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CHARACTERIZING S-FLAT MODULES AND S-VON NEUMANN REGULAR RINGS BY UNIFORMITY

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Abstract. Let R be a ring and S a multiplicative subset of R. An R-module T is called u-S-torsion (u-always abbreviates uniformly) provided that $sT = 0$ for some $s \in S$. The notion of u-S-exact sequences is also introduced from the viewpoint of uniformity. An R-module F is called u-S-flat provided that the induced sequence $0 \to A \otimes_R F \to B \otimes_R F \to C \otimes_R F \to 0$ is u-S-exact for any u-S-exact sequence $0 \to A \to B \to C \to 0$. A ring R is called u-S-von Neumann regular provided there exists an element $s \in S$ satisfying that for any $a \in R$ there exists $r \in R$ such that $sa = ra^2$. We obtain that a ring R is a u-S-von Neumann regular ring if and only if any R-module is u-S-flat. Several properties of u-S-flat modules and u-S-von Neumann regular rings are obtained.

1. Introduction

Throughout this article, R is always a commutative ring with identity and S is always a multiplicative subset of R, that is, $1 \in S$ and $s_1s_2 \in S$ for any $s_1 \in S$, $s_2 \in S$. Let S be a multiplicative subset of R. Recall from [11, Definition 1.6.10] that an R-module M is called an S-*torsion* module if for any $m \in M$, there is an $s \in S$ such that $sm = 0$. S-torsion-free modules can be defined as the right part of the hereditary torsion theory τ_S generated by S-torsion modules (see [10]). Early in 1965, Năstăsescu et al. [9] defined τ_S-*Noetherian rings* as rings R satisfying that for any ideal I of R there is a finitely generated sub-ideal J of I such that I/J is S-torsion. However, to tie together some Noetherian properties of commutative rings and their polynomial rings or formal power series rings, Anderson and Dumitrescu [1] defined S-*Noetherian rings* R, that is, any ideal of R is S-finite in 2002. Recall from [1] that an R-module M is called S-*finite* provided that $sM \subseteq F$ for some $s \in S$ and some finitely generated submodule F of M. One can see that there is some uniformity is hidden in the definition of S-finite modules. In fact, an R-module M is S-finite if and only if $s(M/F) = 0$ for some $s \in S$ and some finitely

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generated submodule F of M. In this article, we introduce the notion of *u-S-torsion* modules T for which there exists $s \in S$ such that $sT = 0$. The notion of u-S-torsion modules is different from that of S-torsion modules (see Example 2.2). In the past few years, the notions of S-analogues of Noetherian rings, coherent rings, almost perfect rings and strong Mori domains are introduced and studied extensively in [1–3, 6–8].

In this article, we introduce the notions of u-S-monomorphisms, u-S-epimorphisms, u-S-isomorphisms and u-S-exact sequences according to the idea of uniformity (see Definition 2.7). Some properties of u-S-torsion modules and S-finite modules with respect to u-S-exact sequences are given in Proposition 2.8 and Proposition 2.9. We say an R-module F is *u-S-flat* provided that the induced sequence $0 \to A \otimes_R F \to B \otimes_R F \to C \otimes_R F \to 0$ is u-S-exact for any u-S-exact sequence $0 \to A \to B \to C \to 0$ (see Definition 3.1). Some basic characterizations of u-S-flat modules are given (see Theorem 3.2). It is well known that an R-module F is flat if and only $\mathrm{Tor}_1^R(R/I, F) = 0$ for any ideal I of R. However, the S-analogue of this result is not true (see Example 3.3). It is also worth remarking that the class of u-S-flat modules is not closed under direct limits and direct sums (see Remark 3.5). If an R-module F is u-S-flat, then F_S is flat over R_S (see Corollary 3.6). However, the converse does not hold (see Remark 3.7). A new local characterization of flat modules is given in Proposition 3.9. A ring R is called a u-S-von Neumann regular ring if there exists an element $s \in S$ satisfies that for any $a \in R$ there exists $r \in R$ such that $sa = ra^2$ (see Definition 3.12). A ring R is *u-S-von Neumann regular* if and only if any R-module is u-S-flat (see Theorem 3.13). Every u-S-von Neumann regular ring is locally von Neumann regular at S (see Corollary 3.14). However, the converse is also not true in general (see Example 3.15). We also give a non-trivial example of u-S-von Neumann regular which is not von Neumann regular (see Example 3.18). Finally, we give a new local characterization of von Neumann regular rings in Proposition 3.19.

2. *u*-*S*-torsion modules

Recall from [11, Definition 1.6.10] that an R-module T is said to be an S-torsion module if for any $t \in T$ there is an element $s \in S$ such that $st = 0$. Note that the choice of s is decided by the element t. In this article, we care more about the uniformity of s on T.

Definition 2.1. Let R be a ring and S a multiplicative subset of R. An R-module T is called a u-S-torsion (abbreviates uniformly S-torsion) module provided that there exists an element $s \in S$ such that $sT = 0$.

Obviously, the submodules and quotients of u-S-torsion modules are also u-S-torsion. Note that finitely generated S-torsion modules are u-S-torsion and any u-S-torsion modules are S-torsion. However, S-torsion modules are not necessary u-S-torsion. We also note that every R-module does not have a maximal u-S-torsion submodule.

Example 2.2. Let $\mathbb{Z}$ be the ring of integers, p a prime in $\mathbb{Z}$ and $S = \{p^n \mid n \geq 0\}$. Let $M = \mathbb{Z}_{(p)}/\mathbb{Z}$ be a $\mathbb{Z}$-module where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at S. Then

(1) M is S-torsion but not u-S-torsion.
(2) M has no maximal u-S-torsion submodule.

Proof. (1) Obviously, M is an S-torsion module. Suppose there is a p^n such that $p^n M = 0$. However, $p^n(\frac{1}{p^{n+1}} + \mathbb{Z}) = \frac{1}{p} + \mathbb{Z} \neq 0 + \mathbb{Z}$ in M. Thus M is not u-S-torsion.

