

3. Quasi hemi-slant submanifolds of Kenmotsu manifolds

In the current section, we study and define the quasi hemi-slant submanifolds of Kenmotsu manifolds and we acquire the necessary and sufficient conditions for the distributions associated with the definition of such submanifolds to be integrable.

Definition 3.1. A submanifold $\mathcal{M}$ of a Kenmotsu manifold $\mathcal{N}$, is said to be a quasi hemi-slant submanifold if there exists three orthogonal distributions D, D^{θ} and $D^{\perp}$ of $\mathcal{M}$, at the point $p \in M$ satisfying the following properties [17]:
(1) $\mathcal{TM}$ possess the following orthogonal direct decomposition

$$\mathcal{TM} = D \oplus D^{\theta} \oplus D^{\perp} \oplus < \xi >,$$

where $< \xi >$ denotes the distribution spanned by ξ,
(2) D is the invariant distribution, i.e., $\phi D = D$,
(3) D^{θ} is the slant distribution with slant angle θ,
(4) $D^{\perp}$ is the $\phi-$ anti-invariant distribution, i.e., $\phi D^{\perp} \subset \mathcal{T}^{\perp}\mathcal{M}$.

Consequently, θ is called as quasi hemi-slant angle of $\mathcal{M}$. Now, consider m, m^{θ} and $m^{\perp}$ be the dimension of the distributions D, D^{θ} and $D^{\perp}$, respectively and then we see the following cases:

- $\mathcal{M}$ is hemi-slant submanifold if $m = 0$, $m^{\theta} \neq 0$ and $m^{\perp} \neq 0$,
- $\mathcal{M}$ is semi-invariant submanifold if $m \neq 0$, $m^{\theta} = 0$, and $m^{\perp} \neq 0$,
- $\mathcal{M}$ is semi-slant submanifold if $m \neq 0$, $m^{\theta} \neq 0$ and $m^{\perp} = 0$,
- we may call $\mathcal{M}$ as proper quasi hemi-slant submanifold if $m \neq 0, m^{\perp} \neq 0$ and $m^{\theta} \neq 0$ with $\theta \neq 0, \frac{\pi}{2}$.

Hence, one can easily observe that the notion of quasi hemi-slant submanifold is a generalization of invariant, anti-invariant, semi-invariant, slant, semi-slant and hemi-slant submanifolds.

Let $\mathcal{M}$ be a quasi hemi-slant submanifold of Kenmotsu manifold $\mathcal{N}$. Suppose that P, P^{θ} and $P^{\perp}$ be the projections of $U \in \Gamma(\mathcal{TM})$ on the distributions D, D^{θ} and $D^{\perp}$, respectively. Then we have the following conditions:

$$X = PX + P^{\theta}X + P^{\perp}X + \eta(X)\xi, \tag{16}$$

for any $X \in \Gamma(\mathcal{TM})$.
Now in view of (10), (16) takes the form

$$\phi X = \nu PX + \omega PX + \nu P^\theta X + \omega P^\theta X + \nu P^\perp X + \omega P^\perp X.$$

But since, $\phi D = D$ and $\phi D^\perp \subset \mathcal{T}^\perp\mathcal{M}$, we have $wPX = 0$ and $\nu P^\theta X = 0$.
Hence, we have

$$\phi X = \nu PX + \nu P^\theta X + \omega P^\theta X + \omega P^\perp X. \tag{17}$$

Since $\phi D = D$, we have $\omega PX = 0$. Therefore, we get

$$\phi X = \nu PX + \nu P_1 X + \omega P_1 X + \nu P_2 X + \omega P_2 X.$$

So, it is obvious that

$$\nu X = \nu PX + \nu P^\theta X,$$

and

$$\omega X = \omega P^\theta X + \omega P^\perp X$$

which implies that

$$\phi(\mathcal{TM}) = D \oplus \nu D^\theta \oplus \omega D^\theta \oplus \omega D^\perp, \tag{18}$$

and

$$\mathcal{T}^\perp\mathcal{M} = \omega D^\theta \oplus \omega D^\perp \oplus \mu, \tag{19}$$

where μ is the orthogonal complement of $\omega D^\theta \oplus \omega D^\perp$ in $\Gamma(\mathcal{T}^\perp\mathcal{M})$ and μ is invariant with respect to ϕ.
Also, for any $N \in \Gamma(\mathcal{T}^\perp\mathcal{M})$, we put

$$\phi N = FN + GN, \tag{20}$$

where $FN \in \Gamma(D^\theta \oplus D^\perp)$ and $GN \in \Gamma(\mu)$.

Lemma 3.2. *Let $\mathcal{M}$ be a quasi hemi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$. Then we have the following observations* [21]*:*
$\nu D = D$, $\quad \nu D^\perp = 0$, $\quad F\omega D^\theta = D^\theta$,
$F\omega D^\perp = D^\perp$, *and* $\nu D^\theta \subset D^\theta$.

