Bull. Korean Math. Soc. **59** (2022), No. 3, pp. 709–723
https://doi.org/10.4134/BKMS.b210426
pISSN: 1015-8634 / eISSN: 2234-3016

FINITENESS AND VANISHING RESULTS ON HYPERSURFACES WITH FINITE INDEX IN $\mathbb{R}^{n+1}$: A REVISION

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Abstract. In this note, we revise some vanishing and finiteness results on hypersurfaces with finite index in $\mathbb{R}^{n+1}$. When the hypersurface is stable minimal, we show that there is no nontrivial L^{2p} harmonic 1-form for some p. The our range of p is better than those in [7]. With the same range of p, we also give finiteness results on minimal hypersurfaces with finite index.

1. Introduction

Let M be an n-dimension complete, noncompact hypersurface in $\mathbb{R}^{n+1}$. If A denotes the second fundamental form of M and $S = |A|^2$, then M is said to satisfy a stable inequality if

$$\int_M |\nabla\psi|^2 - S\psi^2 \geq 0$$

for any compact supported function ψ. In particular, the stable inequality is always true on stable minimal hypersurfaces. Moreover, this notation has a close relationship with the index of a minimally immersed hypersurface in $\mathbb{R}^n$ which is defined to be the limit of the indices of an increasing sequence of exhausting compact domains in M. Note that the index of a domain D is the number of negative eigenvalues of the eigenvalue problem

$$(\Delta + S)f + \lambda f = 0, \ f\,|_{\partial D} = 0.$$

Geometrically, the index of M can be described as the maximum dimension of a linear space of compactly supported deformations that decrease the volume up to second order. This also has the geometric interpretation that there is only a finite dimensional space of normal variations violating the stability inequality.

Received June 1, 2021; Accepted November 8, 2021.
2020 *Mathematics Subject Classification.* Primary 53C24; Secondary 53C40, 53A05.
Key words and phrases. Finite index, finiteness result, harmonic forms, minimal hypersurfaces, rigidity theorem.
This research is funded by the University of Science, Vietnam National University, Hanoi under project number TN.21.04.

In [20], the author proved that if M has finite index, then $M \setminus B_{2R}(p)$ is stable for R large enough. Here $p \in M$ is a fixed point, and B_{2R} is the geodesic ball centered at p with radius $2R$.

The study of stable minimal hypersurfaces can be viewed as an effort to prove a generalized Bernstein's theorem. Bernstein first established that an entire minimal graph in $\mathbb{R}^3$ must be a plane. The validity of Bernstein's theorem in higher dimensions was established for the entire minimal graph in $\mathbb{R}^{n+1}$ by Simons, and many other authors, for the lower dimensional cases. Simons [19] proved for $n \leq 8$, and gave a conjecture that whether or not entire minimal graph in $\mathbb{R}^{n+1}$ is plane for any n. In 1969, Bombieri, De Giorgi, and Guisti [2] proved that Simons's conjecture is not true for higher dimension; in fact, they showed us that there exist complete minimal graphs in $\mathbb{R}^n$ which are not hyperplanes if $n \geq 9$. In 1984, Gulliver [9] studied a yet larger class of submanifolds in $\mathbb{R}^3$. He proved that a complete, oriented, minimally immersed hypersurface with finite index in $\mathbb{R}^3$ must have finite total curvature, which means $\int_M |A|^2 < \infty$. In 1989, J. Tysk [20] showed this result for some larger dimension; he proved that an oriented minimally immersed complete hypersurface M^n in Euclidean space, with $3 \leq n \leq 6$ satisfying the following volume growth condition has finite index if and only if it has finite total curvature; which means $\int_M |A|^n < \infty$. It is worth to note that in that article, Tysk said that his theorem cannot be generalized for higher dimension. In 1997, Cao, Shen, and Zhu [3] proved that a complete, oriented, stable, minimally immersed hypersurface in Euclidean space must have only one end. This theorem was generalized by Li and Wang [14] when they showed that a complete, oriented, minimally immersed hypersurface M^n in $\mathbb{R}^{n+1}$ with finite index must have finitely many ends. In a later paper [15], Li and Wang also generalized their theorem to minimal hypersurfaces with finite index in a complete manifold with nonnegative sectional curvature.

On the other hand, it is well-known that the L^2 harmonic function theory has played an important role in the study of stable minimal hypersurfaces. For example, the constant in the mean value inequality depends only on the lower bound of the Ricci curvature and the radius of the ball is essential in some of the geometric applications. Palmer [17] proved that if there exists a codimension one cycle C in M which does not separate M, then M is unstable. In [13], Li and Tam proved that the number of ends with infinite volume is bounded by the dimension of the space of L^2 harmonic functions. This means that we can apply the theory of L^2 harmonic functions, and in general the theory of L^2 harmonic 1-forms, to investigate the connectedness of manifolds at infinity.