(2) Suppose N is a maximal u-S-torsion submodule of M. Then there is an element $p^n \in S$ such that $p^n N = 0$. Note N is a submodule of $M_n := \{\frac{a}{p^n} + \mathbb{Z} \in M \mid a \in \mathbb{Z}\}$. Since $M_{n+1} := \{\frac{a}{p^{n+1}} + \mathbb{Z} \in M \mid a \in \mathbb{Z}\}$ is a u-S-torsion submodule of M and N is a proper submodule of M_{n+1}, which is a contradiction. □

Proposition 2.3. *Let R be a ring and M an R-module. Let S be a multiplicative subset of R consisting of finite elements. Then M is S-torsion if and only if M is u-S-torsion.*

Proof. If M is u-S-torsion, then M is trivially S-torsion. Let $S = \{s_1, \ldots, s_n\}$ and $s = s_1 \cdots s_n$. Suppose M is an S-torsion module. Then for any $m \in M$, there is an element $s_i \in S$ such that $s_i m = 0$. Thus $sm = 0$ for any $m \in M$. So $sM = 0$. □

Proposition 2.4. *Let R be a ring and S a multiplicative subset of R. If an R-module M has a maximal u-S-torsion submodule, then M has only one maximal u-S-torsion submodule.*

Proof. Let M_1 and M_2 be maximal u-S-torsion submodules of M such that $s_1 M_1 = 0$ and $s_2 M_2 = 0$ for some $s_1, s_2 \in S$. We claim that $M_1 = M_2$. Indeed, otherwise we may assume there is an $m \in M_2 - M_1$. Let M_3 be a submodule of M generated by M_1 and m. Then $s_1 s_2 M_3 = 0$. Thus M_3 is a u-S-torsion submodule properly containing M_1, which is a contradiction. □

Recall from [11, Definition 1.6.10] that an R-module M is said to be an *S-torsion-free module* if $sm = 0$ for some $s \in S$ and $m \in M$ implies that $m = 0$. The classes of S-torsion modules and S-torsion-free modules constitute a hereditary torsion theory (see [10]). From this result it follows immediately the next result (see [11, Theorem 6.1.6]). However we give a direct proof for completeness.

Proposition 2.5. *Let R be a ring and S a multiplicative subset of R. Then an R-module F is S-torsion-free if and only if* $\mathrm{Hom}_R(T, F) = 0$ *for any u-S-torsion module T.*

Proof. Assume that F is an S-torsion-free module and let T be a u-S-torsion module and $f \in \mathrm{Hom}_R(T, F)$. Then there exists $s \in S$ such that $sT = 0$. Thus for any $t \in T$, $sf(t) = f(st) = 0 \in F$. Thus $f(t) = 0$ for any $t \in T$. Conversely suppose that $sm = 0$ for some $s \in S$ and $m \in F$. Set $F_s = \{x \in F \mid sx = 0\}$. Then F_s is a u-S-torsion submodule of F. Thus $\mathrm{Hom}_R(F_s, F) = 0$. It follows that $F_s = 0$ and thus m=0. So F is S-torsion-free. □

Corollary 2.6. *Let R be a ring, S a multiplicative subset of R and T a u-S-torsion module. Then $\mathrm{Tor}_n^R(M,T)$ is u-S-torsion for any R-module M and $n \geq 0$.*

Proof. Let T be a u-S-torsion module with $sT = 0$. If $n = 0$, then for any $\sum a \otimes b \in M \otimes_R T$, we have $s \sum a \otimes b = \sum a \otimes sb = 0$. Thus $s(M \otimes_R T) = 0$. Let $0 \to \Omega(M) \to P \to M \to 0$ be a short exact sequence with P projective. Then $\mathrm{Tor}_1^R(M,T)$ is a submodule of $\Omega(M) \otimes_R T$ which is u-S-torsion. Thus $\mathrm{Tor}_1^R(M,T)$ is u-S-torsion. For $n \geq 2$, we have an isomorphism $\mathrm{Tor}_n^R(M,T) \cong \mathrm{Tor}_1^R(\Omega^{n-1}(M),T)$, where $\Omega^{n-1}(M)$ is the $(n-1)$-th syzygy of M. Since $\mathrm{Tor}_1^R(\Omega^{n-1}(M),T)$ is u-S-torsion by induction, $\mathrm{Tor}_n^R(M,T)$ is u-S-torsion. □

Definition 2.7. Let R be a ring and S a multiplicative subset of R. Let M, N and L be R-modules.

(1) An R-homomorphism $f : M \to N$ is called a u-S-monomorphism (resp., u-S-epimorphism) provided that $\mathrm{Ker}(f)$ (resp., $\mathrm{Coker}(f)$) is a u-S-torsion module.
(2) An R-homomorphism $f : M \to N$ is called a u-S-isomorphism provided that f is both a u-S-monomorphism and a u-S-epimorphism.
(3) An R-sequence $M \xrightarrow{f} N \xrightarrow{g} L$ is called u-S-exact provided that there is an element $s \in S$ such that $s\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$ and $s\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$.

It is easy to verify that $f : M \to N$ is a u-S-monomorphism (resp., u-S-epimorphism) if and only if $0 \to M \xrightarrow{f} N$ (resp., $M \xrightarrow{f} N \to 0$) is u-S-exact.

Proposition 2.8. *Let R be a ring, S a multiplicative subset of R and M an R-module. Then the following assertions hold.*

(1) *Suppose M is u-S-torsion and $f : L \to M$ is a u-S-monomorphism. Then L is u-S-torsion.*
(2) *Suppose M is u-S-torsion and $g : M \to N$ is a u-S-epimorphism. Then N is u-S-torsion.*
(3) *Let $f : M \to N$ be a u-S-isomorphism. If one of M and N is u-S-torsion, so is the other.*
(4) *Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a u-S-exact sequence. Then M is u-S-torsion if and only if L and N are u-S-torsion.*

Proof. We only prove (4) since (1), (2) and (3) are the consequences of (4).

Suppose M is u-S-torsion with $sM = 0$. Since $\mathrm{Ker}(f)$ (resp., $\mathrm{Coker}(g)$) is u-S-torsion with $s_1\mathrm{Ker}(f) = 0$ (resp., $s_2\mathrm{Coker}(g) = 0$) for some $s_1 \in S$ (resp., $s_2 \in S$), it follows that $ss_1L = 0$ (resp., $ss_2N = 0$). Consequently, L (resp., N) is u-S-torsion. Now suppose L and N are u-S-torsion with $s_1L = s_2N = 0$

for some $s_1, s_2 \in S$. Since the u-S-exact sequence is exact at M, there exists $s \in S$ such that $s\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$ and $s\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$. Let $m \in M$. Then $s_2 g(m) = g(s_2 m) = 0$. Thus there exists $l \in L$ such that $ss_2 m = f(l)$. So $s_1 ss_2 m = s_1 f(l) = f(s_1 l) = 0$. So M is u-S-torsion. □

Let R be a ring and S a multiplicative subset of R. Recall from [1] that an R-module M is called *S-finite* provided that there exists $s \in S$ such that $sM \subseteq N \subseteq M$, where N is a finitely generated R-module. Let M be an R-module, $\{m_i\}_{i\in\Lambda} \subseteq M$ and $N = \langle m_i\rangle_{i\in\Lambda}$. We say an R-module M is *S-generated* by $\{m_i\}_{i\in\Lambda}$ provided that $sM \subseteq N$ for some $s \in S$. Thus an R-module M is S-finite provided that M can be S-generated by finite elements.

Proposition 2.9. *Let R be a ring, S a multiplicative subset of R and M an R-module. Then the following assertions hold.*

1. *Let M be an S-finite R-module and $f : M \to N$ a u-S-epimorphism. Then N is S-finite.*
2. *Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a u-S-exact sequence. If L and N are S-finite, so is M.*
3. *Let $f : M \to N$ be a u-S-isomorphism. If one of M and N is S-finite, so is the other.*

Proof. (1) Consider the exact sequence $M \xrightarrow{f} N \to T \to 0$ with $sT = 0$ for some $s \in S$. Let F be a finitely generated submodule of M such that $s'M \subseteq F$ for some $s' \in S$. Then $f(F)$ is a finitely generated submodule of N such that $ss'N \subseteq f(F)$.

