J. Appl. Math. & Informatics Vol. **40***(2022), No. 3 - 4, pp. 445 - 460*
https://doi.org/10.14317/jami.2022.445

β-FUZZY FILTERS OF STONE ALMOST DISTRIBUTIVE LATTICES

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Abstract. In this paper, we studied on β-fuzzy filters of Stone almost distributive lattices. An isomorphism between the lattice of β-fuzzy filters of a Stone ADL A onto the lattice of fuzzy ideals of the set of all boosters of A is established. The fact that any β-fuzzy filter of A is an e-fuzzy filter of A is proved. We discuss on some properties of prime β-fuzzy filters and some topological concepts on the collection of prime β-fuzzy filters of a Stone ADL. Further we show that the collection $\mathcal{T} = \{X^{\beta}(\lambda) : \lambda$ is a fuzzy ideal of $A\}$ is a topology on $\mathcal{F}Spec_{\beta}(A)$ where $X^{\beta}(\lambda) = \{\mu \in \mathcal{F}Spec_{\beta}(A) : \lambda \nsubseteq \mu\}$.

AMS Mathematics Subject Classification : 06D99, 06D05, 06D30.
Key words and phrases : Almost distributive lattices, Stone almost distributive lattices, prime β-fuzzy filter, prime fuzzy ideal, booster, isomorphism.

1. Introduction

The class of distributive lattices has many interesting properties, which lattices, in general, do not have. For this reason, U.M. Swamy and G.C. Rao [12] introduced the concept of an almost distributive lattice(ADL) as a common abstraction of lattice and ring theoretic generalizations of a Boolean algebra. In [12], it was proved that the commutativity of $\vee$, the commutativity of $\wedge$, the right distributivity of $\vee$ over $\wedge$ and the absorption law $(x \wedge y) \vee x = x$ are all equivalent to each other and whenever any one of these properties holds, an ADL A becomes a distributive lattice. Later, U.M. Swamy, G.C. Rao, and G. Nanaji Rao in [13] introduced the concept of pseudo-complementation in an ADL. U.M. Swamy, G.C. Rao, and G. Nanaji Rao in [14] introduced the concept of Stone ADL. It is an ADL with a pseudo-complementation $*$ that satisfies the condition $r^* \vee r^{**}$ is maximal, for all $r \in A$.

Received March 29, 2021. Revised September 15, 2021. Accepted September 22, 2021.
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In [18], the concept of fuzzy set theory as a generalization of classical set theory was introduced by Zadeh. Rosenfield [8] started the pioneering work in the domain of fuzzification of algebraic objects on fuzzy groups. In particular Y. Bo et al [17] and Swamy et al [12] have laid down the foundation for fuzzy ideals of a lattice and an ADL respectively.

C. Santhi Sundar Raj and et al. [10] introduced the concept of fuzzy prime ideals of an ADLs. In 1998, U. M. Swamy and D. Viswanadha Raju [11] introduced the concept of fuzzy ideals and fuzzy congruences of distributive lattices and showed that there is a one-to-one correspondence between the lattice of fuzzy ideals and the lattice of fuzzy congruences of A. U.M. Swamy et al.[16] studied about L-fuzzy filters of an ADL. In [1] Berhanu Assaye Alaba and Gezahagne Mulat Addis studied on fuzzy congruence relations on an ADL A and they give the smallest fuzzy congruence on A such that its quotient is a distributive lattice.

This paper comprises of four sections the first two sections deals on the introductory and preliminary concepts. In section 3, we studied on β-fuzzy filters of stone almost distributive lattices. An isomorphism of the lattice of β-fuzzy filters of a Stone ADL A onto the lattice of fuzzy ideals of $\mathcal{B}_0(A)$ is established. We proved that any β-fuzzy filter of a Stone ADL A is an e-fuzzy filter of A. In section 4, we discuss on some properties of prime β-fuzzy filters and some topological concepts on the collection of prime β-fuzzy filters of a Stone ADL A. Further we show that the collection $\mathcal{T} = \{X^\beta(\lambda) : \lambda \text{ is a fuzzy ideal of } A\}$ is a topology on $\mathcal{F}Spec_\beta(A)$ where $X^\beta(\lambda) = \{\mu \in \mathcal{F}Spec_\beta(A) : \lambda \nsubseteq \mu\}$.

2. Preliminaries

This section devoted on definitions and results which will be used in the sequel.

Definition 2.1. [3] Let L be a lattice. A unary operation C on L is a closure operator if C satisfies the following conitions:

(1) $x \leq y$ implies $C(x) \leq C(y)$ for all $x, y \in L$,
(2) $x \leq C(X)$ for all $x \in L$,
(3) $C(x) = C^2(x)$ for all $x \in L$.

Definition 2.2. [3] The map $\varphi : P_0 \to P_1$ is an isotone map (also called monotone map or order-preserving niap) of the poset P_0 into the poset PP_1 iff $x \leq b$ in P_0, implies that $\varphi(a) \leq \varphi(b)$, in P_1.

Recall that ZORN'S LEMMA: Let A be a set and let X be a nonvoid subset of $P(A)$. Let us assume that X has the following property: If C is a chain in$(X; \subseteq)$, then $\cup(X : X \in C) \in X$. Then X has a maximal member.

Definition 2.3. [11] An algebra $(A, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice if it satisfies the following conditions for all x, y and $z \in A$:

(1) $0 \wedge x = 0$,

(2) $x \vee 0 = x$,
(3) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
(4) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$,
(5) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$,
(6) $(x \vee y) \wedge y = y$.

Let $x, y \in A$, we read x is less than or equal to y and write $x \leq y$ if $x \wedge y = x$, equivalently $x \vee y = y$. If an element m is maximal with respect to the partial ordering $\leq$ on A, then m is said to be maximal.

If $(A, \vee, \wedge, 0)$ is an ADL, for any $x, y \in A$, define $x \leq y$ if and only if $x = x \wedge y$ (or equivalently, $x \vee y = y$), then $\leq$ is a partial ordering on A.

Theorem 2.4. [11] *Let A be an ADL and $m \in A$. Then the following are equivalent:*

(1) *m is maximal with respect to $\leq$*
(2) $m \vee x = m$
(3) $m \wedge x = x$

for all $x \in A$.

Definition 2.5. [12]

Let $(A, \vee, \wedge, 0)$ be an ADL. Then for any $x, y, z \in A$, we have the following:

(1) $x \vee y = x \Leftrightarrow x \wedge y = y$,
(2) $x \vee y = y \Leftrightarrow x \wedge y = x$,
(3) $\wedge$ is associative in A,
(4) $x \wedge y \wedge z = y \wedge x \wedge z$,
(5) $x \wedge t \wedge z = y \wedge x \wedge z$,
(6) $(x \vee y) \wedge z = (y \vee x) \wedge z$
(7) $x \wedge y = 0 \Leftrightarrow y \wedge x = 0$,
(8) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$,
(9) $x \wedge (x \vee y) = x,\ (x \wedge y) \vee y = y$ and$x \vee (y \wedge x) = x$,
(10) $x \leq x \vee y$ and$x \wedge y \leq y$,
(11) $x \wedge x = x$ and $x \vee x = x$,
(12) $0 \vee x = x$ and $x \wedge 0 = 0$,
(13) If $x \leq z, y \leq z$ then $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$.