In this paper, we study the structure of space of harmonic 1-forms with finite L^p energy for some $p > 0$ on hypersurface immersed in $\mathbb{R}^{n+1}$. Using Bochner's technique, we prove some results about the vanish and the finiteness of L^p harmonic 1-form on miminal stable hypersurface M^n immersed in $\mathbb{R}^{n+1}$ for some positive number p. The first result is stated as follows.

Theorem 1.1. *Let M be an n-dimensional complete noncompact, stable, minimal hypersurface in $\mathbb{R}^{n+1}$, with $n \geq 2$. Then, for*

$$\frac{n-\sqrt{2n}}{n-1} < p < \frac{n+\sqrt{2n}}{n-1},$$

there is no nontrivial L^{2p} harmonic 1-form on M.

It is worth to remark that Dung and Seo proved this theorem for hypersurfaces without minimal condition in [7]. However, this theorem is not just a corollary of their results. In fact, by performing a more careful computation, we improve the range of p and n when M is minimal. Indeed, it is easy to see that our range of p and n are wider than those in [7].

Besides the vanishing theorem, we also investigate some finiteness results. We obtain an interesting theorem about minimal hypersurfaces with finite index in Euclidean space.

Theorem 1.2. *Let M be an n-dimensional complete, noncompact, minimal hypersurface in $\mathbb{R}^{n+1}$, with $n \geq 3$. Assume that M has finite index. Then, for*

$$\frac{n-\sqrt{2n}}{n-1} < p < \frac{n+\sqrt{2n}}{n-1},$$

we have

$$\dim L^{2p}(H^1(M)) < \infty.$$

We remark that without minimality, we also can obtain a finiteness result (see Theorem 3.3). However, Theorem 1.2 is not only a consequence of Theorem 3.3. In fact, when the hypersurface is minimal, we can perform a more careful computation, to improve the range of p and n as stated in Theorem 1.2.

The remainder of this paper is organized as follows. In Section 2, we recall some useful fact regarding the estimation of the Ricci curvature on immersed hypersurfaces and the bounded of the number of dimension of harmonic form with finite energies, then we prove vanishing properties. Final, we use Section 3 to verify finiteness results.

2. Vanishing theorem for L^p harmonic 1-forms

To begin with, we need to have an estimation of Ricci curvature as follows.

Lemma 2.1 ([11])**.** *Let M be an n-dimensional submanifold in a Riemannian manifold N with sectional curvature K_N satisfying that $K_N \geq k$ where k is a constant. Then the Ricci curvature of M satisfies*

$$\begin{aligned} Ric \geq\ &(n-1)k + \frac{1}{n^2}\left\{2(n-1)|H|^2 - (n-2)\ \sqrt{n-1}|H|\sqrt{n|A|^2-|H|^2}\right\} \\ &- \frac{n-1}{n}|A|^2. \end{aligned}$$

Now, we give a proof of Theorem 1.1

Proof of Theorem 1.1. Let ω be an L^{2p} harmonic 1-form on M. Then, we have Bochner's formula:

$$|\omega|\Delta|\omega| = |\nabla\omega|^2 - |\nabla|\omega||^2 + Ric(\omega^\sharp, \omega^\sharp).$$

Using the Kato inequality

$$|\nabla|\omega||^2 \leq \frac{n-1}{n}|\nabla\omega|^2$$

and the Ricci curvature estimate in Lemma 2.1

$$Ric(\omega^\sharp, \omega^\sharp) \geq -\frac{n-1}{n}|\omega|^2 S,$$

we have

$$|\omega|\Delta|\omega| \geq \frac{1}{n-1}|\nabla|\omega||^2 - \frac{n-1}{n}|\omega|^2 S.$$

Combining with the identity

$$\begin{aligned}|\omega|^p\Delta|\omega|^p &= \frac{p-1}{p}|\nabla|\omega|^p|^2 + p|\omega|^{2p-2}|\omega|\Delta|\omega|,\\ |\nabla|\omega|^p|^2 &= p^2|\omega|^{2p-2}|\nabla|\omega||^2\end{aligned}$$

we have that

$$|\omega|^p\Delta|\omega|^p \geq \left(\frac{1}{p(n-1)} + \frac{p-1}{p}\right)|\nabla|\omega|^p|^2 - \frac{p(n-1)|\omega|^{2p}S}{n}.$$

Choose a cut off function $f \in C_0^\infty(M)$. Multiplying both sides by f^2, integrating over M, and using the following identity obtained from the divergence theorem

$$\int_M f^2|\omega|^p\Delta|\omega|^p = -\int_M f^2|\nabla|\omega|^p|^2 - 2\int_M f|\omega|^p\langle\nabla f, \nabla|\omega|^p\rangle,$$

we get

$$\begin{aligned}-2\int_M f|\omega|^p\langle\nabla f, \nabla|\omega|^p\rangle \geq &\left(\frac{2p-1}{p} + \frac{1}{p(n-1)}\right)\int_M f^2|\nabla|\omega|^p|^2\\ &- \frac{p(n-1)}{n}\int_M f^2|\omega|^{2p}S.\end{aligned}$$

