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EXISTENCE OF THE CONTINUED FRACTIONS OF $\sqrt{d}$ AND ITS APPLICATIONS

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Abstract. It is well known that the continued fraction expansion of $\sqrt{d}$ has the form $[a_0, \overline{a_1, \ldots, a_{l-1}, 2a_0}]$ and $a_1, \ldots, a_{l-1}$ is a palindromic sequence of positive integers. For a given positive integer l and a palindromic sequence of positive integers $a_1, \ldots, a_{l-1}$, we define the set $S(l; a_1, \ldots, a_{l-1}) := \{d \in \mathbb{Z} \mid d > 0, \sqrt{d} = [a_0, \overline{a_1, \ldots, a_{l-1}, 2a_0}]$, where $a_0 = \lfloor \sqrt{d} \rfloor\}$. In this paper, we completely determine when $S(l; a_1, \ldots, a_{l-1})$ is not empty in the case that l is 4, 5, 6, or 7. We also give similar results for $(1+\sqrt{d})/2$. For the case that l is 4, 5, or 6, we explicitly describe the fundamental units of the real quadratic field $\mathbb{Q}(\sqrt{d})$. Finally, we apply our results to the Mordell conjecture for the fundamental units of $\mathbb{Q}(\sqrt{d})$.

1. Introduction

For a positive square-free integer d, let t_d and u_d be positive integers such that

$$\epsilon_d = \frac{t_d + u_d\sqrt{d}}{z} > 1$$

is the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{d})$, where $z = 2$ if $d \equiv 1 \pmod 4$ and $z = 1$ otherwise. It is well known for the relation between the continued fraction of $\sqrt{d}$ and the fundamental unit of real quadratic field $\mathbb{Q}(\sqrt{d})$ [2,4–6,10,11,17,18]. Let d be a non-square positive integer. We denote the continued fraction of $\sqrt{d}$ by

$$\sqrt{d} = [a_0, a_1, \ldots] = [a_0, \overline{a_1, \ldots, a_{l_d}}],$$

where l_d is the length of the period of the continued fraction expansion. Then the period is palindromic, that is, $a_{l_d - t} = a_t$ for $1 \le t < l_d$ and $a_{l_d} = 2a_0$. On

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the other hand, let d be a non-square positive integer congruent to 1 modulo 4. We denote the continued fraction of $(1+\sqrt{d})/2$ by

$$(1+\sqrt{d})/2 = [a'_0, a'_1, \ldots] = [a'_0, \overline{a'_1, \ldots, a'_{l'_d}}],$$

where l'_d is the length of the period of the continued fraction expansion. Then the continued fraction of $(1+\sqrt{d})/2$ has a similar property with the continued fraction of $\sqrt{d}$. In fact, the period is also palindromic and $a'_{l'_d} = 2a'_0 - 1$. In [18], for a positive square-free integer d congruent to 1 modulo 4, Tomita determined general forms of the continued fraction of $(1+\sqrt{d})/2$ with period 4 or 5. For a positive square-free integer d congruent to 2 or 3 modulo 4, authors [15] determined general forms of the continued fraction of $\sqrt{d}$ with period 4. Furthermore, for a positive square-free integer d congruent to 2 or 3 modulo 4, by determining general forms of the continued fraction of $\sqrt{d}$ with period 6, authors [14] considered some properties of the fundamental units of the real quadratic field $\mathbb{Q}(\sqrt{d})$. In this paper, we exactly determine when $\sqrt{d}$ (resp. $(1+\sqrt{d})/2$) having the forms of the continued fraction presented in [14, 15] (resp. [18]) exists. Let m, n, and l be positive integers. We have the following theorems.