Let J be a non-empty subset of an ADL A. For any $x, y \in J$ and $z \in A$ if $x \vee y \in J (x \wedge y \in J)$ and $x \wedge z \in J (z \vee x \in J)$, then J is said to be an ideal(filter) of A respectively [11]. For any two elements J and K of the set $I(A)$ of all ideals of A, define $J \cap K$ is the infimum and $J \vee K = \{x \vee y : x \in J, y \in K\}$ is the supremum of J and K. Clearly $I(A)$ is a bounded distributive lattice with least element $\{0\}$ and greatest element A under set inclusion. A proper ideal J of A is called a prime ideal if, for any $a, b \in A, a \wedge b \in J \Rightarrow a \in J$ or $b \in J$. Let K be a proper ideal of A. K is said to be maximal if it is not properly contained in any proper ideal of A.

For any $A \subseteq L$, $Ann\{A\} = \{x \in L : a \wedge x = 0$ for all $a \in A\}$ is an ideal of L. We write $Ann\{(a]\}$ for $Ann\{a\}$. Then clearly $Ann\{(0]\} = L$ and $Ann\{L\} = (0]$.

Definition 2.6. [7] Let A be an ADL and $a \in A$. Then define $Ann\{a\} = \{x \in A : a \wedge x = 0\}$. Clearly, $Ann\{a\}$ is an ideal in A and hence an annihilator ideal.

Definition 2.7. [13] Let $(A, \vee, \wedge, 0)$ be an ADL. Then a unary operation $x \to x^*$ on A is called a pseudo-complementation on A if, for any $x, y \in A$, it satisfies the following conditions:

(1) $x \wedge y = 0 \Rightarrow x^* \wedge y = y$,
(2) $x \wedge x^* = 0$,
(3) $(x \vee y)^* = x^* \wedge y^*$,

Then $(A, \vee, \wedge, ^*, 0)$ is called a pseudo-complemented ADL.

Here, the unary operation * is called a pseudo-complementation on A and x^* is called a pseudo-complement of x in A. An element x of a pseudo-complemnted ADL A is called a dense element if $x^* = 0$. Now denote the set of all dense elements of A by D.

Theorem 2.8. [13] *Let A be an ADL and *, a pseudo-complementation on A. Then, for any $x, y \in A$, we have the following:*

(1) 0^* *is a maximal,*
(2) *If x is maximal, then $x^* = 0$,*
(3) $0^{**} = 0$,
(4) $x^{**} \wedge x = x$,
(5) $x^{**} = x$,
(6) $x \leq y \Rightarrow y^* \leq x^*$,
(7) $x^* \wedge y^* = y^* \wedge x^*$,
(8) $(x \wedge y)^{**} = x^{**} \wedge y^{**}$.

Definition 2.9. [14] Let A be an ADL and * a pseudo-complementation on A. Then A is called Stone ADL if, for any $a \in A$, $a^* \vee a^{**} = 0^*$.

Lemma 2.10. [14] *For any two elements x and y of a Stone ADL A the following conditions hold:*

(1) $0^* \wedge x = x$ and $0^* \vee x = 0^*$
(2) $(x \wedge y)^* = x^* \vee y^*$.

Definition 2.11. [7] For any filter F of a Stone ADL A, define an extension of F as the set $F^e = \{x \in A / x^* \in Ann\{a\}$ for some $a \in F\}$.

Definition 2.12. [7] A filter F of a Stone ADL A is called an e-filter of A if $F = F^e$.

Definition 2.13. [9] Let A be a Stone ADL with maximal elements. Then for any $x \in A$, define $(x)^+ = \{y \in A : y \vee x^*$ is a maximal element of $A\}$. We call $(x)^+$ as booster of x.

We denote the set of all boosters of a Stone ADL A by $B_0(A)$.

Definition 2.14. [9] Let A be a Stone ADL. Then the following hold:

(1) For any filter F of A, define an operator β as
$$\beta(F) = \{(x)^+ | x \in F\},$$
(2) For any ideal I of $B_0(A)$, define an operator $\overleftarrow{\beta}$ as $\overleftarrow{\beta}(I) = \{x \in A | (x)^+ \in I\}$.

Definition 2.15. A filter F of A is called a β-filter if $\overleftarrow{\beta}\beta(F) = F$.

Remember that, for any set S a function $\mu : S \longrightarrow ([0,1], \wedge, \vee)$ is called a fuzzy subset of S, where $[0,1]$ is a unit interval, $\alpha \wedge \beta = min\{\alpha, \beta\}$ and $\alpha \vee \beta = max\{\alpha, \beta\}$ for all $\alpha, \beta \in [0,1]$.

Definition 2.16. [16] Let ν be a fuzzy subset of an ADL A. For any $\alpha \in [0,1]$, we denote the level subset by ν_α and defined as
$$\nu_\alpha = \{a \in A : \alpha \leq \nu(a)\}.$$

Theorem 2.17. [16]
For any fuzzy subset ν of an ADL A the following are equivalent.
(1) *ν is a fuzzy filter of A,*
(2) *$\nu(m) = 1$ for all maximal element m and $\nu(s \wedge t) = \nu(s) \wedge \nu(t)$, for all $s, t \in A$,*
(3) *$\nu(m) = 1$ for all maximal element m and $\lambda(s \vee t) \geq \lambda(s) \vee \nu(t)$ and $\nu(s \wedge t) \geq \nu(s) \wedge \nu(t)$, for all $s, t \in A$.*

We define the binary operations " + " and "." on all fuzzy subsets of an ADL A as: $(\mu + \theta)(s) = \sup\{\mu(x) \wedge \theta(y) : x, y \in A, x \vee y = s\}$ and $(\mu.\theta)(s) = \sup\{\mu(x) \wedge \theta(y) : x, y \in A, x \wedge y = s\}$ for any $s \in A$.

The intersection of fuzzy filters of A is a fuzzy filter. However the union of fuzzy filters may not be fuzzy filter. The least upper bound of a fuzzy filters μ and θ of A is denoted as $\mu \vee \theta = \cap\{\sigma \in FF(A) : \mu \cup \theta \subseteq \sigma\}$.

If μ and θ are fuzzy filters of A, then $\mu.\theta = \mu \vee \theta$ and $\mu + \theta = \mu \cap \theta$.

3. β-fuzzy filters in Stone ADLs

Definition 3.1. Let ν be a fuzzy filter of a Stone ADL A and μ be a fuzzy ideal of $\mathcal{B}_0(A)$. Then we define operators β and $\overleftarrow{\beta}$ as follows:
(1) $\beta(\nu)((s)^+) = \sup\{\nu(t) : (s)^+ = (t)^+, t \in A\}$, for any s in A.
(2) $\overleftarrow{\beta}(\mu)(s) = \mu((s)^+)$, for any s in A.

Lemma 3.2. *Let A be a Stone ADL with maximal elements. Then for any fuzzy ideals μ and θ of $\mathcal{B}_0(A)$ and for any fuzzy filters ν and η of A we have the following:*
(1) *$\beta(\nu)$ is a fuzzy ideal of $\mathcal{B}_0(A)$,*
(2) *$\overleftarrow{\beta}(\mu)$ is a fuzzy filter of A,*

(3) $\nu \subseteq \eta$ *implies* $\beta(\nu) \subseteq \beta(\eta)$,
(4) $\mu \subseteq \theta$ *implies* $\overleftarrow{\beta}(\mu) \subseteq \overleftarrow{\beta}(\theta)$.