Using Cauchy's and Young inequalities, and strong stability, we infer

$$\begin{aligned}-2f|\omega|^p\langle\nabla f, \nabla|\omega|^p\rangle &\leq \frac{1}{a}|\nabla f|^2|\omega|^{2p} + af^2|\nabla|\omega|^p|^2,\\ \int_M f^2|\omega|^{2p}S &\leq \int_M |\nabla(f|\omega|^p)|^2\\ &\leq (1+c)\int_M |\omega|^{2p}|\nabla f|^2 + \left(1+\frac{1}{c}\right)\int_M |\nabla|\omega|^p|^2 f^2,\end{aligned}$$

we get that

$$\left[\frac{1}{a} + \frac{p(n-1)}{n}(1+c)\right]\int_M |\omega|^{2p}|\nabla f|^2 \tag{1}$$

$$\geq \left[\frac{2p-1}{p} + \frac{1}{p(n-1)} - a - \frac{p(n-1)}{n}\left(1+\frac{1}{c}\right)\right]\int_M f^2|\nabla|\omega|^p|^2.$$

We have that:

$$\frac{2p-1}{p} + \frac{1}{p(n-1)} - \frac{p(n-1)}{n} > 0,$$

so we can choose positive numbers a small and c large enough such that

$$\frac{2p-1}{p} + \frac{1}{p(n-1)} - a - \frac{p(n-1)}{n}\left(1+\frac{1}{c}\right) > 0.$$

So from (1) we have:

$$A\int_M |\omega|^{2p}|\nabla f|^2 \geq B\int_M f^2|\nabla|\omega|^p|^2 + C\int_M f^2|\omega|^{2p},$$

where A, B, C are positive numbers.

Now, we choose function $f \in C_0^\infty(M)$ such that $0 \leq f \leq 1$, $f = 1$ on $B_R(p)$ for some $p \in M$, $f = 0$ on $M \setminus B_{2R}(p)$, and $|df| \leq \frac{1}{R}$. Then we get:

$$\frac{A}{R^2}\int_{B_{2R}(p)} |\omega|^{2p} \geq B\int_{B_R(p)} |\nabla|\omega|^p|^2 + C\int_{B_R(p)} |\omega|^{2p}.$$

Because $\omega \in L^{2p}(M)$, letting $R \to \infty$, we see that $\omega = 0$, which complete the proof. □

3. Finiteness results

We begin with the following useful facts in order to prove our main theorems.

Lemma 3.1 ([6, 12, 18])**.** *Let K be a finite-dimensional subspace of L^{2p} harmonic q-forms on an m-dimensional complete noncompact Riemannian manifold M for any $p > 0$. Then, there exists $\omega \in K$ such that*

$$(\dim K)^{\min\{1,p\}}\int_{B_x(r)} |\omega|^{2p} \leq \mathrm{Vol}(B_x(r))\min\left\{\binom{m}{q}, \mathrm{dimK}\right\}^{\min\{1,p\}}\sup_{B_x(r)}|\omega|^{2p}$$

for any $x \in M$ and $r > 0$.

We also note that there is a Sobolev inequality on immersed hypersurfaces in $\mathbb{R}^n$. In fact, the following Sobolev inequality was pointed out by Hoffman and Spruck in [10].

Lemma 3.2. *Suppose that M^n is a complete oriented submanifold isometrically immersed in an $(n+p)$-dimensional manifold with non-positive sectional curvatures. Then there exists a positive constant C_s such that*

$$\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C_s\int_M (|\nabla f|^2 + |H|^2 f^2)$$

for any nonnegative C^1-functions $f : M \to \mathbb{R}$ with a compact support. Here $|H|$ stands for the length of the mean curvature of M.

By adapting the argument of [4] and [5], we are now able to prove the following theorem.

Theorem 3.3. *Let M be an n-dimensional complete noncompact submanifold in $\mathbb{R}^{n+m}$, with $3 \leq n \leq 6$. Assume that M has a finite index. Then, for any*

$$\frac{1-\sqrt{1-\frac{n-2}{2\sqrt{n-1}}}}{\frac{\sqrt{n-1}}{2}} < p < \frac{1+\sqrt{1-\frac{n-2}{2\sqrt{n-1}}}}{\frac{\sqrt{n-1}}{2}},$$

we have

$$\dim L^{2p}(H^1(M)) < \infty.$$

Proof. We can assume that there is a ball $B_R(o)$ such that M has the strong stable inequality on $M \setminus B_R(o)$, that mean

$$\int_{M\setminus B_R(o)} f^2|\omega|^{2p}S \leq \int_{M\setminus B_R(o)} |\nabla(f|\omega|^p)|^2.$$

Let ω be an L^{2p} harmonic 1-form on M. By Bochner formula, we have

$$|\omega|\Delta|\omega| = |\nabla\omega|^2 - |\nabla|\omega||^2 + Ric(\omega^\sharp,\omega^\sharp).$$