With the help of the equations (1), (10) and (11), we can obtain the following observations:

Lemma 3.3. *Let $\mathcal{M}$ be a quasi hemi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$ and consider the endomorphism ν, the projection morphism ω, F and G in the tangent bundle of $\mathcal{M}$, then*
(i) $\nu^2 + F\omega = -I + \eta \otimes \xi$ *on* $\mathcal{TM}$,
(ii) $\omega\nu + G\omega = 0$ *on* $\mathcal{TM}$,
(iii) $\omega\nu + G\omega = 0$ *on* $\mathcal{TM}$,
(iv) $\nu F + FG = 0$ *on* $\mathcal{T}^\perp\mathcal{M}$,
where I is an identity operator.

Proof. From the equation (10), we have

$$\phi U = \nu U + \omega U$$

for any $U \in \Gamma(\mathcal{TM})$. Operating ϕ on both sides, we obtain

$$\phi^2 U = \phi\nu U + \phi\omega U.$$

By using equation (1) and comparing tangential and normal parts, we have the required identities (i) and (ii). Similarly, with the help of the equations (11) and (1) for any $N \in \Gamma(\mathcal{T}^\perp\mathcal{M})$, we have the rest two identities (iii) and (iv). □

Lemma 3.4. *Let $\mathcal{M}$ be a quasi hemi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$, then for any $X \in \Gamma(\mathcal{TM})$ and $N \in \Gamma(\mathcal{T}^\perp\mathcal{M})$, we have*

$$\nabla_X FN - A_{GN}X + \nu A_N X - F\nabla^\perp_X N = g(\omega X, N)\xi,$$

$$h(X, FN) + \nabla^\perp_X GN + \omega A_N X - G\nabla^\perp_X N = 0,$$

$$(\overline{\nabla}_X F)N = A_{GN}X - \nu A_N X + g(\omega X, N)\xi,$$

and

$$(\overline{\nabla}_X G)N = -h(X, FN) - \omega A_N X.$$

Proof. For any $X \in \Gamma(\mathcal{TM})$ and $N \in \Gamma(\mathcal{T}^\perp\mathcal{M})$, we have

$$(\overline{\nabla}_X \phi)N = g(\phi X, N)\xi - \eta(N)\phi X.$$

Bu using the equations (7), (8), (10) and (11), we obtain

$$\begin{aligned} \nabla_X FN + h(X, FN) + (-A_{GN}X + \nabla^\perp_X GN) + \nu A_N X \\ + \omega A_N X - F\nabla^\perp_X N - G\nabla^\perp_X N = g(\phi X, N)\xi. \end{aligned}$$

Now comparing the tangential and normal parts, we get the first two assertions. Next with the help of (14) and (15), we obtain the rest two required results. □

In a similar way as Lemma 3.4, using the equations (7), (8), (10), (11), (12) and (13), we also have the following properties:

Proposition 3.5. *Let $\mathcal{M}$ be a quasi hemi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$, then for any $X, Y \in \Gamma(\mathcal{TM})$, we have*

$$\nabla_X \nu Y - A_{\omega Y}X - \nu\nabla_X Y - Fh(X, Y) = g(\nu X, Y)\xi - \eta(Y)\nu X,$$

$$h(X, \nu Y) + \nabla^\perp_X \omega Y - \omega\nabla_X Y - Gh(X, Y) = -\eta(Y)\omega X,$$

$$(\overline{\nabla}_X \nu)Y = A_{\omega Y}X + Fh(X, Y) + g(\nu X, Y)\xi - \eta(Y)\nu X,$$

and

$$(\overline{\nabla}_X \omega)Y = Gh(X, Y) - \eta(Y)\omega X - h(X, \nu Y).$$

For later use, we state the following Lemmas related to the distributions D^θ and $D^\perp$ involved in the definition of hemi-slant submanifold:

Lemma 3.6. *Let $\mathcal{M}$ be a quasi hemi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$, then*
(i) $\nu^2 U = -(cos^2\theta)U$,
(ii) $g(\nu U, \nu V) = (cos^2\theta)g(U, V)$,
(iii) $g(\omega U, \omega V) = (sin^2\theta)g(U, V)$
for any $U, V \in \Gamma(D^\theta)$.

Proof. Proof of this lemma is same as in [22]. □

Lemma 3.7. *Let $\mathcal{M}$ be a quasi hemi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$, then*

$$A_{\phi W}Z = A_{\phi Z}W + \nu([W, Z]),$$

and

$$\nabla^\perp_Z \phi W - \nabla^\perp_W \phi Z = \omega([Z, W])$$

for all $Z, W \in D^\perp$.