Proof. (1) Let ν be a fuzzy filter of A. Then clearly $\beta(\nu)((m)^+) = 1$. For any $(a)^+, (b)^+$ in $\mathcal{B}_0(A)$,

$$\begin{aligned}
\beta(\nu)((a)^+) \wedge \beta(\nu)((b)^+) &= \sup\{\nu(s) : (s)^+ = (a)^+\} \wedge \sup\{\nu(t) : (t)^+ = (b)^+\} \\
&= \sup\{\nu(s) \wedge \nu(t) : (s)^+ = (a)^+, (t)^+ = (b)^+\} \\
&\leq \sup\{\nu(s \wedge t) : (s \wedge t)^+ = (a \wedge b)^+\} \\
&= \beta(\nu)((a \wedge b)^+) = \beta(\nu)((a)^+ \sqcup (b)^+), \\
\beta(\nu)((a)^+) \vee \beta(\nu)((b)^+) &= \sup\{\nu(s) : (s)^+ = (a)^+\} \vee \sup\{\nu(t) : (t)^+ = (b)^+\} \\
&= \sup\{\nu(s) \vee \nu(t) : (s)^+ = (a)^+, (t)^+ = (b)^+\} \\
&\leq \sup\{\nu(s \vee t) : (s \vee t)^+ = (a \vee b)^+\} \\
&= \beta(\nu)((a \vee b)^+) \\
&= \beta(\nu)((a)^+ \cap (b)^+).
\end{aligned}$$

Therefore, $\beta(\nu)$ is a fuzzy ideal of $\mathcal{B}_0(A)$.

(2) For any fuzzy ideal μ of $\mathcal{B}_0(A)$. $\overleftarrow{\beta}(\mu)(m) = \mu((m)^+) = 1$. For any a, b in A,

$$\begin{aligned}
\overleftarrow{\beta}(\mu)(a \wedge b) &= \mu((a \wedge b)^+) \\
&= \mu((a)^+ \sqcup (b)^+) \\
&\geq \mu((a)^+) \wedge \mu((b)^+) \\
&= \overleftarrow{\beta}(\mu)(a) \wedge \overleftarrow{\beta}(\mu)(b) \\
\overleftarrow{\beta}(\mu)(a \vee b) &= \mu((a \vee b)^+) \\
&= \mu((a)^+ \cap (b)^+) \\
&\geq \mu((a)^+) \vee \mu((b)^+) \\
&= \overleftarrow{\beta}(\mu)(a) \vee \overleftarrow{\beta}(\mu)(b)
\end{aligned}$$

This implies $\overleftarrow{\beta}(\mu)$ is a fuzzy filter of A.

(3) Suppose that ν and η are fuzzy filters of A such that $\nu \subseteq \eta$.
$\beta(\nu)((x)^+) = \sup\{\nu(y) : (y)^+ = (x)^+\} \leq \sup\{\eta(y) : (y)^+ = (x)^+\} = \beta(\eta)((x)^+)$. Therefore β is an isotone.

(4) Similar with the proof of (3). $\square$

Lemma 3.3. *Let A be a Stone ADL. Then the map $\eta \mapsto \overleftarrow{\beta}\beta(\eta)$ is a closure operator on fuzzy filter of A. i.e., for any $\eta, \nu \in \mathcal{FF}(A)$,*

(1) $\eta \subseteq \overleftarrow{\beta}\beta(\eta)$,
(2) $\eta \subseteq \nu \Rightarrow \overleftarrow{\beta}\beta(\eta) \subseteq \overleftarrow{\beta}\beta(\nu)$,
(3) $\overleftarrow{\beta}\beta\{\overleftarrow{\beta}\beta(\nu)\} = \overleftarrow{\beta}\beta(\nu)$.

Proof. (1) For any $x \in A$, $\overleftarrow{\beta}\beta(\eta)(x) = \sup\{\eta(y) : (x)^+ = (y)^+\} \geq \eta(x)$. Thus $\eta \subseteq \overleftarrow{\beta}\beta(\eta)$

(2) It is obvious, since β and $\overleftarrow{\beta}$ are isotones.

(3) For any $s \in A$,

$$\begin{aligned} \overleftarrow{\beta}\beta\{\overleftarrow{\beta}\beta(\nu)\}(s) &= \beta\{\overleftarrow{\beta}\beta(\nu)\}((s)^+) \\ &= \sup\{\overleftarrow{\beta}\beta(\nu)(t) : (t)^+ = (s)^+, t \in A\} \\ &= \sup\{\beta(\nu)((t)^+) : (t)^+ = (s)^+, t \in A\} \\ &= \beta(\nu)((s)^+) = \overleftarrow{\beta}\beta(\nu)(s). \end{aligned}$$

□

Theorem 3.4. *Let A be a Stone ADL. Then β is a homomorphism of the lattice of fuzzy filters of A into the lattice of fuzzy ideals of $\mathcal{B}_0(A)$.*

Proof. Let $\mathcal{FF}(A)$ be the set of all fuzzy filters of A and $\mathcal{FIB}_0(A)$ be the set of all fuzzy ideals in $\mathcal{B}_0(A)$. For any $\mu, \theta \in \mathcal{FF}(A)$, $\mu \cap \theta \subseteq \mu$ and $\mu \cap \theta \subseteq \theta$. This implies $\beta(\mu \cap \theta) \subseteq \beta(\mu)$ and $\beta(\mu \cap \theta) \subseteq \beta(\theta)$. We have $\beta(\mu \cap \theta) \subseteq \beta(\theta) \cap \beta(\mu)$. Also,

$$\begin{aligned} (\beta(\mu) \cap \beta(\theta))((x)^+) &= \beta(\mu)((x)^+) \wedge \beta(\theta)((x)^+) \\ &= \sup\{\mu(a) | (a)^+ = (x)^+\} \wedge \\ &\quad \sup\{\theta(b) | (b)^+ = (x)^+\} \\ &\leq \sup\{\mu(a \vee b) : (a \vee b)^+ = (x)^+\} \wedge \\ &\quad \sup\{\theta(a \vee b) : (a \vee b)^+ = (x)^+\} \\ &= \sup\{\mu(a \vee b) \wedge \theta(a \vee b) : (a \vee b)^+ = (x)^+\} \\ &= \sup\{(\mu \cap \theta)(a \vee b) : (a \vee b)^+ = (x)^+\} \\ &= \beta(\mu \cap \theta)((x)^+). \end{aligned}$$

Thus $\beta(\mu \cap \theta) = \beta(\mu) \cap \beta(\theta)$.

Since $\mu \subseteq \mu \vee \theta$ and $\theta \subseteq \mu \vee \theta$, $\beta(\mu) \subseteq \beta(\mu \vee \theta)$ and $\beta(\mu) \subseteq \beta(\mu \vee \theta)$. This gives $\beta(\mu) \sqcup \beta(\theta) \subseteq \beta(\mu \vee \theta)$. Again

$$\begin{aligned} (\beta(\mu \vee \theta))((x)^+) &= \sup\{(\mu \vee \theta)(a) | (a)^+ = (x)^+\} \\ &= \sup\{\sup\{\mu(a_1) \wedge \theta(a_2) | a = a_1 \wedge a_2\} | (a)^+ = (x)^+\} \\ &\leq \sup\{\sup\{\mu(b_1) \wedge \theta(b_2) | (b_1)^+ = (a_1)^+, \\ &\quad (b_2)^+ = (a_2)^+\} | (a_1 \wedge a_2)^+ = (x)^+\} \\ &= \sup\{\sup\{\mu(b_1) | (b_1)^+ = (a_1)^+\} \wedge \\ &\quad \sup\{\theta(b_2) | (b_2)^+ = (a_2)^+\} | (a_1)^+ \sqcup (a_2)^+ = (x)^+\} \\ &= \sup\{\beta(\mu)((a_1)^+) \wedge \beta(\theta)((a_2)^+) | (a_1)^+ \sqcup (a_2)^+ = \\ &\quad (x)^+\} \\ &= (\beta(\mu) \sqcup \beta(\theta))((x)^+) \end{aligned}$$