Note that the Ricci curvature estimate in Lemma [11] infers

$$Ric(\omega^\sharp,\omega^\sharp) \geq \frac{|\omega|^2}{n^2}\left[2(n-1)H^2 - (n-2)|H|\sqrt{(n-1)(nS-H^2)} - n(n-1)S\right].$$

Hence, using the refined Kato inequality (see [6])

$$|\nabla|\omega||^2 \leq \frac{n-1}{n}|\nabla\omega|^2,$$

we have

$$\begin{aligned}
&|\omega|\Delta|\omega| \\
&\geq \frac{1}{n-1}|\nabla|\omega||^2 + \frac{|\omega|^2}{n^2}\left[2(n-1)H^2 - (n-2)|H|\sqrt{(n-1)(nS-H^2)} - n(n-1)S\right].
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
|\omega|^p\Delta|\omega|^p &= \frac{p-1}{p}|\nabla|\omega|^p|^2 + p|\omega|^{2p-2}|\omega|\Delta|\omega|,\\
|\nabla|\omega|^p|^2 &= p^2|\omega|^{2p-2}|\nabla|\omega||^2.
\end{aligned}$$

This implies

$$\begin{aligned}
(2)\quad &|\omega|^p\Delta|\omega|^p \\
&\geq \left(\frac{1}{p(n-1)} + \frac{p-1}{p}\right)|\nabla|\omega|^p|^2 \\
&\quad + \frac{p|\omega|^{2p}}{n^2}\left[2(n-1)H^2 - (n-2)|H|\sqrt{(n-1)(nS-H^2)} - n(n-1)S\right].
\end{aligned}$$

Let $f \in C_0^\infty(M)$. Multiplying both sides of the above inequality by f^2 and integrate over M, we get

$$\begin{aligned}
&-2\int_M f|\omega|^p\langle\nabla f,\nabla|\omega|^p\rangle\\
\geq\ &\left(\frac{2p-1}{p}+\frac{1}{p(n-1)}\right)\int_M f^2|\nabla|\omega|^p|^2+\frac{2p(n-1)}{n^2}\int_M f^2|\omega|^{2p}H^2\\
&-\frac{p(n-2)\sqrt{n-1}}{n^2}\int_M f^2|\omega|^{2p}|H|\sqrt{nS-H^2}-\frac{p(n-1)}{n}\int_M f^2|\omega|^{2p}S,
\end{aligned}$$

where we used the divergence theorem

$$\int_M f^2|\omega|^p\Delta|\omega|^p=-\int_M f^2|\nabla|\omega|^p|^2-2\int_M f|\omega|^p\langle\nabla f,\nabla|\omega|^p\rangle.$$

For any positive numbers a,b, using Cauchy inequalities

$$-2f|\omega|^p\langle\nabla f,\nabla|\omega|^p\rangle\leq a|\nabla f|^2|\omega|^{2p}+\frac{1}{a}f^2|\nabla|\omega|^p|^2,$$

$$2f^2|\omega|^{2p}|H|\sqrt{nS-H^2}\leq bf^2|\omega|^{2p}H^2+\frac{1}{b}f^2|\omega|^{2p}(nS-H^2),$$

we have

$$\begin{aligned}
(3)\qquad &a\int_M|\omega|^{2p}|\nabla f|^2\\
\geq\ &\left(-\frac{1}{a}+\frac{2p-1}{p}+\frac{1}{p(n-1)}\right)\int_M f^2|\nabla|\omega|^p|^2\\
&+\left(\frac{2p(n-1)}{n^2}-\frac{p(n-2)\sqrt{n-1}b}{2n^2}+\frac{p(n-2)\sqrt{n-1}}{2n^2b}\right)\int_M f^2|\omega|^{2p}H^2\\
&-\left(\frac{p(n-2)\sqrt{n-1}}{2nb}+\frac{p(n-1)}{n}\right)\int_M f^2|\omega|^{2p}S.
\end{aligned}$$

Since M has a finite index, there exists a compact subset $\Omega\subset M$ such that $M\setminus\Omega$ is stable (see [8, 20] for example). Hence, there exists R large enough such that

$$\int_{M\setminus B_R(o)}f^2|\omega|^{2p}S\leq\int_{M\setminus B_R(o)}|\nabla(f|\omega|^p)|^2$$

for any $f\in\mathcal{C}_0^\infty(M\setminus B_R(o))$. Therefore, for such smooth f, Young inequality implies

$$\begin{aligned}
&\int_{M\setminus B_R(o)}f^2|\omega|^{2p}S\\
\leq\ &\int_{M\setminus B_R(o)}|\nabla(f|\omega|^p)|^2\\
\leq\ &(1+c)\int_{M\setminus B_R(o)}|\omega|^{2p}|\nabla f|^2+\left(1+\frac{1}{c}\right)\int_{M\setminus B_R(o)}|\nabla|\omega|^p|^2f^2
\end{aligned}$$