Proof. Let $Z, W \in D^\perp$, then

$$(\overline{\nabla}_Z \phi)W = \overline{\nabla}_Z(\phi W) - \phi(\overline{\nabla}_Z W).$$

By making use of the equations (5), (7) and (8), we obtain

$$\begin{aligned} g(\phi Z, W)\xi - \eta(W)\phi Z = &- A_{\phi W}Z + \nabla^\perp_Z \phi W - \nu(\nabla_Z W) \\ &- \omega(\nabla_Z W) - Fh(Z, W) - Gh(Z, W). \end{aligned}$$

Now, comparing tangential and normal parts, we get

$$0 = -A_{\phi W}Z - \nu(\nabla_Z W) - Fh(Z, W), \tag{21}$$

and

$$\nabla^\perp_Z \phi W - \omega(\nabla_Z W) - Gh(Z, W) = 0. \tag{22}$$

Now, interchanging Z and W in the equations (21) and (22), we get

$$0 = -A_{\phi Z}W - \nu(\nabla_W Z) - Fh(W, Z), \tag{23}$$

and

$$\nabla^\perp_Z \phi Z - \omega(\nabla_W Z) - Gh(W, Z) = 0. \tag{24}$$

Now, subtracting equation (23) from (21) and equation (24) from (22), respectively and using the fact that h is symmetric, we obtain the required results. □

4. Integrability of distributions

In this section we will derive some results on involved distributions D, D^θ and $D^\perp$ of quasi hemi-slant submanifold, which play a crucial role from a geometrical point of view.

Theorem 4.1. *Let $\mathcal{M}$ be a quasi hemi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$, then the distribution $D \oplus D^\theta \oplus D^\perp$ is integrable.*

Proof. For $X, Y \in D \oplus D^\theta \oplus D^\perp$, we have

$$\begin{aligned} g([X,Y],\xi) &= g(\overline{\nabla}_X Y, \xi) - g(\overline{\nabla}_Y X, \xi) \\ &= Xg(Y,\xi) - g(Y, \overline{\nabla}_X \xi) - Yg(X,\xi) + g(X, \overline{\nabla}_Y \xi) \\ &= g(X, Y - \eta(Y)\xi) - g(Y, X - \eta(X)\xi) \\ &= 0. \end{aligned}$$

Since, $\mathcal{TM} = D \oplus D^\theta \oplus D^\perp \oplus < \xi >$, therefore $[X,Y] \in D \oplus D^\theta \oplus D^\perp$. So, $D \oplus D^\theta \oplus D^\perp$ is integrable. □

Theorem 4.2. *Let $\mathcal{M}$ be a quasi hemi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$. Then the invariant distribution D is integrable if and only if*

$$g(\nabla_U \nu V - \nabla_V \nu U, \nu P^\theta Z) = g(h(V, \nu U) - h(U, \nu V), \omega Z) \tag{25}$$

for any $U, V \in \Gamma(D)$ and $Z = P^\theta Z + P^\perp Z \in \Gamma(D^\theta \oplus D^\perp)$.

Proof. As we know that the invariant distribution D is integrable on $\mathcal{M}$ if and only if $g([U,V],\xi) = 0$ and $g([U,V],Z) = 0$ for any $U, V \in \Gamma(D), Z \in \Gamma(D^\theta \oplus D^\perp)$ and $\xi \in \Gamma(\mathcal{TM})$.
Since $\mathcal{M}$ is a quasi hemi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$, therefore, immediately we have

$$\begin{aligned} g([U,V],\xi) &= g(\overline{\nabla}_U V, \xi) - g(\overline{\nabla}_V U, \xi) \\ &= Ug(V,\xi) - g(V, \overline{\nabla}_U \xi) - Vg(U,\xi) + g(U, \overline{\nabla}_V \xi) \\ &= g(U, V - \eta(V)\xi) - g(V, U - \eta(U)\xi) \\ &= 0. \end{aligned}$$

Thus, the invariant distribution D is integrable iff $g([U,V],Z) = 0$.

Now, for any $U, V \in \Gamma(D)$ and $Z = P^\theta Z + P^\perp Z \in \Gamma(D^\theta \oplus D^\perp)$, with the help of the equations (2) and (4), we have

$$\begin{aligned} g([U,V],Z) &= g(\phi([U,V]), \phi Z) + \eta([U,V])\eta(Z) \\ &= g(\phi(\overline{\nabla}_U V), \phi Z) - g(\phi(\overline{\nabla}_V U), \phi Z) \\ &= g(\overline{\nabla}_U \phi V - (\overline{\nabla}_U \phi)V, \phi Z) - g(\overline{\nabla}_V \phi U - (\overline{\nabla}_V \phi)U, \phi Z) \\ &= g(\overline{\nabla}_U \phi V, \phi Z) - g((\overline{\nabla}_U \phi)V, \phi Z) - g(\overline{\nabla}_V \phi U, \phi Z) + g((\overline{\nabla}_V \phi)U, \phi Z). \end{aligned}$$