This implies $\beta(\mu \vee \theta) \subseteq \beta(\mu) \sqcup \beta(\mu)$. Therefore, $\beta(\mu \vee \theta) = \beta(\mu) \sqcup \beta(\theta)$ and clearly $\chi_{\{1\}}, \chi_A$ are the smallest and the largest fuzzy filters of A respectively and also $\beta(\chi_{\{1\}}), \beta(\chi_A)$ are the smallest and the greatest fuzzy ideals of $\mathcal{B}_0(A)$ respectively. Therefore β is a homomorphism from $\mathcal{FF}(A)$ into $\mathcal{FIB}_0(A)$. □

Corollary 3.5. *For any two fuzzy filter μ and θ of a Stone ADL A, we have $\overleftarrow{\beta}\beta(\mu \cap \theta) = \overleftarrow{\beta}\beta(\mu) \cap \overleftarrow{\beta}\beta(\theta)$.*

Proof. By Theorem 3.4 , $\beta(\mu \cap \theta) = \beta(\mu) \cap \beta(\theta)$. Thus for any $t \in A$, we get

$$\begin{aligned} \overleftarrow{\beta}\beta(\mu \cap \theta)(t) &= \beta(\mu \cap \theta)((t)^+) \\ &= \beta(\mu)((t)^+) \wedge \beta(\theta)((t)^+) \\ &= \overleftarrow{\beta}\beta(\mu)((t)) \wedge \overleftarrow{\beta}\beta(\theta)((t)) \end{aligned}$$

Therefore $\overleftarrow{\beta}\beta(\mu \cap \theta) = \overleftarrow{\beta}\beta(\mu) \cap \overleftarrow{\beta}\beta(\theta)$. □

Now we introduce the notion of β-fuzzy filters in stone ADL.

Definition 3.6. A fuzzy filter μ of a Stone ADL A is called a β-fuzzy filter if $\overleftarrow{\beta}\beta(\mu) = \mu$.

Example 3.7. Let $A = \{0, a, b, c\}$. Define the binary operations $\vee$ and $\wedge$ on A as follows:

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	a	b	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

and define $x^* = 0$ if $x \neq 0$ and $0^* = a$. Then $(A, \vee, \wedge, 0)$ is a Stone ADL with 0 and $x \mapsto x^*$ is a pseudo-complementation on A. For fuzzy subsets μ and λ of A, define $\mu(0) = 0.5$, $\mu(a) = \mu(b) = \mu(c) = 1$, $\lambda(0) = 0.5$, $\lambda(a) = \lambda(b) = 1$ and $\lambda(c) = 0.7$.

It is easy to verify that μ is a β-fuzzy filter of A and λ is not β-fuzzy filter of A.

In the following Theorem, we characterize β-fuzzy filters in terms of level subsets and characteristic functions.

Theorem 3.8. *Let μ be a proper fuzzy subset of a Stone ADL A. Then μ is a β-fuzzy filter if and only if μ_α is a β-filter of A, $\forall \alpha \in [0, 1]$.*

Proof. Suppose that μ is a β-fuzzy filter of A. Then $(\overleftarrow{\beta}\beta(\mu))_\alpha = \mu_\alpha$.
To prove each level subset of μ is a β-filter of A, it is enough to show $\overleftarrow{\beta}\beta(\mu_\alpha) = \mu_\alpha$. Clearly $\mu_\alpha \subseteq \overleftarrow{\beta}\beta(\mu_\alpha)$. Next, let $x \in \overleftarrow{\beta}\beta(\mu_\alpha)$. Then $(x)^+ \in \beta(\mu_\alpha)$. This implies there exists $y \in \mu_\alpha$ such that $(x)^+ = (y)^+$, and so $\mu(y) \geq \alpha$ such that

$(x)^+ = (y)^+$. This gives $\beta(\mu)((x)^+) = \sup\{\mu(y) : (x)^+ = (y)^+\} \geq \alpha$ and so $\overleftarrow{\beta}\beta(\mu)(x) \geq \alpha$. We have $x \in (\overleftarrow{\beta}\beta(\mu))_\alpha = \mu_\alpha$. Thus $\overleftarrow{\beta}\beta(\mu_\alpha) \subseteq \mu_\alpha$. Therefore, $\overleftarrow{\beta}\beta(\mu_\alpha) = \mu_\alpha$.
Conversely, from Lemma 3.3 we get $\mu \subseteq \overleftarrow{\beta}\beta(\mu)$. Next, let $\alpha = \overleftarrow{\beta}\beta(\mu)(x) = \sup\{\mu(y) : (y)^+ = (x)^+\}$. Then for each $\epsilon > 0$, there is $a \in A, (a)^+ = (x)^+$ such that $\mu(a) > \alpha - \epsilon$. Since ϵ is arbitrary then $\mu(a) \geq \alpha$ such that $(a)^+ = (x)^+$. This implies $a \in \mu_\alpha$ for $(a)^+ = (x)^+$. This implies $(x)^+ = (a)^+ \in \beta(\mu)$. Thus $x \in \overleftarrow{\beta}\beta(\mu_\alpha) = \mu_\alpha$. Hence $\mu(x) \geq \alpha = \overleftarrow{\beta}\beta(\mu)(x)$. Therefore, $\mu = \overleftarrow{\beta}\beta(\mu)$. □

Corollary 3.9. *For a nonempty subset F of a Stone ADL A, F is a β-filter if and only if χ_F is β-fuzzy filter of A.*

In the following Theorem, the class of all β-fuzzy filters of an MS-algebra can be characterized in terms of boosters.

Theorem 3.10. *A fuzzy filter μ of a Stone ADL A is a β-fuzzy filter if and only if for all $x, y \in A, (x)^+ = (y)^+$ implies $\mu(x) = \mu(y)$.*

Proof. Suppose that μ is a β-fuzzy filter of A. Then $\mu(x) = \overleftarrow{\beta}\beta(\mu)(x), \forall x \in A$. For any $x, y \in A$ assume that $(x)^+ = (y)^+$. This implies $\mu(x) = \overleftarrow{\beta}\beta(\mu)((x) = \beta(\mu)((x)^+) = \beta(\mu)((y)^+) = \overleftarrow{\beta}\beta(\mu)(y) = \mu(y)$.
Conversely, suppose that $\forall x, y \in A, (x)^+ = (y)^+$ implies $\mu(x) = \mu(y)$. Now $\overleftarrow{\beta}\beta(\mu)(x) = \sup\{\mu(y) : (y)^+ = (x)^+\} = \mu(x)$. Therefore, $\overleftarrow{\beta}\beta(\mu) = \mu$. □

Theorem 3.11. *Let $\{\mu_i : i \in \Omega\}$ be a family of β-fuzzy filters in A. Then $\cap_{i\in\Omega}\mu_i$ is a β-fuzzy filter of A.*

It can be observed that β-fuzzy filters are simply the closed elements with respect to the closure operation of Lemma 3.3

Corollary 3.12. *Let A be a Stone ADL with maximal elements. Then the set $\mathcal{FF}_\beta(A)$ of all β-fuzzy filters of A is a complete distributive lattice with relation $\subseteq$. The supremum and infimum of any subfamily $\{\mu_i | i \in \Omega\}$ of β-fuzzy filters are $\overleftarrow{\beta}\beta(\bigvee_{i\in\Omega}\mu_i)$ and $\cap_{i\in\Omega}\mu_i$ respectively, where $\bigvee_{i\in\Omega}\mu_i$ is their supremum in the lattice of fuzzy filters of A.*

Proof. By Theorem 3.11, $\cap_{i\in\Omega}\mu_i$ is the greatest lower bound of any sub family $\{\mu_i : i \in \Omega\}$ of β-fuzzy filters of A.
Clearly $\overleftarrow{\beta}\beta(\bigvee_{i\in\Omega}\mu_i)$ is an upper bound of $\{\mu_i : i \in \Omega\}$. Let E be any β-fuzzy filter such that $\mu_i \subseteq E$ for all $i \in \Omega$.

$$\begin{aligned} &\Rightarrow \bigvee_{i\in\Omega}\mu_i \subseteq E \\ &\Rightarrow \overleftarrow{\beta}\beta(\bigvee_{i\in\Omega}\mu_i) \subseteq \overleftarrow{\beta}\beta(E) = E \end{aligned}$$

This implies $\overleftarrow{\beta}\beta(\bigvee\mu_i)$ is least upper bound of $\{\mu_i : i \in \Omega\}$.