for any positive number c. Plugging this inequality into (3), it turns out that

$$\left[a+(1+c)\left(\frac{p(n-2)\sqrt{n-1}}{2nb}+\frac{p(n-1)}{n}\right)\right]\int_M |\omega|^{2p}|\nabla f|^2 \tag{4}$$
$$\geq \left[-\frac{1}{a}+\frac{2p-1}{p}+\frac{1}{p(n-1)}-\left(\frac{p(n-2)\sqrt{n-1}}{2nb}+\frac{p(n-1)}{n}\right)\left(1+\frac{1}{c}\right)\right]\int_M f^2|\nabla|\omega|^p|^2$$
$$+\left[\frac{2p(n-1)}{n^2}-\frac{p(n-2)\sqrt{n-1}b}{2n^2}+\frac{p(n-2)\sqrt{n-1}}{2n^2b}\right]\int_M f^2|\omega|^{2p}H^2.$$

Now, we observe that if $0<b<\frac{2\sqrt{n-1}+n}{n-2}$, then

$$\frac{2p(n-1)}{n^2}-\frac{p(n-2)\sqrt{n-1}b}{2n^2}+\frac{p(n-2)\sqrt{n-1}}{2n^2b}>0.$$

By the assumption on p, we have that:

$$\frac{2p-1}{p}+\frac{1}{p(n-1)}-\frac{p(n-1)}{n}>\frac{p(n-2)\sqrt{n-1}}{2n\frac{2\sqrt{n-1}+n}{n-2}}.$$

Combining the above two observations, we conclude that there is a positive number b satisfying

$$\frac{2p-1}{p}+\frac{1}{p(n-1)}-\frac{p(n-1)}{n}>\frac{p(n-2)\sqrt{n-1}}{2nb}.$$

Hence, we can choose a and c large enough such that

$$-\frac{1}{a}+\frac{2p-1}{p}+\frac{1}{p(n-1)}-\left(\frac{p(n-2)\sqrt{n-1}}{2nb}+\frac{p(n-1)}{n}\right)\left(1+\frac{1}{c}\right)>0.$$

In conclusion, we have show that there exist positive numbers A,B,C such that

$$A\int_{M\setminus B_R(o)}|\nabla f|^2|\omega|^{2p}\geq B\int_{M\setminus B_R(o)}f^2|\nabla|\omega|^p|^2+C\int_{M\setminus B_R(o)}f^2|\omega|^{2p}H^2. \tag{5}$$

On the other hand, applying the Sobolev inequality and Young's inequality, we get

$$\begin{aligned}
&\frac{1}{C_s}\left(\int_{M\setminus B_R(o)}|f|\omega|^p|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\\
&\leq \int_{M\setminus B_R(o)}|\nabla(f|\omega|^p)|^2+\int_{M\setminus B_R(o)}H^2f^2|\omega|^{2p}\\
&\leq (1+\epsilon)\int_{M\setminus B_R(o)}f^2|\nabla|\omega|^p|^2+\left(1+\frac{1}{\epsilon}\right)\int_{M\setminus B_R(o)}|\nabla f|^2|\omega|^{2p}\\
&\quad+\int_{M\setminus B_R(o)}H^2f^2|\omega|^{2p}
\end{aligned}$$

for any positive number ϵ. This together with (5) implies

$$\frac{1}{C_s}\left(\int_{M\setminus B_R(o)}|f|\omega|^p|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \left(1+\frac{1}{\epsilon}+(1+\epsilon)\frac{A}{B}\right)\int_{M\setminus B_R(o)}|\nabla f|^2|\omega|^{2p} + \left(1-\frac{(1+\epsilon)C}{B}\right)\int_{M\setminus B_R(o)}H^2f^2|\omega|^{2p}.$$

Because $\frac{C}{B}$ is positive number, so we can choose ϵ large enough such that

$$1-\frac{(1+\epsilon)C}{B}<0.$$

Hence, there exists a positive constant C_1 such that

$$\left(\int_{M\setminus B_R(o)}(f|\omega|^p)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C_1\int_{M\setminus B_R(o)}|\nabla f|^2|\omega|^{2p}. \tag{6}$$

Now, given $r>R+1$, we choose $f\in\mathcal{C}_0^\infty(M\setminus B_R(o))$ satisfying $0\leq f\leq 1$ and

$$\begin{cases} f=1 & \text{on } B_r(o)\setminus B_{R+1}(o);\\ f=0 & \text{on } B_R(o)\cup(M\setminus B_{2r}(o));\\ |\nabla f|\leq C_2 & \text{on } B_{R+1}(o)\setminus B_R(o);\\ |\nabla f|\leq \frac{C_2}{r} & \text{on } B_{2r}(o)\setminus B_r(o)\end{cases}$$

for some constant $C_2>0$. Applying this test function to (6), we infer that

$$\left(\int_{B_r(o)\setminus B_{R+1}(o)}|\omega|^{\frac{2pn}{n-2}}\right)^{\frac{n-2}{n}} \leq C_3\int_{B_{R+1}(o)\setminus B_R(o)}|\omega|^{2p}+\frac{C_3}{r^2}\int_{B_{2r}(o)\setminus B_r(o)}|\omega|^{2p}.$$