Now using the fact that $\omega U = 0$ and $\omega V = 0$ for any $U, V \in \Gamma(D)$, we obtain

$$g([U,V],\xi) = g(\overline{\nabla}_U(\nu V), \phi Z) - g(\overline{\nabla}_V(\nu U), \phi Z).$$

By using the equation (7), we have

$$g([U,V],Z)=g(\nabla_U(\nu V)-\nabla_V(\nu U),\phi Z)+g(h(U,\nu V)-h(V,\nu U),\phi Z).$$

Again using the equation (10) for any $Z=P^\theta Z+P^\perp Z\in\Gamma(D^\theta\oplus D^\perp)$, we get

$$\begin{aligned}g([U,V],Z)=&g(\nabla_U\nu V-\nabla_V\nu U,\nu P^\theta Z+\nu P^\perp Z)+g(h(U,\nu V)\\&-h(V,\nu U),\omega P^\theta Z+\omega P^\perp Z).\end{aligned}$$

This proves the assertion. □

Theorem 4.3. *Let $\mathcal{M}$ be a proper quasi hemi-slant submanifold of a Kenmotsu $\mathcal{N}$. Then the slant distribution D^θ is integrable if and only if*

$$\begin{aligned}g(A_{\omega V_1}U_1-A_{\omega U_1}V_1,\nu PZ)=&g(A_{\omega\nu V_1}U_1-A_{\omega\nu U_1}V_1,Z)\\&+g(\nabla^\perp_{U_1}\omega V_1-\nabla^\perp_{V_1}\omega U_1,\omega P^\perp Z)\end{aligned}$$

for any $U_1,V_1\in\Gamma(D^\theta)$ and $Z\in\Gamma(D\oplus D^\perp)$.

Proof. For any $U_1,V_1\in\Gamma(D^\theta)$ and $Z=PZ+P^\perp Z\in\Gamma(D\oplus D^\perp)$, the distribution D^θ is integrable on $\mathcal{M}$ iff $g([U_1,V_1],\xi)=0$ and $g([U_1,V_1],Z)=0$, where $\xi\in\Gamma(\mathcal{TM})$. Now, the first case is obvious as in Theorem 4.2. So, the slant distribution D^θ is integrable iff $g([U_1,V_1],Z)=0$.
Now, for any $U_1,V_1\in\Gamma(D^\theta)$ and $Z=PZ+P^\perp Z\in\Gamma(D\oplus D^\perp)$, using the equation (3) we get

$$\begin{aligned}g([U_1,V_1],Z)&=g(\phi[U_1,V_1],\phi Z)+\eta([U_1,V_1])\eta(Z)\\&=g(\phi(\overline{\nabla}_{U_1}V_1),\phi Z)-g(\phi(\overline{\nabla}_{V_1}U_1),\phi Z).\end{aligned}$$

Now using the equations (5) and (10), we get

$$\begin{aligned}g([U_1,V_1],Z)=&g(\overline{\nabla}_{U_1}(\nu V_1+\omega V_1),\phi Z)-g(\overline{\nabla}_{V_1}(\nu U_1+\omega U_1),\phi Z)\\&-\{g(\phi U_1,V_1)g(\xi,\phi Z)-\eta(V_1)g(\phi U_1,\phi Z)\}\\&+\{g(\phi V_1,U_1)g(\xi,\phi Z)-\eta(U_1)g(\phi V_1,\phi Z)\}\\=&g(\overline{\nabla}_{U_1}(\nu V_1),\phi Z)+g(\overline{\nabla}_{U_1}(\omega V_1),\phi Z)-g(\overline{\nabla}_{V_1}(\nu U_1),\phi Z)\\&-g(\overline{\nabla}_{V_1}(\omega U_1),\phi Z)\end{aligned}$$

which in light of the equations (5) and (8) and the fact that $(\overline{\nabla}_U\phi)V=\overline{\nabla}_U\phi V-\phi\overline{\nabla}_UV$, we obtain the following condition:

$$\begin{aligned}g([U_1,V_1],Z)=&-g(\overline{\nabla}_{U_1}\phi(\nu V_1)-(\overline{\nabla}_{U_1}\phi)\nu V_1,Z)+g(\overline{\nabla}_{V_1}\phi(\nu U_1)-(\overline{\nabla}_{V_1}\phi)\nu U_1,Z)\\&+g(-A_{\omega V_1}U_1+\nabla^\perp_{U_1}\omega V_1,\phi Z)-g(-A_{\omega U_1}V_1+\nabla^\perp_{V_1}\omega U_1,\phi Z)\\=&-g(\overline{\nabla}_{U_1}\phi(\nu V_1),Z)+g(\overline{\nabla}_{V_1}\phi(\nu U_1),Z)+g(-A_{\omega V_1}U_1+\nabla^\perp_{U_1}\omega V_1,\phi Z)\\&-g(-A_{\omega U_1}V_1+\nabla^\perp_{V_1}\omega U_1,\phi Z).\end{aligned}$$