Theorem 1.1. *Let l_d be 4. If both m and n are even, there exists a positive integer a_0 such that $\sqrt{d} = [a_0, \overline{m, n, m, 2a_0}]$. If m is odd, there always exists d having the form of the continued expansion $\sqrt{d} = [a_0, \overline{m, n, m, 2a_0}]$ for every n. If m is even and n is odd, there does not exist d having the form of the continued expansion $\sqrt{d} = [a_0, \overline{m, n, m, 2a_0}]$.*

Theorem 1.2. *Let l_d be 5. If both m and n are odd, there exists a positive integer a_0 such that $\sqrt{d} = [a_0, \overline{m, n, n, m, 2a_0}]$. If m is even, there always exists d having the form of the continued expansion $\sqrt{d} = [a_0, \overline{m, n, n, m, 2a_0}]$ for every n. If m is odd and n is even, there does not exist d having the form of the continued expansion $\sqrt{d} = [a_0, \overline{m, n, n, m, 2a_0}]$.*

Theorem 1.3. *Let l_d be 6. If mn is even or mn is odd and l is even, there always exists a positive integer a_0 such that $\sqrt{d} = [a_0, \overline{m, n, l, n, m, 2a_0}]$. For the other cases, there does not exist such d.*

Theorem 1.4. *Let l_d be 7. Then the continued expansion $\sqrt{d}$ has the form $\sqrt{d} = [a_0, \overline{m, n, l, l, n, m, 2a_0}]$ and there exists a positive integer a_0 such that $\sqrt{d} = [a_0, \overline{m, n, l, l, n, m, 2a_0}]$ only for the following three cases*: (i) *m is even, l is even, and n is any positive integer,* (ii) *m is odd, n is odd, and l is any positive integer,* (iii) *m is odd, n is even, and l is odd.*

2. Continued expansion of $\sqrt{d}$

In this section, we recall the expression of the continued fractions of $\sqrt{d}$ (cf. [8, 16]). Let $\sqrt{d} = [a_0, a_1, \ldots]$. We define the sequences $\{p_n\}$, $\{q_n\}$, and

$\{r_n\}$ by

$$p_{-1} = 1, \quad p_0 = a_0, \quad p_n = a_n p_{n-1} + p_{n-2} \ (n \geq 1),$$
$$q_{-1} = 0, \quad q_0 = 1, \quad q_n = a_n q_{n-1} + q_{n-2} \ (n \geq 1), \tag{2.1}$$
$$r_{-1} = 1, \quad r_0 = 0, \quad r_n = a_n r_{n-1} + r_{n-2} \ (n \geq 1).$$

Note that

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n], \quad \lim_{n\to\infty} \frac{p_n}{q_n} = \sqrt{d},$$

and

$$\frac{q_n}{r_n} = [a_1, a_2, \ldots, a_n].$$

It is well known for the following recurrence relations for the sequences $\{p_n\}$, $\{q_n\}$, and $\{r_n\}$:

$$q_n r_{n-1} - r_n q_{n-1} = (-1)^n, \tag{2.2}$$

$$q_n r_{n-2} - r_n q_{n-2} = (-1)^{n-1} a_n, \tag{2.3}$$

$$p_n - a_0 q_n = r_n. \tag{2.4}$$

Now, in order to obtain the expression of $\sqrt{d}$ in term of the sequences $\{p_n\}$, $\{q_n\}$, and $\{r_n\}$, we consider the following equation:

$$\begin{aligned} \sqrt{d} &= [a_0, a_1, \ldots, a_{l_d-1}, [2a_0, \overline{a_1, \ldots, a_{l_d}}]] \\ &= [a_0, a_1, \ldots, a_{l_d-1}, a_0 + \sqrt{d}\,] \\ &= \left[a_0, \frac{(a+\sqrt{d})q_{l_d-1} + q_{l_d-2}}{(a+\sqrt{d})r_{l_d-1} + r_{l_d-2}} \right] \\ &= a_0 + \frac{(a+\sqrt{d})r_{l_d-1} + r_{l_d-2}}{(a+\sqrt{d})q_{l_d-1} + q_{l_d-2}}. \end{aligned} \tag{2.5}$$

That is,

$$dq_{l_d-1} + q_{l_d-2}\sqrt{d} = a_0^2 q_{l_d-1} + a_0 q_{l_d-2} + a_0 r_{l_d-1} + r_{l_d-1}\sqrt{d} + r_{l_d-2}. \tag{2.6}$$