Letting $r\to\infty$ and using the fact that $\omega\in L^{2p}$, we obtain

$$\left(\int_{M\setminus B_{R+1}(o)}|\omega|^{\frac{2pn}{n-2}}\right)^{\frac{n-2}{n}} \leq C_3\int_{B_{R+1}(o)\setminus B_R(o)}|\omega|^{2p}.$$

Moreover, by Hölder inequality, we have

$$\int_{B_{R+2}(o)\setminus B_{R+1}(o)}|\omega|^{2p} \leq (\mathrm{Vol}(B_{R+2}(o)\setminus B_{R+1}(o)))^{\frac{2}{n}}\left(\int_{B_{R+2}(o)\setminus B_{R+1}(o)}|\omega|^{\frac{2pn}{n-2}}\right)^{\frac{n-2}{n}}.$$

Therefore, the above two inequalities imply

$$\int_{B_{R+2}(o)\setminus B_{R+1}(o)}|\omega|^{2p}\leq C_3(\mathrm{Vol}(B_{R+2}(o)\setminus B_{R+1}(o)))^{\frac{2}{n}}\int_{B_{R+1}(o)\setminus B_R(o)}|\omega|^{2p}.$$

By adding $\int_{B_{R+1}(o)\setminus B_R(o)} |\omega|^{2p}$ to both sides of the last inequality, we get

$$\begin{aligned}&\int_{B_{R+2}(o)\setminus B_R(o)} |\omega|^{2p}\\ &\leq \left(C_3(\mathrm{Vol}(B_{R+2}(o)\setminus B_{R+1}(o)))^{\frac{2}{n}}+1\right)\int_{B_{R+1}(o)\setminus B_R(o)} |\omega|^{2p}.\end{aligned}$$

Again, adding $\int_{B_R(o)} |\omega|^{2p}$ to both sides of the above inequality, it turns out that

$$\int_{B_{R+2}(o)} |\omega|^{2p} \leq C_4 \int_{B_{R+1}(o)} |\omega|^{2p}, \tag{7}$$

where $C_4 = C_3(Vol(B_{R+2}(o)\setminus B_{R+1}(o)))^{\frac{2}{n}}+1$.

On the other hand, by (2), we have

$$|\omega|\Delta|\omega| \geq \frac{1}{n-1}|\nabla|\omega||^2 - T|\omega|^2,$$

where $T = \frac{1}{n^2}|2(n-1)H^2-(n-2)|H|\sqrt{(n-1)(nS-H^2)}-n(n-1)S|$.

Fix $o\in M$, and take $f\in C_0^1(B_r(o))$ with sufficiently large r. Multiplying both sides of the above inequality by $f^2|\omega|^{s-2}$, with $s\geq 2p$ then integrating by parts, we obtain

$$\begin{aligned}-2\int_{B_r(o)} f|\omega|^{s-1}\langle\nabla f,\nabla|\omega|\rangle \geq \left(s-1+\frac{1}{n-1}\right)&\int_{B_r(o)} |\omega|^{s-2}f^2|\nabla|\omega||^2\\ -&\int_{B_r(o)} Tf^2|\omega|^s.\end{aligned}\tag{8}$$

Note that for any positive constant a, Young inequality infers

$$-2f|\omega|^{s-1}\langle\nabla f,\nabla|\omega|\rangle \leq a|\omega|^s|\nabla f|^2+\frac{1}{a}|\omega|^{s-2}f^2|\nabla|\omega||^2.$$

Applying this inequality in (8), we have

$$\begin{aligned}&\left(s-1+\frac{1}{(n-1)}-a\right)\int_{B_r(o)} |\omega|^{s-2}f^2|\nabla|\omega||^2\\ \leq &\int_{B_r(o)} Tf^2|\omega|^s+\frac{1}{a}\int_{B_r(o)} |\omega|^s|\nabla f|^2.\end{aligned}$$

Again, applying the Young inequality, it turns out that

$$\int_{B_r(o)} |\nabla(f|\omega|^{\frac{s}{2}})|^2 \leq 2\int_{B_r(o)} |\omega|^s|\nabla f|^2+\frac{s^2}{2}\int_{B_r(o)} |\omega|^{s-2}f^2|\nabla|\omega||^2.$$

Combining the above two inequalities, we obtain

$$\int_{B_r(o)} |\nabla(f|\omega|^{\frac{s}{2}})|^2 \leq \int_{B_r(o)} ATf^2|\omega|^s+B|\omega|^s|\nabla f|^2, \tag{9}$$

where

$$A = \frac{s^2}{2}\left(s - 1 + \frac{1}{n-1} - a\right)^{-1},$$
$$B = \frac{s^2}{2}.\frac{1}{a}.\left(s - 1 + \frac{1}{n-1} - a\right)^{-1} + 2.$$