Now for any $Z=PZ+P^\perp Z\in\Gamma(D\oplus D^\perp)$, from the equation (10) we obtain

$$g([U_1,V_1],Z)=-g(\overline{\nabla}_{U_1}\nu^2V_1,Z)-g(\overline{\nabla}_{U_1}\omega\nu V_1,Z)+g(\overline{\nabla}_{V_1}\nu^2U_1,Z)$$

$$+ g(\overline{\nabla}_{V_1}\omega\nu U_1, Z) - g(A_{\omega V_1}U_1 - A_{\omega U_1}V_1, \nu Z + \omega Z)$$
$$+ g(\nabla^{\perp}_{U_1}\omega V_1 - \nabla^{\perp}_{V_1}\omega U_1, \phi PZ + \phi P^{\perp}Z).$$

With the help of Lemma 3.6 and well-known fact $\omega PZ = 0$, the above equation leads to

$$\begin{aligned} g([U_1, V_1], Z) =& cos^2\theta g(\overline{\nabla}_{U_1}V_1 - \overline{\nabla}_{V_1}U_1, Z) - g(\overline{\nabla}_{U_1}\omega\nu V_1 - \overline{\nabla}_{V_1}\omega\nu U_1, Z) \\ & - g(A_{\omega V_1}U_1 - A_{\omega U_1}V_1, \nu PZ) + g(\nabla^{\perp}_{U_1}\omega V_1 - \nabla^{\perp}_{V_1}\omega U_1, \omega P^{\perp}Z) \\ =& cos^2\theta g([U_1, V_1], Z) \\ & - g(-A_{\omega\nu V_1}U_1 + \nabla^{\perp}_{U_1}\omega\nu V_1 + A_{\omega\nu U_1}V_1 - \nabla^{\perp}_{V_1}\omega\nu U_1, Z) \\ & - g(A_{\omega V_1}U_1 - A_{\omega U_1}V_1, \nu PZ) + g(\nabla^{\perp}_{U_1}\omega V_1 - \nabla^{\perp}_{V_1}\omega U_1, \omega P^{\perp}Z) \\ =& cos^2\theta g([U_1, V_1], Z) - g(-A_{\omega\nu V_1}U_1 + A_{\omega\nu U_1}V_1, Z) \\ & - g(A_{\omega V_1}U_1 - A_{\omega U_1}V_1, \nu PZ) + g(\nabla^{\perp}_{U_1}\omega V_1 - \nabla^{\perp}_{V_1}\omega U_1, \omega P^{\perp}Z) \end{aligned}$$

which implies that

$$\begin{aligned} sin^2\theta g([U_1, V_1], Z) =& g(A_{\omega\nu V_1}U_1 - A_{\omega\nu U_1}V_1, Z) - g(A_{\omega V_1}U_1 - A_{\omega U_1}V_1, \nu PZ) \\ & + g(\nabla^{\perp}_{U_1}\omega V_1 - \nabla^{\perp}_{V_1}\omega U_1, \omega P^{\perp}Z). \end{aligned}$$

Thus the statement of the Theorem 4.3 follows. □

Now, with the help of the above result, the sufficient condition for D^{θ} to be integrable is given by the following theorem:

Theorem 4.4. *Let $\mathcal{M}$ be a proper quasi hemi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$. If*

$$A_{\omega V_1}U_1 - A_{\omega U_1}V_1 \in D^{\perp} \oplus D^{\theta}$$

$$A_{\omega\nu V_1}U_1 - A_{\omega\nu U_1}V_1 \in D^{\theta}$$

and

$$\nabla^{\perp}_{U_1}\omega V_1 - \nabla^{\perp}_{V_1}\omega U_1 \in \omega D^{\theta} \oplus \mu$$

for any $U_1, V_1 \in \Gamma(D^{\theta})$, then the slant distribution D^{θ} is integrable.

Theorem 4.5. *Let $\mathcal{M}$ be a proper quasi hemi-slant submanifold of a Kenmotsu $\mathcal{N}$. Then the anti-invariant distribution $D^{\perp}$ is integrable if and only if*

$$g(\nu([U, V]), \nu X) = g(\nabla^{\perp}_{V}\omega U - \nabla^{\perp}_{U}\omega V, \omega P^{\theta}X)$$

for any $U, V \in \Gamma(D^{\perp})$, and $X \in \Gamma(D \oplus D^{\theta})$.