(2) Suppose $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a u-S-exact sequence. Let L_1 and N_1 be finitely generated submodules of L and N such that $s_L L \subseteq L_1$ and $s_N N \subseteq N_1$ for some $s_L, s_N \in S$, respectively. Let M_1 be a finitely generated submodule of M generated by the finite images of generators of L_1 and the finite pre-images of finite generators of N_1. Then for any $m \in M$, $s_N g(m) \in N_1$. Thus there exists $m_1 \in M_1$ such that $s_N g(m) = g(m_1)$. We have $s_N m - m_1 \in \mathrm{Ker}(g)$. Since there exists $s \in S$ such that $s\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$. So there exists $l \in L$ such that $s(s_N m - m_1) = f(l)$. Then there exists $l_1 \in L_1$ such that $s_L l = l_1$. Thus $s_L s(s_N m - m_1) = s_L f(l) = f(s_L l) = f(l_1)$. Consequently, $s_L ss_N m = s_L sm_1 + sf(l_1) \in M_1$. So $s_L ss_N M \subseteq M_1$. Since M_1 is finitely generated, we have M is S-finite.

(3) It is a consequence of (2). □

3. u-S-flat modules and u-S-von Neumann regular rings

Recall from [11] that an R-module F is called *flat* provided that for any short exact sequence $0 \to A \to B \to C \to 0$, the induced sequence $0 \to A \otimes_R F \to B \otimes_R F \to C \otimes_R F \to 0$ is exact. Now, we give an S-analogue of flat modules.

Definition 3.1. Let R be a ring, S a multiplicative subset of R. An R-module F is called u-S-flat (abbreviates uniformly S-flat) provided that for any u-S-exact sequence $0 \to A \to B \to C \to 0$, the induced sequence $0 \to A \otimes_R F \to B \otimes_R F \to C \otimes_R F \to 0$ is u-S-exact.

Recall from [11] that an R-module F is flat if and only if $\mathrm{Tor}_1^R(M,F)=0$ for any R-module M if and only if $\mathrm{Tor}_n^R(M,F)=0$ for any R-module M and $n \geq 1$. We give an S-analogue of this result.

Theorem 3.2. *Let R be a ring, S a multiplicative subset of R and F an R-module. The following statements are equivalent*:

(1) *F is u-S-flat*;
(2) *For any short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the induced sequence $0 \to A \otimes_R F \xrightarrow{f\otimes_R F} B \otimes_R F \xrightarrow{g\otimes_R F} C \otimes_R F \to 0$ is u-S-exact*;
(3) *$\mathrm{Tor}_1^R(M,F)$ is u-S-torsion for any R-module M*;
(4) *$\mathrm{Tor}_n^R(M,F)$ is u-S-torsion for any R-module M and $n \geq 1$.*

Proof. (1) ⇒ (2), (3) ⇒ (2) and (4) ⇒ (3): Trivial.

(2) ⇒ (3): Let $0 \to L \to P \to M \to 0$ be a short exact sequence with P projective. Then there exists a long exact sequence

$$0 \to \mathrm{Tor}_1^R(M,F) \to F \otimes L \to P \otimes F \to M \otimes F \to 0.$$

Thus $\mathrm{Tor}_1^R(M,F)$ is u-S-torsion by (2).

(3) ⇒ (4): Let M be an R-module. Denote the $(n-1)$-th syzygy of M by $\Omega^{n-1}(M)$. Then $\mathrm{Tor}_n^R(M,F) \cong \mathrm{Tor}_1^R(\Omega^{n-1}(M),F)$ is u-S-torsion by (3).

(2) ⇒ (1): Let F be an R-module satisfies (2). Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a u-S-exact sequence. Then there is an exact sequence $B \xrightarrow{g} C \to T \to 0$, where $T = \mathrm{Coker}(g)$ is u-S-torsion. Tensoring F over R, we have an exact sequence

$$B \otimes_R F \xrightarrow{g\otimes_R F} C \otimes_R F \to T \otimes_R F \to 0.$$

Then $T \otimes_R F$ is u-S-torsion by Corollary 2.6. Thus $0 \to A \otimes_R F \xrightarrow{f\otimes_R F} B \otimes_R F \xrightarrow{g\otimes_R F} C \otimes_R F \to 0$ is u-S-exact at $C \otimes_R F$.

There are naturally two short exact sequences: $0 \to \mathrm{Ker}(f) \to A \to \mathrm{Im}(f) \to 0$, $0 \to \mathrm{Im}(f) \to B \to \mathrm{Coker}(f) \to 0$, where $\mathrm{Ker}(f)$ is u-S-torsion. Consider the induced exact sequences

$$\to \mathrm{Ker}(f) \otimes_R F \xrightarrow{i_{\mathrm{Ker}(f)}\otimes_R F} A \otimes_R F \to \mathrm{Im}(f) \otimes_R F \to 0,$$

$$\to \mathrm{Im}(f) \otimes_R F \xrightarrow{i_{\mathrm{Im}(f)}\otimes_R F} B \otimes_R F \to \mathrm{Coker}(f) \otimes_R F \to 0,$$

where $\mathrm{Ker}(i_{\mathrm{Im}(f)} \otimes_R F)$ and $\mathrm{Ker}(i_{\mathrm{Ker}(f)} \otimes_R F)$ are u-S-torsion. We have the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Im}(i_{\mathrm{Ker}(f)} \otimes_R F) & = & \mathrm{Im}(i_{\mathrm{Ker}(f)} \otimes_R F) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Y & \longrightarrow & A \otimes_R F & \longrightarrow & \mathrm{Im}(i_{\mathrm{Im}(f)} \otimes_R F) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{Ker}(i_{\mathrm{Im}(f)} \otimes_R F) & \longrightarrow & \mathrm{Im}(f) \otimes_R F & \longrightarrow & \mathrm{Im}(i_{\mathrm{Im}(f)} \otimes_R F) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $\mathrm{Ker}(f)$ is u-S-torsion, so is $\mathrm{Ker}(f) \otimes_R F$ by Corollary 2.6. Hence $\mathrm{Im}(i_{\mathrm{Ker}(f)} \otimes_R F)$ is u-S-torsion, and thus Y is also u-S-torsion by Proposition 2.8. So the composition $f \otimes_R F : A \otimes_R F \twoheadrightarrow \mathrm{Im}(i_{\mathrm{Im}(f)} \otimes_R F) \rightarrowtail B \otimes_R F$ is a u-S-monomorphism. Thus $0 \to A \otimes_R F \xrightarrow{f \otimes_R F} B \otimes_R F \xrightarrow{g \otimes_R F} C \otimes_R F \to 0$ is u-S-exact at $A \otimes_R F$.