Next, we show distributivity of A. Let $\mu, \theta, \nu \in \mathcal{FF}_\beta(A)$,

$$\begin{aligned}
\mu \cap (\theta \sqcup \nu) &= \overleftarrow{\beta}\beta(\mu) \cap \overleftarrow{\beta}\beta(\theta \vee \nu) \\
&= \overleftarrow{\beta}\beta(\mu \cap (\theta \vee \nu)) \\
&= \overleftarrow{\beta}\beta(\mu \cap \theta) \vee (\mu \vee \nu)) \\
&= (\mu \cap \theta) \sqcup (\mu \vee \nu).
\end{aligned}$$

This implies the set of all β-fuzzy filters $\mathcal{FF}_\beta(A)$ of A is a complete distributive lattice. □

Lemma 3.13. *For any fuzzy ideal μ of $\mathcal{B}_0(A)$, $\beta\overleftarrow{\beta}(\mu) = \mu$.*

Proof. Let $(x)^+ \in \mathcal{B}_0(A)$. Now $\beta\overleftarrow{\beta}(\mu)((x)^+) = \sup\{\overleftarrow{\beta}(\mu)(y) : (y)^+ = (x)^+\} = \sup\{\mu((y)^+) : (y)^+ = (x)^+\} = \mu((x)^+)$. Therefore $\beta\overleftarrow{\beta}(\mu) = \mu$. □

Using Corollary 3.12 and Lemma 3.13 , we prove that the lattice of β-fuzzy filters of L is isomorphic to the lattice of fuzzy ideals of $\mathcal{B}_0(A)$.

Theorem 3.14. *Let A be a Stone ADL with maximal elements. Then there is an isomorphism of the lattice of β-fuzzy filters of A onto the lattice of fuzzy ideals of $\mathcal{B}_0(A)$.*

Proof. Let $\mathcal{FF}_\beta(A)$ be the set of all β-fuzzy filters of A, $\mathcal{FIB}_0(A)$ be the set of all fuzzy ideals of $\mathcal{B}_0(A)$ and $f : \mathcal{FF}_\beta(A) \to \mathcal{FIB}_0(A)$ be a mapping defined by $f(\mu) = \beta(\mu)$, for any $\mu \in \mathcal{FF}_\beta(A)$. Then clearly f is one-to-one. Let μ be any fuzzy ideal of $\mathcal{B}_0(A)$. Then $\overleftarrow{\beta}(\mu)$ is a fuzzy filter of A. By Lemma 3.13, $\overleftarrow{\beta}\beta(\overleftarrow{\beta}(\mu)) = \overleftarrow{\beta}(\beta\overleftarrow{\beta}(\mu)) = \overleftarrow{\beta}(\mu)$. Thus $\overleftarrow{\beta}(\mu)$ is a β-fuzzy filter of A. Now $f(\overleftarrow{\beta}(\mu)) = \beta(\overleftarrow{\beta}(\mu)) = \mu$. This gives f is onto. Let μ, θ be any two β-fuzzy filters of A. Then clearly $f(\mu \cap \theta) = \beta(\mu \cap \theta) = \beta(\mu) \cap \beta(\theta)$. Again $f(\overleftarrow{\beta}\beta(\mu \vee \theta)) = \beta(\overleftarrow{\beta}\beta(\mu \vee \theta)) = \beta(\mu \vee \theta) = \beta(\mu) \sqcup \beta(\theta)$. Therefore f is an isomorphism of the lattice of β-fuzzy filters of A onto the lattice of fuzzy ideals of $\mathcal{B}_0(A)$. □

In the following Theorem, we show that the relation between e-fuzzy filter and β-fuzzy filter

Theorem 3.15. *Any β-fuzzy filter of a Stone ADL A is an e-fuzzy filter of A.*

Proof. Suppose that μ is β-fuzzy filter of A.

$$\begin{aligned}
\mu(x) &= \beta\overleftarrow{\beta}(\mu) \\
&= \sup\{\mu(y) : (x)^+ = (y)^+, \text{ for some } y \in A\} \\
&\geq \mu(x^{**}) \; as \; (x)^+ = (x^{**})^+
\end{aligned}$$

Clearly $\mu(x) \leq \mu(x^{**})$. Hence $\mu(x^{**}) = \mu(x)$ for all $x \in A$. Therefore every β-fuzzy filter of A is and e-fuzzy filter of A. □

4. Prime β-Fuzzy Filters and the space of prime β- fuzzy filters of a Stone Almost Distributive Lattice

In this section, we have discussed some properties of prime β-fuzzy filters and some topological concepts on the collection of prime β-fuzzy filters of a stone ADL.

Corollary 4.1. *Let A be a stone ADL. Then the prime β-fuzzy filters of A are one to one correspondence with the prime fuzzy ideals of $\mathcal{B}_0(A)$.*

Proof. From Theorem 3.14, we have seen that β-fuzzy filters of A are one to one correspondence with the fuzzy ideals of $\mathcal{B}_0(A)$. Now we prove that if μ is a prime β-fuzzy filter of A, then $\beta(\mu)$ is a prime fuzzy ideal of $\mathcal{B}_0(A)$ and vice versa. Let μ be a prime β-fuzzy filter of A. Then $\beta(\mu)$ is a fuzzy ideal of $\mathcal{B}_0(A)$. Let θ and ν be any fuzzy ideals of $\mathcal{B}_0(A)$. Then there exist β-fuzzy filters of A, say ϕ and ψ such that $\theta = \beta(\phi)$ and $\nu = \beta(\psi)$. Assume $\beta(\phi) \cap \beta(\psi) \subseteq \beta(\mu)$. Then $\beta(\phi \cap \psi) \subseteq \beta(\mu)$ and so $\phi \cap \psi \subseteq \mu$. Since μ is a prime β-filter of A, then $\phi \subseteq \mu$ or $\psi \subseteq \mu$. This gives $\beta(\phi) \subseteq \beta(\mu)$ or $\beta(\psi) \subseteq \beta(\mu)$.

Conversely, let μ be a prime fuzzy ideal of $\mathcal{B}_0(A)$. Then there exists a β-fuzzy filter η of A such that $\mu = \beta(\eta)$. Let ϕ and ψ be any fuzzy filters of A such that $\phi \cap \psi \subseteq \eta$. Then $\beta(\phi \cap \psi) = \beta(\phi) \cap \beta(\psi) \subseteq \beta(\eta)$. Since $\beta(\eta)$ is a prime ideal of A, then $\beta(\phi) \subseteq \beta(\eta)$ or $\beta(\psi) \subseteq \beta(\eta)$ and so $\phi \subseteq \eta$ or $\psi \subseteq \eta$. This implies η is a prime β-fuzzy filter of A. Thus prime β-fuzzy filters of A are one to one correspondence with the prime fuzzy ideals of $\mathcal{B}_0(A)$. ☐

In the following Theorem we prove the existence of prime β-fuzzy filters in stone ADL.