Proof. The anti-invariant distribution is integrable iff for any $U, V \in \Gamma(D^{\perp})$ and $X \in \Gamma(D \oplus D^{\theta})$, we have $g([U, V], X) = 0$, where $X = PX + P^{\theta}X \in \Gamma(D \oplus D^{\theta})$. Now from the equation (2), we have

$$\begin{aligned} g([U, V], X) &= g(\phi([U, V]), \phi X) + \eta([U, V])\eta(X) \\ &= g(\phi([U, V]), \phi X) \end{aligned}$$

$$= g(\phi\overline{\nabla}_U V, \phi X) - g(\phi\overline{\nabla}_V U, \phi X)$$
$$= g(\overline{\nabla}_U \phi V, \phi X) - g(\overline{\nabla}_V \phi U, \phi X).$$

In view of the equation (8) and the fact that $\omega PX = 0$, the last equation takes the form

$$\begin{aligned} g([U,V],X) &= g(-A_{\phi V}U + A_{\phi U}V, \phi X) + g(\nabla^{\perp}_U \phi V - \nabla^{\perp}_V \phi U, \phi X) \\ &= g(-A_{\phi V}U + A_{\phi U}V, \nu PX + \nu P^{\theta}X) + g(\nabla^{\perp}_U \phi V - \nabla^{\perp}_V \phi U, \omega P^{\theta}X). \end{aligned}$$

In the account of Lemma 3.7, we get the following consequences:

$$g([U,V],X) = g(\nu([U,V], \nu X) + g(\nabla^{\perp}_U \omega V - \nabla^{\perp}_V \omega U, \omega P^{\theta}X).$$

This shows that anti-invariant distribution $D^{\perp}$ is integrable if and only if

$$g(\nu([U,V]), \nu X) = g(\nabla^{\perp}_V \omega U - \nabla^{\perp}_U \omega V, \omega P^{\theta}X).$$

Hence, the theorem is proved. □

We also have the following necessary and sufficient condition for anti-invariant distribution $D^{\perp}$ to be integrable:

Theorem 4.6. *Let $\mathcal{M}$ be a quasi hemi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$, then the anti-invariant distribution $D^{\perp}$ is integrable if and only if*

$$A_{\phi W}Z = A_{\phi Z}W$$

for any $Z, W \in D^{\perp}$.

Proof. With the help of Lemma 3.7, for any $Z, W \in D^{\perp}$ we have

$$A_{\phi W}Z = A_{\phi Z}W + \nu([W,Z]).$$

Now if $D^{\perp}$ is integrable then for any $Z, W \in \Gamma(D^{\perp})$, we have $[W,Z] \in \Gamma(D^{\perp})$, which implies that $\nu([W,Z]) = 0$.
Hence, from the above equation we have

$$A_{\phi W}Z = A_{\phi Z}W.$$

Conversely, if $A_{\phi W}Z = A_{\phi Z}W$, then we have

$$\nu([W,Z]) = 0 \implies [W,Z] \in D^{\perp}$$

for any $Z, W \in D^{\perp}$. Hence, $D^{\perp}$ is integrable. □

5. Totally geodesic foliations

Now, we give some results on totally geodesicness property of quasi hemi-slant submanifolds of a Kenmotsu manifold.

Theorem 5.1. *Let $\mathcal{M}$ be a proper quasi hemi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$. Then $\mathcal{M}$ defines totally geodesic foliation if and only if*

$$g(h(Z,PW)+cos^2\theta h(Z,P^\theta W),V)=g(\nabla^\perp_Z\omega\nu P^\theta W,V)$$
$$+g(A_{\omega P^\theta W}Z+A_{\omega P^\perp W}Z,FV)-g(\nabla^\perp_Z\omega W,GV)$$

for any $Z,W\in\Gamma(\mathcal{TM})$ and $V\in\Gamma(T^\perp M)$.

Proof. By using $(\overline{\nabla}_X\phi)Y=\overline{\nabla}_X\phi Y-\phi(\overline{\nabla}_XY)$ and the equation (2), we have the following conditions

$$\begin{aligned}g(\overline{\nabla}_ZW,V)=&g(\overline{\nabla}_ZPW,V)+g(\overline{\nabla}_ZP^\theta W,V)+g(\overline{\nabla}_ZP^\perp W,V)\\=&g(\overline{\nabla}_ZPW,V)+g(\phi\overline{\nabla}_ZP^\theta W,\phi V)+\eta(\overline{\nabla}_ZP^\theta W)\eta(V)\\&+g(\phi\overline{\nabla}_ZP^\perp W,\phi V)+\eta(\overline{\nabla}_ZP^\perp W)\eta(V).\end{aligned}$$