Since the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is u-S-exact at B, there exists $s_1 \in S$ such that $s_1\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$ and $s_1\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$. By (2), there are two exact sequences $0 \to T_1 \to s_1\mathrm{Ker}(g) \otimes_R F \to \mathrm{Im}(f) \otimes_R F$ with $s_2T_1 = 0$ for some $s_2 \in S$, and $0 \to T_2 \to s_1\mathrm{Im}(f) \otimes_R F \to \mathrm{Ker}(g) \otimes_R F$ with $s_3T_2 = 0$ for some $s_3 \in S$. Consider the induced sequence $0 \to T \to \mathrm{Ker}(g) \otimes_R F \to B \otimes_R F \to \mathrm{Coker}(g) \otimes_R F \to 0$ with $s_4T = 0$ for some $s_4 \in S$. Set $s = s_1s_2s_3s_4$, we will show $s\mathrm{Ker}(g \otimes_R F) \subseteq \mathrm{Im}(f \otimes_R F)$ and $s\mathrm{Im}(f \otimes_R F) \subseteq \mathrm{Ker}(g \otimes_R F)$. Consider the following exact sequence

$$0 \to T \to \mathrm{Ker}(g) \otimes_R F \xrightarrow{i_{\mathrm{Ker}(g)} \otimes_R F} B \otimes_R F \xrightarrow{g \otimes_R F} C \otimes_R F.$$

Then $\mathrm{Im}(i_{\mathrm{Ker}(g)} \otimes_R F) = \mathrm{Ker}(g \otimes_R F)$. Thus $s\mathrm{Ker}(g \otimes_R F) = s_1s_2s_3s_4\mathrm{Ker}(g \otimes_R F) = s_1s_2s_3s_4\mathrm{Im}(i_{\mathrm{Ker}(g)} \otimes_R F) \subseteq s_1s_2s_3\mathrm{Ker}(g) \otimes_R F \subseteq s_3\mathrm{Im}(f) \otimes_R F = s_3\mathrm{Im}(f \otimes_R F) \subseteq \mathrm{Im}(f \otimes_R F)$, and $s\mathrm{Im}(f \otimes_R F) = s_1s_2s_3s_4\mathrm{Im}(f) \otimes_R F \subseteq s_2s_4\mathrm{Ker}(g) \otimes_R F \subseteq s_2\mathrm{Im}(i_{\mathrm{Ker}(g)} \otimes_R F) = s_2\mathrm{Ker}(g \otimes_R F) \subseteq \mathrm{Ker}(g \otimes_R F)$. Thus $0 \to A \otimes_R F \to B \otimes_R F \to C \otimes_R F \to 0$ is u-S-exact at $B \otimes_R F$. □

By Corollary 2.6 and Theorem 3.2, flat modules and u-S-torsion modules are u-S-flat. And u-S-flat modules are flat provided that any element in S is a unit. Moreover, if any element in S is regular and all u-S-flat modules are flat, then any element in S is a unit. Indeed, for any $s \in S$, we have $R/\langle s\rangle$ is u-S-flat and thus flat. So $\langle s\rangle$ is a pure ideal of R. By [5, Theorem 1.2.15], there exists $r \in R$ such that $s(1 - rs) = 0$. Since s is regular, s is a unit.

The following example shows that the condition "$\mathrm{Tor}_1^R(M, F)$ is u-S-torsion for any R-module M" in Theorem 3.2 can not be replaced by "$\mathrm{Tor}_1^R(R/I, F)$ is u-S-torsion for any ideal I of R".

Example 3.3. Let $\mathbb{Z}$ be the ring of integers, p a prime in $\mathbb{Z}$ and $S = \{p^n \mid n \geq 0\}$ as in Example 2.2. Let $M = \mathbb{Z}_{(p)}/\mathbb{Z}$. Then $\mathrm{Tor}_1^R(R/I, M)$ is u-S-torsion for any ideal I of R. However, M is not u-S-flat.

Proof. Let $\langle n\rangle$ be an ideal of $\mathbb{Z}$. It follows from [4, Chapter I, Lemma 6.2(a)] that $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/\langle n\rangle, M) \cong \{m \in M \mid nm = 0\} = \{\frac{b}{p^a} + \mathbb{Z} \in \mathbb{Z}_{(p)}/\mathbb{Z} \mid a, b \text{ satisfies } p^a \mid nb\}$. Write $n = p^k m$ where $(p, m) = 1$. If $k = 0$, then $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/\langle n\rangle, M) = 0$. If $k \geq 1$, then $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/\langle n\rangle, M) = \{\frac{b}{p^k} + \mathbb{Z} \in \mathbb{Z}_{(p)}/\mathbb{Z} \mid a, b \in \mathbb{Z}\}$. Thus $p^k \cdot \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/\langle n\rangle, M) = 0$. So $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/\langle n\rangle, M)$ is u-S-torsion for any ideal $\langle n\rangle$ of $\mathbb{Z}$. However, $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}_{(p)}/\mathbb{Z}) \cong t(\mathbb{Z}_{(p)}/\mathbb{Z}) = \mathbb{Z}_{(p)}/\mathbb{Z}$ by [4, Chapter I, Lemma 6.2(b)]. Since $\mathbb{Z}_{(p)}/\mathbb{Z}$ is not u-S-torsion by Example 2.2, $M = \mathbb{Z}_{(p)}/\mathbb{Z}$ is not u-S-flat. □

Proposition 3.4. *Let R be a ring and S a multiplicative subset of R. Then the following statements hold.*

(1) *Any pure quotient of u-S-flat modules is u-S-flat.*
(2) *Any finite direct sum of u-S-flat modules is u-S-flat.*
(3) *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a u-S-exact sequence. If A and C are u-S-flat modules, so is B.*
(4) *Let $A \to B$ be a u-S-isomorphism. If one of A and B is u-S-flat, so is the other.*
(5) *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a u-S-exact sequence. If B and C are u-S-flat, then A is u-S-flat.*

Proof. (1) Let $0 \to A \to B \to C \to 0$ be a pure exact sequence with B u-S-flat. Let M be an R-module. Then there is an exact sequence $\mathrm{Tor}_1^R(M, B) \to \mathrm{Tor}_1^R(M, C) \to 0$. Since $\mathrm{Tor}_1^R(M, B)$ is u-S-torsion, $\mathrm{Tor}_1^R(M, C)$ also is u-S-torsion. Thus C is u-S-flat.

(2) Let $F_1, \ldots, F_n$ be u-S-flat modules. Let M be an R-module. Then there exists $s_i \in S$ such that $s_i\mathrm{Tor}_1^R(M, F_i) = 0$. Set $s = s_1 \cdots s_n$. Then $s\mathrm{Tor}_1^R(M, \bigoplus\limits_{i=1}^n F_i) \cong \bigoplus\limits_{i=1}^n s\mathrm{Tor}_1^R(M, F_i) = 0$. Thus $\bigoplus\limits_{i=1}^n F_i$ is u-S-flat.