Theorem 4.2. *Let $\alpha \in [0, 1)$, μ be a β-fuzzy filter and σ be a fuzzy ideal of a stone ADL A such that $\mu \cap \sigma \leq \alpha$. Then there exists a prime β-fuzzy filter η such that $\mu \subseteq \eta$ and $\eta \cap \sigma \leq \alpha$.*

Proof. Put $\xi = \{\theta \in \mathcal{FF}_\beta(A) : \mu \subseteq \theta,\ \theta \cap \sigma \leq \alpha\}$. Clearly $\mu \in \xi$, $\xi \neq \emptyset$, and $(\xi, \subseteq)$ is a poset. Let $Q = \{\mu_i : i \in \Omega\}$ be a chain in ξ. We prove that $\cup_{i\in\Omega}\mu_i \in \xi$. Clearly $(\cup_{i\in\Omega}\mu_i)(1) = 1$. For any $x, y \in A$,

$$\begin{aligned}(\cup_{i\in\Omega}\mu_i)(x) \wedge (\cup_{i\in\Omega}\mu_i)(y) &= \sup\{\mu_i(x) : i \in \Omega\} \wedge \sup\{\mu_j(y) : j \in \Omega\}\\ &= \sup\{\mu_i(x) \wedge \mu_j(y) : i, j \in \Omega\}\\ &\leq \sup\{(\mu_i \cup \mu_j)(x) \wedge (\mu_i \cup \mu_j)(y) : i, j \in \Omega\}\end{aligned}$$

Since Q is a chain, $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$. Without loss of generality, assume $\mu_j \subseteq \mu_i$. This implies $\mu_i \cup \mu_j = \mu_i$. This shows,

$$\begin{aligned}(\cup_{i\in\Omega}\mu_i)(x) \wedge (\cup_{i\in\Omega}\mu_i)(y) &\leq \sup\{\mu_i(x) \wedge \mu_i(y),\ i \in \Omega\}\\ &= \sup\{\mu_i(x \wedge y),\ i \in \Omega\}\\ &= (\cup_{i\in\Omega}\mu_i)(x \wedge y)\end{aligned}$$

Again $(\cup_{i\in\Omega}\mu_i)(x) = \sup\{\mu_i(x) : i \in \Omega\} \leq \sup\{\mu_i(x\vee y) : i \in \Omega\} = (\cup_{i\in\Omega}\mu_i)(x\vee y)$. Similarly $(\cup_{i\in\Omega}\mu_i)(y) \leq (\cup_{i\in\Omega}\mu_i)(x\vee y)$. This implies $(\cup_{i\in\Omega}\mu_i)(x)\vee(\cup_{i\in\Omega}\mu_i)(y) \leq (\cup_{i\in\Omega}\mu_i)(x\vee y)$. Hence $\cup_{i\in\Omega}\mu_i$ is a fuzzy filter of A. Now prove that $(\cup_{i\in\Omega}\mu_i)$ is a β-fuzzy filter.

$$\begin{aligned}
\overleftarrow{\beta}\beta(\cup_{i\in\Omega}\mu_i)(x) &= \sup\{(\cup_{i\in\Omega}\mu_i)(a) : (x)^+ = (a)^+,\ a \in L\}\\
&= \sup\{\sup\{(\mu_i)(a) :,\ i \in \Omega\} : (x)^+ = (a)^+,\ a \in L\}\\
&= \sup\{\sup\{(\mu_i)(a) : (x)^+ = (a)^+,\ a \in L\},\ i \in \Omega\}\\
&= \sup\{\overleftarrow{\beta}\beta(\mu_i)(x), i \in \Omega\} = \sup\{\mu_i(x), i \in \Omega\}\\
&= (\cup_{i\in\Omega}\mu_i)(x)
\end{aligned}$$

Thus $\cup_{i\in\Omega}\mu_i$ is a β-fuzzy filter of A. Since $\mu_i \cap \sigma \leq \alpha$ for each $i \in \Omega$,

$$\begin{aligned}
((\cup_{i\in\Omega}\mu_i) \cap \sigma)(x) &= (\cup_{i\in\Omega}\mu_i)(x) \wedge \sigma(x)\\
&= \sup\{\mu_i(x),\ i \in \Omega\} \wedge \sigma(x)\\
&= \sup\{\mu_i(x) \wedge \sigma(x),\ i \in \Omega\}\\
&= \sup\{(\mu_i \wedge \sigma)(x),\ i \in \Omega\} \leq \alpha
\end{aligned}$$

Thus $(\cup_{i\in\Omega}\mu_i) \cap \sigma) \leq \alpha$. Hence $\cup_{i\in\Omega}\mu_i \in \xi$. By applying Zorn's Lemma, we get a maximal element, say δ, i.e., δ is a β-fuzzy filter of A such that $\mu \subseteq \delta$ and $\delta \cap \sigma \leq \alpha$. Next we show that δ is a prime β-fuzzy filter of A. Assume that δ is not a prime β-fuzzy filter. Let $\lambda_1,\ \lambda_2 \in FF(A)$, and $\lambda_1 \cap \lambda_2 \subseteq \delta$ such that$\lambda_1 \nsubseteq \delta$ and $\lambda_2 \nsubseteq \delta$. If we put $\delta_1 = \overleftarrow{\beta}\beta(\lambda_1 \vee \delta)$ and $\delta_2 = \overleftarrow{\beta}\beta(\lambda_2 \vee \delta)$, then both δ_1, δ_2 are β-fuzzy filters of A properly containing δ. Since δ is a maximal in ξ, we get $\delta_1, \delta_2 \notin \xi$. This indicates $\delta_1 \cap \sigma \nleq \alpha$ and $\delta_2 \cap \sigma \nleq \alpha$. This implies there exist $x, y \in A$ such that $(\delta_1 \cap \sigma)(x) > \alpha$ and $(\delta_2 \cap \sigma)(y) > \alpha$. We have $(\delta_1 \cap \sigma)(x \vee y) \wedge (\delta_2 \cap \sigma)(x \vee y) \geq (\delta_1 \cap \sigma)(x) \wedge (\delta_2 \cap \sigma)(y) > \alpha$, which implies

$$\begin{aligned}
\alpha &< (\delta_1 \cap \sigma)(x \vee y) \wedge (\delta_2 \cap \sigma)(x \vee y)\\
&= ((\delta_1 \cap \theta) \cap (\delta_2 \cap \sigma))(x \vee y)\\
&= ((\delta_2 \cap \delta_2) \cap \sigma)(x \vee y)\\
&= ((\overleftarrow{\beta}\beta(\lambda_1 \vee \delta) \cap \overleftarrow{\beta}\beta(\lambda_2 \vee \delta)) \cap \sigma)(x \vee y)\\
&= (\overleftarrow{\beta}\beta((\lambda_1 \cap \lambda_2) \vee \delta) \cap \sigma)(x \vee y)\\
&= (\overleftarrow{\beta}\beta(\delta \cap \sigma)(x \vee y) \text{ as } \lambda_1 \subseteq \delta \text{ and } \lambda_2 \subseteq \delta\\
&= (\delta \cap \sigma)(x \vee y)
\end{aligned}$$