With the help of the equations (2),(5), (10), (11) and Lemma 3.7, we obtain

$$\begin{aligned}g(\overline{\nabla}_ZW,V)=&g(\overline{\nabla}_ZPW,V)+g(\overline{\nabla}_Z\nu P^\theta W,\phi V)+g(\overline{\nabla}_Z\omega P^\theta W,\phi V)\\&+g(\overline{\nabla}_Z\omega P^\perp W,\phi V)\\=&g(\overline{\nabla}_ZPW,V)-g(\overline{\nabla}_Z\nu^2P^\theta W+\overline{\nabla}_Z\omega\nu P^\theta W,V)\\&+g(\overline{\nabla}_Z\omega P^\theta W,\phi V)+g(\overline{\nabla}_Z\omega P^\perp W,\phi V)\\=&g(\overline{\nabla}_ZPW,V)+cos^2\theta g(\overline{\nabla}_ZP^\theta W,V)-g(\overline{\nabla}_Z\omega\nu P^\theta W,V)\\&+g(\overline{\nabla}_Z\omega P^\theta W,\phi V)+g(\overline{\nabla}_Z\omega P^\perp W,\phi V).\end{aligned}$$

Using the equations (7) and (8), we have

$$\begin{aligned}g(\overline{\nabla}_ZW,V)=&g(h(Z,PW),V)+cos^2\theta g(h(Z,P^\theta W),V)-g(\nabla^\perp_Z\omega\nu P^\theta W,V)\\&+g(-A_{\omega P^\theta W}Z+\nabla^\perp_Z\omega P^\theta W,\phi V)+g(-A_{\omega P^\perp W}Z+\nabla^\perp_Z\omega P^\perp W,\phi V)\\=&g(h(Z,PW)+cos^2\theta h(Z,P^\theta W),V)-g(A_{\omega P^\theta W}Z+A_{\omega P^\perp W}Z,FV)\\&+g(\nabla^\perp_Z\omega W,GV)-g(\nabla^\perp_Z\omega\nu P^\theta W,V)\end{aligned}$$

and hence we have the required result.

□

Theorem 5.2. *Let $\mathcal{M}$ be a quasi hemi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$. Then the invariant distribution D does not define totally geodesic foliation on $\mathcal{M}$.*

Proof. The invariant distribution D defines a totally geodesic foliation on $\mathcal{M}$ iff $g(\overline{\nabla}_UV,\xi)=0, g(\overline{\nabla}_UV,Z)=0$ and $g(\overline{\nabla}_UV,W)=0$, for any $U,V\in\Gamma(D), Z=P^\theta Z+P^\perp Z\in\Gamma(D^\theta\oplus D^\perp)$ and $W\in\Gamma(T^\perp M)$. Since, $g(\overline{\nabla}_UV,\xi)=Ug(V,\xi)-g(V,\overline{\nabla}_U\xi)=-g(V,\overline{\nabla}_U\xi)$. With the help of the equation (6), we have

$$\begin{aligned}g(\overline{\nabla}_UV,\xi)&=-g(V,U-\eta(U)\xi)\\&=-g(V,U)+\eta(U)g(V,\xi)\end{aligned}$$

$$= -g(V, U)$$
$$\neq 0$$

for some $U, V \in \Gamma(D)$.
As $g(\overline{\nabla}_U V, \xi) \neq 0$, therefore invariant distribution D does not define totally geodesic foliation on $\mathcal{M}$. □

Similarly as above we have the following theorems for slant distribiution D^θ and anti-invariant $D^\perp$:

Theorem 5.3. *Let $\mathcal{M}$ be a quasi hemi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$. Then the slant distribution D^θ with slant angle θ does not define totally geodesic foliation on $\mathcal{M}$.*

Theorem 5.4. *Let $\mathcal{M}$ be a quasi hemi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$. Then the anti-invariant distribution $D^\perp$ does not define totally geodesic foliation on $\mathcal{M}$.*

6. Examples

Let us consider an 11-dimensional manifold

$$\overline{\mathcal{M}} = \{(x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5, z) \in \mathbb{R}^{11} : z \neq 0\},$$

where $(x_i, y_i, z), i = 1, 2, 3, 4, 5$ are standard coordinates in $\mathbb{R}^{11}$. We choose the vector fields

$\epsilon_i = e^{-z}\frac{\partial}{\partial x_i}$, $\quad \epsilon_{5+i} = e^{-z}\frac{\partial}{\partial y_i}$, $\quad \epsilon_{11} = \frac{\partial}{\partial z}$,

where $i = 1, 2, 3, 4, 5$, which are linearly independent at each points of $\overline{\mathcal{M}}$. Let g be the Riemannian metric defined by

$$g = e^{2z}(dx \otimes dx + dy \otimes dy) + \eta \otimes \eta,$$

where η is the 1- form defined by

$$\eta(X) = g(X, \epsilon_{11}),$$

$$g(\epsilon_i, \epsilon_j) = 0 \quad \text{and} \quad g(\epsilon_i, \epsilon_i) = 1$$

for any vector field $X \in \Gamma(\mathcal{T}\overline{\mathcal{M}})$ and $\forall\, i, j = 1, 2, ..., 10$. Hence $\{\epsilon_1, \epsilon_2, ..., \epsilon_{11}\}$ is an orthonormal basis of $\overline{\mathcal{M}}$. We define (1,1) tensor field ϕ as

$$\phi\{\textstyle\sum_{i=1}^{5}(X^i\frac{\partial}{\partial x_i} + Y^i\frac{\partial}{\partial y_i}) + Z\frac{\partial}{\partial z}\} = \textstyle\sum_{i=1}^{5}(X^i\frac{\partial}{\partial y_i} - Y^i\frac{\partial}{\partial x_i}).$$