(3) Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a u-S-exact sequence. Then there are three short exact sequences: $0 \to \mathrm{Ker}(f) \to A \to \mathrm{Im}(f) \to 0$, $0 \to \mathrm{Ker}(g) \to B \to \mathrm{Im}(g) \to 0$ and $0 \to \mathrm{Im}(g) \to C \to \mathrm{Coker}(g) \to 0$. Then $\mathrm{Ker}(f)$ and $\mathrm{Coker}(g)$ are all u-S-torsion and $s\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$ and $s\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$ for some $s \in S$. Let M be an R-module. Suppose A and C are u-S-flat. Then

$$\mathrm{Tor}_1^R(M, A) \to \mathrm{Tor}_1^R(M, \mathrm{Im}(f)) \to M \otimes_R \mathrm{Ker}(f)$$

is exact. Since $\mathrm{Ker}(f)$ is u-S-torsion and A is u-S-flat, it follows that $\mathrm{Tor}_1^R(M, \mathrm{Im}(f))$ is u-S-torsion. Note

$$\mathrm{Tor}_2^R(M, \mathrm{Coker}(g)) \to \mathrm{Tor}_1^R(M, \mathrm{Im}(g)) \to \mathrm{Tor}_1^R(M, C)$$

is exact. Since $\mathrm{Coker}(g)$ is u-S-torsion, $\mathrm{Tor}_2^R(M, \mathrm{Coker}(g))$ is u-S-torsion by Corollary 2.6. Thus $\mathrm{Tor}_1^R(M, \mathrm{Im}(g))$ is u-S-torsion as $\mathrm{Tor}_1^R(M, C)$ is u-S-torsion. We also note that

$$\mathrm{Tor}_1^R(M, \mathrm{Ker}(g)) \to \mathrm{Tor}_1^R(M, B) \to \mathrm{Tor}_1^R(M, \mathrm{Im}(g))$$

is exact. Thus to verify $\mathrm{Tor}_1^R(M, B)$ is u-S-torsion, we just need to show $\mathrm{Tor}_1^R(M, \mathrm{Ker}(g))$ is u-S-torsion. Set $N = \mathrm{Ker}(g) + \mathrm{Im}(f)$. Consider the following two exact sequences

$$0 \to \mathrm{Ker}(g) \to N \to N/\mathrm{Ker}(g) \to 0 \text{ and } 0 \to \mathrm{Im}(f) \to N \to N/\mathrm{Im}(f) \to 0.$$

Then it is easy to verify $N/\mathrm{Ker}(g)$ and $N/\mathrm{Im}(f)$ are all u-S-torsion. Consider the following induced two exact sequences

$$\mathrm{Tor}_2^R(M, N/\mathrm{Im}(f)) \to \mathrm{Tor}_1^R(M, \mathrm{Ker}(g)) \to \mathrm{Tor}_1^R(M, N) \to \mathrm{Tor}_1^R(M, N/\mathrm{Im}(f)),$$

$$\mathrm{Tor}_2^R(M, N/\mathrm{Ker}(g)) \to \mathrm{Tor}_1^R(M, \mathrm{Im}(f)) \to \mathrm{Tor}_1^R(M, N) \to \mathrm{Tor}_1^R(M, N/\mathrm{Ker}(g)).$$

Thus $\mathrm{Tor}_1^R(M, \mathrm{Ker}(g))$ is u-S-torsion if and only if $\mathrm{Tor}_1^R(M, \mathrm{Im}(f))$ is u-S-torsion. Consequently, B is u-S-flat since $\mathrm{Tor}_1^R(M, \mathrm{Im}(f))$ is proved to be u-S-torsion as above.

(4) It can be certainly deduced from (3).

(5) Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a u-S-exact sequence. Then, as in the proof of (3), there are three short exact sequences: $0 \to \mathrm{Ker}(f) \to A \to \mathrm{Im}(f) \to 0$, $0 \to \mathrm{Ker}(g) \to B \to \mathrm{Im}(g) \to 0$ and $0 \to \mathrm{Im}(g) \to C \to \mathrm{Coker}(g) \to 0$. Then $\mathrm{Ker}(f)$ and $\mathrm{Coker}(g)$ are all u-S-torsion and $s\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$ and $s\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$ for some $s \in S$. Let M be an R-module. Note that

$$\mathrm{Tor}_1^R(M, \mathrm{Ker}(f)) \to \mathrm{Tor}_1^R(M, A) \to \mathrm{Tor}_1^R(M, \mathrm{Im}(f)) \to M \otimes_R \mathrm{Ker}(f)$$

is exact. Since $\mathrm{Ker}(f)$ is u-S-torsion, $\mathrm{Tor}_1^R(M, \mathrm{Ker}(f))$ and $M \otimes_R \mathrm{Ker}(f)$ are u-S-torsion by Corollary 2.6. It just need to verify $\mathrm{Tor}_1^R(M, \mathrm{Im}(f))$ is u-S-torsion. By the proof of (3), we just need to show $\mathrm{Tor}_1^R(M, \mathrm{Ker}(g))$ is u-S-torsion. Since

$$\mathrm{Tor}_2^R(M, \mathrm{Im}(g)) \to \mathrm{Tor}_1^R(M, \mathrm{Ker}(g)) \to \mathrm{Tor}_1^R(M, B)$$

is exact and $\mathrm{Tor}_1^R(M, B)$ is u-S-torsion, we just need to show $\mathrm{Tor}_2^R(M, \mathrm{Im}(g))$ is u-S-torsion. Note that

$$\mathrm{Tor}_3^R(M, \mathrm{Coker}(g)) \to \mathrm{Tor}_2^R(M, \mathrm{Im}(g)) \to \mathrm{Tor}_2^R(M, C)$$

is exact. Since $\mathrm{Coker}(g)$ is u-S-torsion and C is u-S-flat, we have

$$\mathrm{Tor}_3^R(M, \mathrm{Coker}(g)) \text{ and } \mathrm{Tor}_2^R(M, C)$$

are u-S-torsion. So $\mathrm{Tor}_2^R(M, \mathrm{Im}(g))$ is u-S-torsion. □

Remark 3.5. It is well known that any direct limit of flat modules is flat. However, every direct limit of u-S-flat modules is not u-S-flat. Let $\mathbb{Z}$ be the ring of integers, p a prime in $\mathbb{Z}$ and $S = \{p^n \mid n \geq 0\}$ as in Example 3.3. Let $F_n = \mathbb{Z}/\langle p^n \rangle$ be a $\mathbb{Z}$-module. Then F_n is u-S-torsion, and thus u-S-flat. Note that each F_n is isomorphic to $M_n = \{\frac{a}{p^n} + \mathbb{Z} \in \mathbb{Z}_{(p)}/\mathbb{Z} \mid a \in \mathbb{Z}\}$. It is easy to verify $\mathbb{Z}_{(p)}/\mathbb{Z} = \bigcup_{i=1}^{\infty} M_n \cong \varinjlim F_n$. However, $\mathbb{Z}_{(p)}/\mathbb{Z}$ is not u-S-flat (see Example 3.3).