By simple calculations, we see that

$$s - 1 + \frac{1}{n-1} \geq 2p - 1 + \frac{1}{n-1}$$
$$> \frac{2\frac{n-2}{n-1}}{1 + \sqrt{1 - \frac{n-2}{2\sqrt{n-1}}}} - 1 + \frac{1}{n-1} = \frac{\frac{(n-2)^2}{2(n-1)^{\frac{3}{2}}}}{\left(1 + \sqrt{1 - \frac{n-2}{2\sqrt{n-1}}}\right)^2}.$$

Choosing $a = \frac{1}{2}\left(s - 1 + \frac{1}{n-1}\right)$, and note that $n \geq 3$, so

$$s - 1 + \frac{1}{n-1} - a > \frac{\frac{(n-2)^2}{4(n-1)^{\frac{3}{2}}}}{\left(1 + \sqrt{1 - \frac{n-2}{2\sqrt{n-1}}}\right)^2} > \frac{1}{16n^2}.$$

Therefore, for such constant a, we have

$$A < 8n^2 s^2 \text{ and } B < 128n^4 s^2 + 2.$$

Since

$$s \geq 2p > \frac{2\frac{n-2}{n-1}}{1 + \sqrt{1 - \frac{n-2}{2\sqrt{n-1}}}} > \frac{n-2}{n-1},$$

we infer $n^4 s^2 > 2$. In particular, $A < 129n^4 s^2$ and $B < 129n^4 s^2$.

Using Sobolev inequality and (9), we have

$$C_s^{-1}\left(\int_{B_r(o)} (f|\omega|^{\frac{s}{2}})^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \int_{B_r(o)} |\nabla(f|\omega|^{\frac{s}{2}})|^2 + (H^2+1)(f|\omega|^{\frac{s}{2}})^2$$
$$\leq \int_{B_r(o)} ((AT + H^2 + 1)f^2 + B|\nabla f|^2)|\omega|^s.$$

For simplicity, we write

$$\left(\int_{B_r(o)} (f|\omega|^{\frac{s}{2}})^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq 129n^4 s^2 C_s \int_{B_r(o)} (Gf^2 + |\nabla f|^2)|\omega|^s, \tag{10}$$

where $G = T + H^2 + 1$.

Given an integer $k \geq 0$, we set $s_k = \frac{2pn^k}{(n-2)^k}$ and $\eta_k = R + 1 + \frac{1}{2^k}$. Take a function $f_k \in C_0^\infty(B_{\eta_k}(o))$ satisfying: $0 \leq f_k \leq 1$, $f_k = 1$ in $B_{\eta_{k+1}}(o)$ and

$|\nabla f_k| \le 2^{k+3}$. Using (10) with $s = s_k$ and $f = f_k$, we obtain

$$\begin{aligned}\left(\int_{B_{\eta_{k+1}}(o)} |\omega|^{s_{k+1}}\right)^{\frac{1}{s_{k+1}}} &\le (129 n^4 s_k^2 C_s)^{\frac{1}{s_k}} \left(\int_{B_{\eta_k}(o)} (4^{k+3} + G)|\omega|^{s_k}\right)^{\frac{1}{s_k}} \\ &\le (129 n^4 s_k^2 C_s (4^{k+3} + \sup_{B_{R+2}(o)} G))^{\frac{1}{s_k}} \left(\int_{B_{\eta_k}(o)} |\omega|^{s_k}\right)^{\frac{1}{s_k}} \\ &\le (s_k)^{\frac{2}{s_k}} (4^{k+k_0})^{\frac{1}{s_k}} \left(\int_{B_{\eta_k}(o)} |\omega|^{s_k}\right)^{\frac{1}{s_k}},\end{aligned}$$

where k_0 is an integer such that $129 n^4 C_s (4^3 + \frac{1}{4^k} \sup_{B_{R+2}(o)} G) \le 4^{k_0}$. By recurrence, this implies

$$||\omega||_{L^{s_{k+1}}(B_{R+1+\frac{1}{2^{k+1}}}(o))} \le \prod_{l=0}^{k} s_l^{\frac{2}{s_l}} 4^{\frac{l}{s_l}} 4^{\frac{k_0}{s_l}} ||\omega||_{L^{2p}(B_{R+2}(o))}.$$

Notice that $s_l^{\frac{2}{s_l}}, 4^{\frac{l}{s_l}} \le C^{lc^l}$, and $4^{\frac{k_0}{s_l}} \le C^{dc^l}$, where $c = \frac{n-2}{n}$ and C, d are suitable positive constants. Thus,

$$\prod_{l=0}^{\infty} s_l^{\frac{1}{s_l}} 4^{\frac{l}{s_l}} 4^{\frac{k_0}{s_l}} \le C^{\sum_l c^l (l+d)} \le D,$$

where $D > 0$ depends only on n and $\sup_{B_1(x)} G$. Taking $k \to \infty$, we obtain

$$||\omega||_{L^\infty(B_{R+1}(o))} \le D ||\omega||_{L^{2p}(B_{R+2}(o))}. \tag{11}$$

Now, take $y \in \bar{B}_{R+1}(o)$ so that $\sup_{B_{R+1}(o)} |\omega|^{2p} = |\omega(y)|^{2p}$. Since $B_1(y) \subset B_{R+2}(o)$, using (11), we obtain

$$\sup_{B_{R+1}(o)} |\omega|^{2p} \le D ||\omega||_{L^{2p}(B_1(o))} \le D ||\omega||_{L^{2p}(B_{R+2}(o))}$$

for some positive number D. Together with (7) this yields

$$\sup_{B_{R+1}(o)} |\omega|^{2p} \le E ||\omega||_{L^{2p}(B_{R+1}(o))} \tag{12}$$

for some positive constant E.