This shows $(\delta \cap \sigma)(x \vee y) > \alpha$, which is a contradiction $\delta \cap \sigma \leq \alpha$. This δ is a prime β-fuzzy filter of A. □

Corollary 4.3. *Let μ be a fuzzy β-filter and σ be a fuzzy ideal of A such that $\mu \cap \sigma = 0$. Then there exists a prime β-fuzzy filter η such that $\mu \subseteq \eta$ and $\eta \cap \sigma = 0$.*

Corollary 4.4. *Let $\alpha \in [0,1)$, μ be a β-fuzzy filter of A and $\mu(x) \leq \alpha$. Then there exists a prime β-fuzzy filter θ of A such that $\mu \subseteq \theta$ and $\theta(x) \leq \alpha$.*

Proof. Put $\xi = \{\theta \in \mathcal{FF}_\beta(A) : \mu \subseteq \theta \text{ and } \theta(x) \leq \alpha\}$. Clearly $\mu \in \xi$, $\xi \neq \emptyset$, and $(\xi, \subseteq)$ is a poset. Let $Q = \{\mu_i : i \in \Omega\}$ be a chain in ξ. We prove that $\cup_{i\in\Omega}\mu_i \in \xi$. By Theorem 4.2, $(\cup_{i\in\Omega}\mu_i)$ is a β-fuzzy filter of A. Since $\mu_i \subseteq \theta$ for each $i \in \Omega$ and $\theta(x) \leq \alpha$.

$(\cup_{i\in\Omega}\mu_i)(x) = \sup\{\mu_i(x),\ i \in \Omega\} \leq \theta(x) \leq \alpha$.

Hence $\cup_{i\in\Omega}\mu_i \in \xi$. By applying Zorn's Lemma, we get a maximal element of ξ, say δ, i.e., δ is an β-fuzzy filter of A such that $\mu \subseteq \delta$ and $\delta(x) \leq \alpha$. Next we show that δ is a prime β-fuzzy filter of A. Assume that δ is not a prime β-fuzzy filter. Let $\lambda_1,\ \lambda_2 \in FF(A)$, and $\lambda_1 \cap \lambda_2 \subseteq \delta$ such that $\lambda_1 \nsubseteq \delta$ and $\lambda_2 \nsubseteq \delta$. If we put $\delta_1 = \overleftarrow{\beta}\beta(\lambda_1 \vee \delta)$ and $\delta_2 = \overleftarrow{\beta}\beta(\lambda_2 \vee \delta)$, then both δ_1, δ_2 are β-fuzzy filters of A properly containing δ. Since δ is maximal in ξ, we get $\delta_1, \delta_2 \notin \xi$. Thus we show that $\delta_1(x) \nleq \alpha$ and $\delta_2(x) \nleq \alpha$. This implies $\delta_1(x) > \alpha$ and $\delta_2(x) > \alpha$. We get $\delta_1(x) \wedge \delta_2(x) = (\delta_1 \cap \delta_2)(x) > \alpha$, which implies

$$\begin{aligned} \alpha &< \delta_1(x) \wedge \delta_2(x) \\ &= (\overleftarrow{\beta}\beta(\lambda_1 \vee \delta) \cap \overleftarrow{\beta}\beta(\lambda_2 \vee \delta))(x) \\ &= (\overleftarrow{\beta}\beta((\lambda_1 \cap \lambda_2) \vee \delta))(x) \\ &= \overleftarrow{\beta}\beta(\delta)(x) \text{ because } \lambda_1 \subseteq \delta \text{ and } \lambda_2 \subseteq \delta \\ &= \delta(x) \end{aligned}$$

This shows $\delta(x) > \alpha$, which is a contradiction $\delta(x) \leq \alpha$. Thus δ is a prime β-fuzzy filter of A. □

Corollary 4.5. *Every proper β-fuzzy filters of a Stone ADL A is the intersection of all prime β-fuzzy filters containing it.*

Proof. Let μ be a proper β-fuzzy filter of A. Put $\eta = \cap\{\theta : \theta \text{ is a prime } \beta\text{-fuzzy filter such that } \mu \subseteq \theta\}$. Now, we prove that $\mu = \eta$. Clearly $\mu \subseteq \eta$. Suppose $\mu(a) < \eta(a)$ for some $a \in A$. Put $\alpha = \mu(a)$ for some $a \in A$. This implies $\mu \subseteq \mu$ and $\mu(a) \leq \alpha$. Thus by the Corollary 4.4, there exists a prime β-fuzzy filter δ such that $\mu \subseteq \delta$ and $\delta(a) \leq \alpha$, which is contradicts $\mu(a) < \eta(a)$. Thus $\eta \subseteq \mu$. Hence $\mu = \eta$. This implies every proper β-fuzzy filters of A is the intersection of all prime β-fuzzy filters containing it. □

Let $\mathcal{F}Spec_\beta(A)$ denotes the set of all prime β fuzzy filters of A. For a fuzzy subset λ of A, define $H^\beta(\lambda) = \{\mu \in \mathcal{F}Spec_\beta(A) : \lambda \subseteq \mu\}$, and $X^\beta(\lambda) = \{\mu \in \mathcal{F}Spec_\beta(A) : \lambda \nsubseteq \mu\}$, We let $\lambda_* = \lambda_1$ i.e., $\lambda_* = \{x \in A : \lambda(x) = 1\}$.

Lemma 4.6. *For any fuzzy ideals μ and θ of a Stone ADL A, we have the following:*

(1) *$\mu \subseteq \theta$ if and only if $X^\beta(\mu) \subseteq X^\beta(\theta)$,*

(2) $\mu \subseteq \theta \Rightarrow H^\beta(\theta) \subseteq H^\beta(\mu)$,
(3) $X^\beta(\mu) \cap X^\beta(\theta) = X^\beta(\mu \cap \theta)$,
(4) $X^\beta(\mu) \cup X^\beta(\theta) = X^\beta(\mu \vee \theta)$.

Theorem 4.7. *The collection $\mathcal{T} = \{X^\beta(\lambda) : \lambda$ is a fuzzy ideal of $A\}$ is a topology on $\mathcal{F}Spec_\beta(A)$.*

Corollary 4.8. *For any $x, y \in A$ and $\gamma \in (0, 1]$, the following condition hold:*
(1) *If $x \le y$, then $X^\beta(x_\gamma) \subseteq X^\beta(y_\gamma)$*
(2) $X^\beta(x_\gamma) \cup X^\beta(y_\gamma) = X^\beta((x \wedge y)_\gamma)$
(3) $X^\beta(x_\gamma) \cap X^\beta(y_\gamma) = X^\beta((x \vee y)_\gamma)$
(4) $\bigcup_{x\in L,\ \gamma\in(0,1])} X^\beta(x_\gamma) = \mathcal{F}Spec_\beta(A)$
(5) $X^\beta([x_\gamma)) = X^\beta(x_\gamma)$,
(6) *$X^\beta(x_\gamma) = \emptyset \Leftrightarrow x$ is maximal.*

Corollary 4.9. *Let $\mathcal{B} = \{X^\beta(x_\gamma) : x \in A, \gamma \in (0, 1]\}$. Then $\mathcal{B}$ forms a base for topology on τ.*

Corollary 4.10. *$\mathcal{F}Spec_\beta(A)$ is a compact space.*

Theorem 4.11. *The space $\mathcal{F}Spec_\beta(A)$ is a T_0-space.*

Proof. Let $\lambda, \nu \in \mathcal{F}Spec_\beta(A)$ such that $\lambda \ne \nu$. Then either $\lambda \nsubseteq \nu$ or $\nu \nsubseteq \lambda$. Without loss of generality we can assume that $\lambda \nsubseteq \nu$. Then $\nu \in X^\beta(\lambda)$ and $\lambda \notin X^\beta(\lambda)$. Thus $\mathcal{F}Spec_\beta(A)$ is a T_0-space. □