It is also worth noting infinite direct sums of u-S-flat modules need not be u-S-flat. Let $M_n = \{\frac{a}{p^n} + \mathbb{Z} \in \mathbb{Z}_{(p)}/\mathbb{Z} \mid a \in \mathbb{Z}\}$ as above. Then M_n is u-S-flat. Set $N = \bigoplus_{n=1}^{\infty} M_n$. Then N is a torsion module. Thus $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, N) = N$ by [4, Chapter I, Lemma 6.2(b)]. It can similarly be deduced from the proof of Example 2.2 that N is not u-S-torsion. Thus N is not u-S-flat.

Corollary 3.6. *Let R be a ring and S a multiplicative subset of R. If F is u-S-flat over a ring R, then F_S is flat over R_S.*

Proof. Let I_S be a finitely generated ideal of R_S, where I is a finitely generated ideal of R. Then there exists $s \in S$ such that $s\mathrm{Tor}_1^R(R/I, F) = 0$. Thus $0 = \mathrm{Tor}_1^R(R/I, F)_S \cong \mathrm{Tor}_1^{R_S}(R_S/I_S, F_S)$. So F_S is flat over R_S. □

Remark 3.7. Note that the converse of Corollary 3.6 does not hold. Consider $\mathbb{Z}$-module $M = \mathbb{Z}_{(p)}/\mathbb{Z}$ in Example 2.2. Let $S = \{p^n \mid n \geq 0\}$. Then $M_S = 0$ and thus is flat over $\mathbb{Z}_S$. However, M is not u-S-flat over $\mathbb{Z}$ (see Example 3.3).

Proposition 3.8. *Let R be a ring and F an R-module. Let S be a multiplicative subset of R consisting of finite elements. Then F is u-S-flat over a ring R if and only if F_S is flat over R_S.*

Proof. We just need to show that if F_S is flat over R_S, then F is u-S-flat over a ring R. Let $0 \to A \xrightarrow{f} B \to C \to 0$ be a short exact sequence over R. By tensoring F, we have an exact sequence $0 \to T \to A \otimes_R F \xrightarrow{f \otimes_R F} B \otimes_R F \to C \otimes_R F \to 0$, where T is the kernel of $f \otimes_R F$. By tensoring R_S, we have an exact sequence $0 \to T_S \to A_S \otimes_{R_S} F_S \to B_S \otimes_{R_S} F_S \to C_S \otimes_{R_S} F_S \to 0$ over R_S. Since F_S is flat over R_S, $T_S = 0$. Thus T is S-torsion. By Proposition 2.3, T is u-S-torsion. So F is u-S-flat over a ring R. □

Let $\mathfrak{p}$ be a prime ideal of R. We say an R-module F is *u-$\mathfrak{p}$-flat* shortly provided that F is u-$(R \setminus \mathfrak{p})$-flat.

Proposition 3.9. *Let R be a ring and F an R-module. Then the following statements are equivalent*:

(1) *F is flat*;
(2) *F is u-$\mathfrak{p}$-flat for any $\mathfrak{p} \in \mathrm{Spec}(R)$*;
(3) *F is u-$\mathfrak{m}$-flat for any $\mathfrak{m} \in \mathrm{Max}(R)$.*

Proof. (1) ⇒ (2) ⇒ (3) : Trivial.

(3) ⇒ (1) : Let M be an R-module. Then $\mathrm{Tor}_1^R(M, F)$ is $(R \setminus \mathfrak{m})$-torsion. Thus for any $\mathfrak{m} \in \mathrm{Max}(R)$, there exists $s_{\mathfrak{m}} \in S$ such that $s_{\mathfrak{m}}\mathrm{Tor}_1^R(M, F) = 0$. Since the ideal generated by all $s_{\mathfrak{m}}$ is R, $\mathrm{Tor}_1^R(M, F) = 0$. So F is flat. □

Recall that a ring R is called *von Neumann regular* provided that for any $a \in R$, there exists $r \in R$ such that $a = ra^2$. One of the main topics is the S-analogue of von Neumann regular rings. In order to study further, we will characterize when a ring R_S is von Neumann regular in the next result.

Proposition 3.10. *Let R be a ring and S a multiplicative subset of R. The following statements are equivalent*:

(1) *R_S is a von Neumann regular ring*;
(2) *Any principal ideal of R is S-generated by an idempotent*;
(3) *Any S-finite ideal of R is S-generated by an idempotent*;
(4) *For any $a \in R$, there exist $s \in S$ and $r \in R$ such that $sa = ra^2$*;
(5) *Fny R_S-module is flat over R_S.*

Proof. (1) $\Leftrightarrow$ (5) : It is well known. (3) $\Rightarrow$ (2) : Trivial.

(1) $\Rightarrow$ (4) : Let $a \in R$. Then there exists $\frac{r_1}{s_1} \in R_S$ such that $\frac{a}{1} = \frac{r_1}{s_1}\frac{a^2}{1}$. Thus there exists $s_2 \in S$ such that $s_1 s_2 a = s_2 r_1 a^2$. Set $s = s_1 s_2$ and $r = s_2 r_1$, (4) holds naturally.

(4) $\Rightarrow$ (1) : Let $\frac{a}{s}$ be an element in R_S. Then there are $s' \in S$ and $x \in R$ such that $s'a = xa^2$. Thus $\frac{a}{s} = \frac{sx}{s'}(\frac{a}{s})^2$. So R_S is a von Neumann regular ring.

(4) $\Rightarrow$ (2) : Let $\langle a \rangle$ be a principal ideal of R. Then there exists $s \in S$ such that $sa = ra^2$ for some $r \in R$. Set $e = ra$. Then $se = e^2$ and $e \in \langle a \rangle$. Since $sa = ea \in \langle e \rangle$, we have $s\langle a \rangle \subseteq \langle e \rangle \subseteq \langle a \rangle$.

(2) $\Rightarrow$ (3) : Let K be an S-finite ideal and $I = Ra_1 + \cdots + Ra_n$ be a finitely generated sub-ideal of I such that $s'K \subseteq I$ for some $s' \in S$. By (2), for each i there is an idempotent $e_i \in Ra_i$ such that $s_i\langle a_i \rangle \subseteq \langle e_i \rangle$ for some $s_i \in S$ $(i = 1, \ldots, n)$. Set $s = s's_1 \cdots s_n$. Then $s\langle a_i \rangle \subseteq \langle e_i \rangle$. Set $J = Re_1 + \cdots + Re_n$. Then J is a sub-ideal of I (thus of K) such that $sK \subseteq s_1 \cdots s_n I \subseteq J$. Claim that J is generated by an idempotent. Indeed, for any $x \in J$, we have $x = r_1 e_1 + \cdots + r_n e_n = r_1 e_1^2 + \cdots + r_n e_n^2 \in J^2$. Thus $J^2 = J$. Since J is finitely generated, $J = \langle e \rangle$ for some idempotent $e \in I$ by [11, Theorem 1.8.22].