Thus, we get

$\phi(\epsilon_1) = \epsilon_6$, $\quad \phi(\epsilon_2) = \epsilon_7$, $\quad \phi(\epsilon_3) = \epsilon_8$, $\quad \phi(\epsilon_4) = \epsilon_9$,

$\phi(\epsilon_5) = \epsilon_{10}$, $\quad \phi(\epsilon_6) = -\epsilon_1$, $\quad \phi(\epsilon_7) = -\epsilon_2$, $\quad \phi(\epsilon_8) = -\epsilon_3$,

$\phi(\epsilon_9) = -\epsilon_4$, $\quad \phi(\epsilon_{10}) = -\epsilon_5$, $\quad \phi(\epsilon_{11}) = 0$.

The linearity property of g and ϕ yields that

$$\eta(\epsilon_{11}) = g(\epsilon_{11}, \epsilon_{11}) = 1,$$
$$\phi^2 X = -X + \eta(X)\epsilon_{11},$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields X, Y on $\overline{\mathcal{M}}$. Thus for $\epsilon_{11} = \xi$, $\overline{\mathcal{M}}(\phi, \xi, \eta, g)$ defines an almost contact metric manifold.

Now by using well-known Koszul's formula for the Riemannian connection ∇ is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) =& Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) \\ &- g([Y, Z], X) + g([Z, X], Y) \end{aligned}$$

for any vector fields X, Y, Z on $\overline{\mathcal{M}}$, we can easily verify the equations (5) and (6) for any vector fields X, Y, Z on $\overline{\mathcal{M}}$. Therefore, $\overline{\mathcal{M}}(\phi, \xi, \eta, g)$ is a Kenmotsu manifold. Moreover, we have

$$[\epsilon_i, \xi] = \epsilon_i, \qquad [\epsilon_i, \epsilon_j] = 0, \qquad \forall i, j = 1, 2, ..., 10.$$

Using Koszul's formula, we can easily find that

$$\nabla_{\epsilon_i}\epsilon_i = -\xi, \qquad \nabla_{\epsilon_i}\epsilon_j = 0, \qquad \text{for } i \neq j,$$

$$\nabla_{\epsilon_i}\xi = \epsilon_i, \qquad \nabla_{\xi}\epsilon_i = 0, \qquad \forall i, j = 1, 2, ..., 10.$$

Now, let $\mathcal{M}$ be a subset of $\overline{\mathcal{M}}$ and consider the immersion $f : \mathcal{M} \to \overline{\mathcal{M}}$ defined as

$$f(u, v, w, r, s, t, z) = (u, 0, w, 0, s, v cos\theta, v sin\theta, 0, r, t, z),$$

where, $0 < \theta < \frac{\pi}{2}$. If we take

$$X_1 = \epsilon_1, \qquad X_2 = cos\theta\epsilon_6 + sin\theta\epsilon_7, \qquad X_3 = \epsilon_3, \qquad X_4 = \epsilon_9,$$

$$X_5 = \epsilon_5, \qquad X_6 = \epsilon_{10}, \qquad X_7 = \xi = \epsilon_{11},$$

then the restriction of $X_1, X_2, ..., X_7$ to $\mathcal{M}$ forms an orthonormal frame of the tangent bundle $\mathcal{TM}$. Obviously, we get

$$\phi X_1 = \epsilon_6, \qquad \phi X_2 = -cos\theta\epsilon_1 - sin\theta\epsilon_2, \qquad \phi X_3 = \epsilon_8, \qquad \phi X_4 = -\epsilon_4,$$

$$\phi X_5 = \epsilon_{10}, \qquad \phi X_6 = -\epsilon_5, \qquad \phi X_7 = 0.$$

Let us put $D^\theta = span\{X_1, X_2\}, D^\perp = span\{X_3, X_4\}$ and $D = span\{X_5, X_6\}$. Then obviously $D^\theta, D^\perp$ and D satisfy the definition of quasi hemi-slant submanifold of a Kenmotsu manifold. Hence, submanifold $\mathcal{M}$ defined by f is proper quasi hemi-slant submanifold of $\mathbb{R}^{11}$ with slant angle $0 < \theta < \frac{\pi}{2}$.

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