Theorem 4.12. *For any fuzzy ideal λ of a Stone ADL A , $X^\beta(\lambda) = X^\beta(\lambda^\beta)$.*

Proof. Clearly $\lambda \subseteq \lambda^\beta$ for any fuzzy ideal λ of A. Then $X^\beta(\lambda) \subseteq X^\beta(\lambda^\beta)$. Conversely, let $\nu \in X^\beta(\lambda^\beta)$. Then $\lambda^\beta \nsubseteq \nu$. Suppose that $\nu \notin X^\beta(\lambda)$, then $\lambda \subseteq \nu$. This implies $\lambda^\beta \subseteq \nu^\beta = \nu$. Which is impossible. Thus $\nu \in X^\beta(\lambda)$ and so $X^\beta(\lambda^\beta) \subseteq X^\beta(\lambda)$. Hence $X^\beta(\lambda) = X^\beta(\lambda^\beta)$. □

Theorem 4.13. *For any fuzzy ideal λ of a Stone ADL A $X^\beta(\lambda) = \cup_{x_\gamma\in\lambda} X^\beta(x_\gamma)$.*

Proof. Let $x_\gamma \in \lambda$. Then $x_\gamma \subseteq \lambda$. This implies $X^\beta(x_\gamma) \subseteq X^\beta(\lambda)$ and so $\cup_{x_\gamma\in\lambda} X^\beta(x_\gamma) \subseteq X^\beta(\lambda)$. Conversely, $\nu \in X^\beta(\lambda)$. Then $\lambda \nsubseteq \nu$. This implies there exist $x_\gamma \notin \nu$ for some $x_\gamma \in \lambda$. This implies $\lambda \in X^\beta(x_\gamma)$ for some $x_\gamma \in \lambda$. This implies $\nu \in \cup_{x_\gamma\in\lambda} X^\beta(x_\gamma)$. Hence $X^\beta(\lambda) \subseteq \cup_{x_\gamma\in\lambda} X^\beta(x_\gamma)$. Thus $X^\beta(\lambda) = \cup_{x_\gamma\in\mu} X^\beta(x_\gamma)$. □

Theorem 4.14. *The lattice $\mathcal{FF}_\beta(A)$ is isomorphic with the lattice of all open sets $FSpec_\beta(A)$.*

Proof. The lattice of all open sets in $FSpec_\beta(A)$ is $(\mathcal{T}, \cap, \cup)$.
Define the mapping $f : \mathcal{FF}_\beta(A) \to \mathcal{T}$ by $f(\lambda) = X^\beta(\lambda)$ for all $\lambda \in FSpec_\beta(A)$. Let $\lambda, \nu \in \mathcal{FF}_\beta(A)$. Then $f(\lambda \sqcup \nu) = f((\lambda \vee \nu)^\beta) = X^\beta((\lambda \vee \nu)^\beta) = X^\beta(\lambda \vee \nu) = X^\beta(\lambda) \cup X^\beta(\nu) = f(\lambda) \cup f(\nu)$, and $f(\lambda \cap \nu) = X^\beta(\lambda \cap \nu) = X^\beta(\lambda) \cap X^\beta(\nu) = f(\lambda) \cap f(\nu)$. This shows f is homomorphism. Since $X^\beta(\lambda) = X^\beta(\lambda^\beta)$ and $\lambda^\beta \in$

$\mathcal{FF}_\beta(A)(A)$, $\forall X^\beta(\lambda) \in T$, there exists $\lambda^\beta \in \mathcal{FF}_\beta(A)$ such that $f(\lambda^\beta) = X^\beta(\lambda)$. Hence f is onto. Next we prove that f is one to one. Let $f(\lambda) = f(\nu)$. Suppose that $\lambda \neq \nu$, then there exists $x \in A$ such that either $\lambda(x) < \nu(x)$ or $\nu(x) < \lambda(x)$. Without loss of generality, we can assume that $\lambda(x) < \nu(x)$. Put $\lambda(x) = \gamma$, then by Corollary 4.4 , we can find a prime fuzzy ideal δ of A such that $\lambda \subseteq \delta$ and $\delta(x) \leq \gamma$. This implies $\delta \notin X^\beta(\lambda)$ and $\nu \nsubseteq \delta$. This show that $\delta \nsubseteq X^\beta(\lambda)$ and $\delta \in X^\beta(\nu)$. Which is a contradiction $f(\lambda) = f(\nu)$. Thus $\lambda = \lambda$. Hence f is an isomorphism. □

For any fuzzy subset ν of A, $X^\beta(\nu) = \{\lambda \in FSpec_\beta(A) : \nu \nsubseteq \lambda\}$ is open set of $\mathcal{F}Spec_\beta(A)$ and $H^\beta(\nu) = FSpec_\beta(A) - X^\beta(\nu)$ is a closed set of $\mathcal{F}Spec_\beta(A)$. Also every closed set in $\mathcal{F}Spec_\beta(A)$ is the form of $H^\beta(\nu)$ for all fuzzy subset of A. Then we have the following:

Theorem 4.15. *The closure of any $B \subseteq \mathcal{F}Spec_\beta(A)$ is given by $\overline{B} = H^\beta(\cap_{\lambda\in B}\lambda)$.*

Proof. Let $B \subseteq \mathcal{F}Spec_\beta(A)$ and $\eta \in B$. Then $\cap_{\lambda\in B}\lambda \subseteq \eta$. Thus $\eta \in H^\beta(\eta) \subseteq H^\beta(\cap_{\lambda\in B}\lambda)$. Therefore, $H^\beta(\cap_{\lambda\in B}\lambda)$ is a closed set containing B. Let C be any closed set containing B in $\mathcal{F}Spec_\beta(A)$. Then $C = H^\beta(\nu)$ for some fuzzy subset ν of A. Since $B \subseteq C = H^\beta(\nu)$, we have $\nu \subseteq \lambda$ for all $\lambda \in B$. Hence $\lambda \subseteq \cap_{\lambda\in B}\lambda$. Therefore, $H^\beta(\cap_{\lambda\in B}\lambda) \subseteq H^\beta(\nu) = C$. Hence $H^\beta(\cap_{\nu\in B}\lambda)$ is the smallest closed set containing B. Therefore, $\overline{B} = H^\beta(\cap_{\lambda\in B}\lambda)$. □

5. Conclusion and Future Work

In this paper, we studied on β-fuzzy filters of Stone almost distributive lattices and their properties. An isomorphism between the lattice of β-fuzzy filters of a Stone ADL A onto the lattice of fuzzy ideals of the set of all boosters of A is established. We discuss on some properties of prime β-fuzzy filters and some topological concepts on the collection of prime β-fuzzy filters of a Stone ADL. Further we show that the collection $\mathcal{T} = \{X^\beta(\lambda) : \lambda \text{ is a fuzzy ideal of } A\}$ is a topology on $\mathcal{F}Spec_\beta(A)$ where $X^\beta(\lambda) = \{\mu \in \mathcal{F}Spec_\beta(A) : \lambda \nsubseteq \mu\}$. We proved that any β-fuzzy filter of A is an e-fuzzy filter of A. However the converse of it is an open problem. In addition to these in the future we will study, soft β- filters of Stone almost distributive lattices, soft β-filters of Stone almost distributive lattices, soft e-fuzzy filters of Stone almost distributive lattices and soft β- fuzzy filters of Stone almost distributive lattices.

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