(2) $\Rightarrow$ (4) : Let $a \in R$. Then there is an idempotent e such that $s\langle a \rangle \subseteq \langle e \rangle \subseteq \langle a \rangle$. If $e = ba$ for some $b \in R$, then $e = e^2 = b^2a^2$. Thus $sa = ce = cb^2a^2$ for some $cb^2 \in R$. So (4) holds. □

Recall from [3] that a ring R is called *c-S*-coherent if any S-finite ideal I is *c-S*-finitely presented, that is, there exists a finitely presented sub-ideal J of I such that $sI \subseteq J \subseteq I$. By Proposition 3.10, the following result holds since any ideal generated by an idempotent is projective, and thus is finitely presented.

Corollary 3.11. *Let R be a ring and S a multiplicative subset of R. If R_S is a von Neumann regular ring, then R is c-S-coherent.*

It is certain that for a ring R such that R_S is von Neumann regular, the element $s \in S$ such that $sa = ra^2$ for some $r \in R$ depends on $a \in R$ by

Proposition 3.10. Now we give the definition of u-S-von Neumann regular ring for which the element $s \in S$ is uniform on any element $a \in R$.

Definition 3.12. Let R be a ring and S a multiplicative subset of R. R is called a u-S-von Neumann regular ring (abbreviates uniformly S-von Neumann regular ring) provided there exists an element $s \in S$ satisfying that for any $a \in R$ there exists $r \in R$ such that $sa = ra^2$.

Let $\{M_j\}_{j\in\Gamma}$ be a family of R-modules. Let $\{m_{i,j}\}_{i\in\Lambda_j} \subseteq M_j$ for each $j \in \Gamma$ and $N_j = \langle m_{i,j}\rangle_{i\in\Lambda_j}$. We say a family of R-modules $\{M_j\}_{j\in\Gamma}$ is *u-S-generated* by $\{\{m_{i,j}\}_{i\in\Lambda_j}\}_{j\in\Gamma}$ provided that there exists an element $s \in S$ such that $sM_j \subseteq N_j$ for each $j \in \Gamma$. It is well known that a ring R is a von Neumann regular ring if and only if every R-module is flat if and only if any principal (finitely generated) ideal is generated by an idempotent (see [11, Theorem 3.6.3]). Now we give an S-analogue of this result.

Theorem 3.13. *Let R be a ring and S a multiplicative subset of R. The following statements are equivalent*:

(1) *R is a u-S-von Neumann regular ring*;
(2) *For any R-module M and N, there exists $s \in S$ such that $s\mathrm{Tor}_1^R(M,N) = 0$*;
(3) *There exists $s \in S$ such that $s\mathrm{Tor}_1^R(R/I, R/J) = 0$ for any ideals I and J of R*;
(4) *There exists $s \in S$ such that $s\mathrm{Tor}_1^R(R/I, R/J) = 0$ for any S-finite ideals I and J of R*;
(5) *There exists $s \in S$ such that $s\mathrm{Tor}_1^R(R/\langle a\rangle, R/\langle a\rangle) = 0$ for any element $a \in R$*;
(6) *Any R-module is u-S-flat*;
(7) *The class of all principal ideals of R is u-S-generated by idempotents*;
(8) *The class of all finitely generated ideals of R is u-S-generated by idempotents.*

Proof. (1) $\Leftrightarrow$ (5) : It follows from the equivalences: $s\mathrm{Tor}_1^R(R/\langle a\rangle, R/\langle a\rangle) = 0$ if and only if $\frac{s\langle a\rangle}{\langle a^2\rangle} = 0$, if and only if there exists $r \in R$ such that $sa = ra^2$.

(2) $\Leftrightarrow$ (6), (8) $\Rightarrow$ (7) and (3) $\Rightarrow$ (4) $\Rightarrow$ (5) : Trivial.

(2) $\Rightarrow$ (3): Set $M = N = \bigoplus_{I\lhd R} R/I$. Then (3) holds naturally.

(3) $\Rightarrow$ (2) : Suppose M is generated by $\{m_i \,|\, i \in \Gamma\}$ and N is generated by $\{n_i \,|\, i \in \Lambda\}$. Let Γ and Λ be well-ordered sets. Set $M_0 = 0$ and $M_\alpha = \langle m_i \,|\, i < \alpha\rangle$ for each $\alpha \le \Gamma$. Then M have a continuous filtration $\{M_\alpha \,|\, \alpha \le \Gamma\}$ with $M_{\alpha+1}/M_\alpha \cong R/I_{\alpha+1}$ and $I_\alpha = \mathrm{Ann}_R(m_\alpha + M_\alpha \cap Rm_\alpha)$. Similarly N has a continuous filtration $\{N_\beta \,|\, \beta \le \Lambda\}$ with $N_{\beta+1}/N_\beta \cong R/J_{\beta+1}$ and $J_\beta = \mathrm{Ann}_R(n_\beta + N_\beta \cap Rn_\beta)$. Since $s\mathrm{Tor}_1^R(R/I_\alpha, R/J_\beta) = 0$ for each $\alpha \le \Gamma$ and $\beta \le \Lambda$, it is easy to verify $s\mathrm{Tor}_1^R(M,N) = 0$ by transfinite induction on both positions of M and N.

(5) $\Rightarrow$ (3) : By [11, Exercise 3.20], we have $s\mathrm{Tor}_1^R(R/I, R/J) = \frac{s(I\cap J)}{IJ}$ for any ideals I and J of R. So we just need to show $s(I \cap J) \subseteq IJ$. Let $a \in I \cap J$. Since $s\mathrm{Tor}_1^R(R/\langle a\rangle, R/\langle a\rangle) = \frac{s\langle a\rangle}{\langle a^2\rangle} = 0$, it follows that $sa \in s\langle a\rangle \subseteq \langle a^2\rangle \subseteq IJ$. Thus $s\mathrm{Tor}_1^R(R/I, R/J) = 0$.

(1) $\Rightarrow$ (7) : Let s be an element in S such that $sa = ra^2$ for some $r \in R$ and any $a \in R$. Set $e = ra$. Then $se = e^2$ and $e \in \langle a\rangle$. Since $sa = ea \in \langle e\rangle$, we have $s\langle a\rangle \subseteq \langle e\rangle \subseteq \langle a\rangle$ for any $a \in R$.