To prove that $\dim \mathcal{H}^1(L^{2p}(M)) < \infty$, let us consider any finite dimensional subspace $K \subset \mathcal{H}^1(L^{2p}(M))$. It suffices to show that the dimension of K is bounded above by some constant that is independent of K. According to Lemma 3.1, we can see that there exists an L^{2p} harmonic 1-form $\omega \in K$ such that

$$\begin{aligned}&(\dim K)^{\min\{1,p\}} \int_{B_{R+1}(o)} |\omega|^{2p} \\ &\le Vol(B_{R+1}(o)) \min\{n, \dim K\}^{\min\{1,p\}} . \sup_{B_{R+1}(o)} |\omega|^{2p}.\end{aligned}$$

From (12), it follows that

$$(\dim K)^{\min\{1,p\}} \leq F \left(\int_{B_{R+1}(o)} |\omega|^{2p} \right)^{\frac{1}{2p}-1}$$

for some positive constant F. This implies that $\dim K$ is bounded by a fixed constant. Because K is an arbitrary subspace of finite dimension, we obtain the desired conclusion. ☐

Remark 3.4. Note that since $\frac{1}{p(n-1)} + \frac{p-1}{p}$ may not positive, we can not use the argument as in [6] (Lemma 2.1) in our proof. Therefore, we need to use the Moser iteration to conclude that

$$||\omega||_{L^\infty(B_{R+1}(o))} \leq D||\omega||_{L^{2p}(B_{R+2}(o))}.$$

We note that if the hypersurface is δ-stable for some $0 < \delta < 1$, then Dung and Seo proved a vanishing result for L^p harmonic 1 forms with the same value of p in [7]. When $H = 0$, namely the manifold is minimal, we can improve the bounds of p and n as in the statement of Theorem 1.2. Now, we will give a proof of this theorem.

Proof of Theorem 1.2. Since M has a finite index, as in the proof of Theorem 3.3, we can assume that there is a ball $B_R(o)$ such that M has the strong stable inequality on $M \setminus B_R(o)$, that means

$$\int_{M\setminus B_R(o)} f^2|\omega|^{2p} S \leq \int_{M\setminus B_R(o)} |\nabla(f|\omega|^p)|^2$$

for any smooth function $f \in \mathcal{C}_0^\infty(M \setminus B_R(o))$. We use the same arguments of Theorem 3.3, then the inequality (4) becomes

$$\begin{aligned}&\left[a + (1+c)\frac{p(n-1)}{n}\right] \int_{M\setminus B_R(o)} |\omega|^{2p}|\nabla f|^2 \\ \geq &\left[-\frac{1}{a} + \frac{2p-1}{p} + \frac{1}{p(n-1)} - \frac{p(n-1)}{n}\left(1+\frac{1}{c}\right)\right] \int_{M\setminus B_R(o)} f^2|\nabla|\omega|^p|^2.\end{aligned}$$

By the assumption on the range of p, we see that

$$\frac{2p-1}{p} + \frac{1}{p(n-1)} - \frac{p(n-1)}{n} > 0.$$

Hence, we can choose a and c large enough such that

$$-\frac{1}{a} + \frac{2p-1}{p} + \frac{1}{p(n-1)} - \frac{p(n-1)}{n}\left(1+\frac{1}{c}\right) > 0.$$

Therefore, we can conclude that there exist positive constants A, B, C such that

$$A\int_{M\setminus B_R(o)} |\omega|^{2p}|\nabla f|^2 \geq B\int_{M\setminus B_R(o)} f^2|\nabla|\omega|^p|^2 + C\int_{M\setminus B_R(o)} f^2|\omega|^{2p}.$$

The rest of the proof now is similar with the proof of Theorem 3.3, we omit the details. □

We remark that our results also hold true for submanifolds of higher dimension. In fact, if the manifold M is a submanifolds of $\mathbb{R}^{n+m}$ and M satisfies a super stable inequality then Theorems 1.1-1.2 are still valid. Moreover, the proof of such results in this setting are the same with our proof here. The interested readers are referred to [1, 16, 21] for further discussion on super stable manifolds.

Acknowledgement. The author would like to express his thanks to Prof. Nguyen Thac Dung for drawing his attention to this topic and useful discussion during the preparation of this work. The author was partially supported by Vingroup Innovation Foundation VINIF under grant number VINIF.2019. ThS.18.

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