(7) $\Rightarrow$ (8) : Let $\{I_j = Ra_{1,j} + \cdots + Ra_{n_j,j} \mid j \in \Gamma\}$ be the family of all finitely generated ideals of R. By (3), there exists an element $s \in S$ such that for each $j \in \Gamma$ and $i = 1, \ldots, n_j$ there is an idempotent $e_{i,j} \in Ra_{i,j}$ such that $s\langle a_{i,j}\rangle \subseteq \langle e_{i,j}\rangle$. Set $J_j = Re_{1,j} + \cdots + Re_{n_j,j}$. Then J_j is a sub-ideal of I_j such that $sJ_j \subseteq I_j \subseteq J_j$. Claim that J_j is generated by an idempotent. Indeed, for any $x \in J_j$, we have $x = r_1e_1 + \cdots + r_ne_n = r_1e_1^2 + \cdots + r_ne_n^2 \in J_j^2$. Thus $J_j^2 = J_j$. Since J_j is finitely generated, $J_j = \langle e_j\rangle$ for some idempotent $e_j \in I_j$ by [11, Theorem 1.8.22]. So $\{I_j \mid j \in \Gamma\}$ is u-S-generated by $\{\{e_j\} \mid j \in \Gamma\}$.

(7) $\Rightarrow$ (1) : There are an element $s \in S$ and a family of idempotents $\{e_a \mid a \in R\}$ such that $s\langle a\rangle \subseteq \langle e_a\rangle \subseteq \langle a\rangle$ for any $a \in R$. Write $e_a = ba$ for some $b \in R$. Then $e_a = e_a^2 = b^2a^2$. Thus $sa = ce_a = cb^2a^2$ for some $cb^2 \in R$. So R is u-S-von Neumann regular. □

Corollary 3.14. *Let R be a ring and S a multiplicative subset of R. If R is a u-S-von Neumann regular ring, then R_S is a von Neumann regular ring. Consequently, any u-S-von Neumann regular ring is c-S-coherent.*

Proof. It follows from Proposition 3.10, Corollary 3.11 and Theorem 3.13. □

Note that a ring R such that R_S is von Neumann regular is not necessary u-S-von Neumann regular.

Example 3.15. Let $\mathbb{Z}$ be the ring of all integers, $S = \mathbb{Z} \setminus \{0\}$. Then $\mathbb{Z}_S = \mathbb{Q}$ is a von Neumann regular ring. Let p be a prime in $\mathbb{Z}$ and $M = \mathbb{Z}_{(p)}/\mathbb{Z}$. Then $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}_{(p)}/\mathbb{Z}) \cong \mathbb{Z}_{(p)}/\mathbb{Z}$ by [4, Chapter I, Lemma 6.2(b)]. It is easy to verify that $n\mathbb{Z}_{(p)}/\mathbb{Z} \neq 0$ for any $n \in S$. Thus M is not u-S-torsion, and so $\mathbb{Z}$ is not a u-S-von Neumann regular ring.

Corollary 3.16. *Let R be a ring. Let S be a multiplicative subset of R consisting of finite elements. Then R is a u-S-von Neumann regular ring if and only if R_S is a von Neumann regular ring.*

Proof. We just need to show that if R_S is a von Neumann regular ring, then R is a u-S-von Neumann regular ring. Let $S = \{s_1, \ldots, s_n\}$. Set $s = s_1 \cdots s_n$. By Proposition 3.10, for any $a \in R$, there exist $s_i \in S$ and $r_a \in R$ such that $s_ia = r_aa^2$. Thus $sa = ra^2$ for any $a \in R$ and some $r \in R$. □

Since every flat module is u-S-flat, von Neumann regular rings are u-S-von Neumann regular. The following result shows u-S-von Neumann regular rings are always von Neumann regular provided S is a regular multiplicative set, i.e., the multiplicative set S is composed of non-zero-divisors.

Proposition 3.17. *Let R be a ring and S a regular multiplicative subset of R. Then R is u-S-von Neumann regular if and only if R is von Neumann regular.*

Proof. We just need to show if R is u-S-von Neumann regular, then R is von Neumann regular. Suppose R is a u-S-von Neumann regular ring. Then there exists $s \in S$ such that for any $a \in R$ there exists $r \in R$ satisfying $sa = ra^2$. Taking $a = s^2$, we have $s^3 = rs^4$. Since s is a non-zero-divisor of R, we have $1 = sr$. Thus s is a unit. So for any $a \in R$ there exists $r \in R$ such that $a = (s^{-1}r)a^2$. It follows that R is a von Neumann regular ring. □

However, the condition that "any element in S is a non-zero-divisor" in Proposition 3.17 cannot be removed. Let R be any ring and S a multiplicative subset of R containing a nilpotent element. Then R is a u-S-von Neumann regular ring. Indeed, let s be a nilpotent element in R with nilpotent index n. Then $0 = s^n \in S$. Thus for any $a \in R$, we have $0a = 0a^2 = 0$. So R is u-S-von Neumann regular. If the multiplicative subset S of R does not contain 0, the condition that "any element in S is a non-zero-divisor" in Corollary 3.17 also cannot be removed.

Example 3.18. Let $T = \mathbb{Z}_2 \times \mathbb{Z}_2$ be a semi-simple ring and $s = (1, 0) \in T$. Then any element $a \in T$ satisfies $a^2 = a$ and $2a = 0$. Let $R = T[x]/\langle sx, x^2\rangle$ with x the indeterminate and $S = \{1, s\}$ be a multiplicative subset of R. Then R is a u-S-von Neumann regular ring, but R is not von Neumann regular. Indeed, let $r = a + b\overline{x}$ be any element in R, where $\overline{x}$ is the residual element of x in R and $a, b \in T$. Then $sr = s(a + b\overline{x}) = sa = sa^2 = s(a^2 + 2ab\overline{x} + b^2\overline{x}^2) = s(a+b\overline{x})^2 = sr^2$. Thus R is u-S-von Neumann regular. However, since R is not reduced, R is not von Neumann regular by [11, Theorem 3.6.16(2), Exercise 3.48].

Let $\mathfrak{p}$ be a prime ideal of R. We say a ring R is a u-$\mathfrak{p}$-*von Neumann regular ring* shortly provided R is a u-$(R \setminus \mathfrak{p})$-von Neumann regular ring. The final result gives a new local characterization of von Neumann regular rings.

Proposition 3.19. *Let R be a ring. Then the following statements are equivalent*:

(1) *R is a von Neumann regular ring*;
(2) *R is a u-$\mathfrak{p}$-von Neumann regular ring for any $\mathfrak{p} \in \mathrm{Spec}(R)$*;
(3) *R is a u-$\mathfrak{m}$-von Neumann regular ring for any $\mathfrak{m} \in \mathrm{Max}(R)$.*

Proof. (1) $\Rightarrow$ (2) : Let F be an R-module and $\mathfrak{m} \in \mathrm{Max}(R)$. Then F is flat, and thus u-$\mathfrak{m}$-flat. So R is a u-$\mathfrak{m}$-von Neumann regular ring.

(2) $\Rightarrow$ (3) : Trivial.

(3) $\Rightarrow$ (1) : Let M be an R-module. Then M is $\mathfrak{m}$-flat for any $\mathfrak{m} \in \mathrm{Max}(R)$. Thus M is flat by Proposition 3.9. So R is a von Neumann regular ring. □

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