Thus

$$r_{l_d-1} = q_{l_d-2}. \tag{2.7}$$

From (2.2) and (2.7), we have

$$q_{l_d-1} r_{l_d-2} - q_{l_d-2}^2 = (-1)^{l_d-1}. \tag{2.8}$$

By (2.6) and (2.7),

$$d - a_0^2 = \frac{2a_0 q_{l_d-2} + r_{l_d-2}}{q_{l_d-1}} \tag{2.9}$$

which implies $d = a_0^2 + \frac{2a_0 q_{l_d-2} + r_{l_d-2}}{q_{l_d-1}}$. Here, a_0 is a solution of

$$2q_{l_d-2}x + r_{l_d-2} \equiv 0 \pmod{q_{l_d-1}}. \tag{2.10}$$

Moreover,

$$S(l;a_1,\dots,a_{l-1}) = \{a_0^2 + \frac{2a_0q_{l-2}+r_{l-2}}{q_{l-1}} \mid a_0 \text{ is a solution of } 2q_{l-2}x + r_{l-2} \equiv 0 \pmod{q_{l-1}},\ a_0 \geq 1\}. \tag{2.11}$$

3. Proof of theorems

In this section, we investigate if there exists d having given continued fraction expansion.

Proposition 3.1. *The set*

$S(l;a_1,\dots,a_{l-1}) := \{d \in \mathbb{Z} \mid d > 0, \sqrt{d} = [a_0, \overline{a_1,\dots,a_{l-1},2a_0}]$ *where* $a_0 = \lfloor \sqrt{d} \rfloor\}$

is nonempty if and only if one of the following two cases holds:

(1) q_{l-1} *is odd*;

(2) *Both* q_{l-1} *and* r_{l-2} *are even, and* q_{l-2} *is odd.*

Proof. By (2.11), $S(l;a_1,\dots,a_{l-1})$ is nonempty if and only if

$$2q_{l-2}x + r_{l-2} \equiv 0 \pmod{q_{l-1}} \tag{3.1}$$

is solvable. Since q_{l-2} and q_{l-1} are relatively prime, (3.1) is solvable if and only if GCD$(2, q_{l-1})$ divides r_{l-2}.

If q_{l-1} is odd, (3.1) is always solvable. Thus $S(l;a_1,\dots,a_{l-1})$ is nonempty. In the case that q_{l-1} is even, (3.1) is solvable if and only if r_{l-2} is even. By (2.8), q_{l-2} is odd, which implies that (3.1) is solvable. It completes the proof. □

If l_d is 4, the continued fraction of $\sqrt{d}$ has the form $[a_0, \overline{m,n,m,2a_0}]$. We can consider the following Table 1.

Table 1.

k	-1	0	1	2	3
a_k		a_0	m	n	m
p_k	1	a_0	ma_0+1	$a_0(mn+1)+n$	$(a_0m+1)(mn+2)-1$
q_k	0	1	m	$mn+1$	$m(mn+2)$
r_k	1	0	1	n	$mn+1$

If both m and n are odd, since $q_3 = m(mn+2)$ is odd, by Proposition 3.1, $S(4;m,n,m)$ is nonempty. If m is odd and n is even, then q_3 is even. In this case, $q_2 = mn+1$ is odd and $r_2 = n$ is even, which means that $S(4;m,n,m)$ is also nonempty. It can be similarly done in the remaining case. It completes the proof of Theorem 1.1.

If l_d is 5 (resp. 6), the continued fraction of $\sqrt{d}$ has the form

$$[a_0, \overline{m,n,n,m,2a_0}] \text{ (resp. } [a_0, \overline{m,n,l,n,m,2a_0}]).$$

We also have Table 2 for $l_d = 5$ (we omit the table for $l_d = 6$). It follows the

TABLE 2.

k	-1	0	1	2	3	4
a_k		a_0	m	n	n	m
q_k	0	1	m	$mn+1$	$m(n^2+1)+n$	$m^2(n^2+1)+2mn+1$
r_k	1	0	1	n	n^2+1	$m(n^2+1)+n$

results of Theorems 1.2 and 1.3 through a similar computation as above.

Finally, if l_d is 7, the continued fraction of $\sqrt{d}$ has the form

$$[a_0, \overline{m, n, l, l, n, m, 2a_0}].$$

By direct calculation, we know that

$$q_6 \equiv (m^2n^2+1)(l^2+1)+m^2 \equiv (mn+1)(l+1)+m \pmod 2, \tag{3.2}$$

$$q_5 \equiv m+n+mn^2+l+nl^2+mn^2l^2 \equiv n(m+1)(l+1)+m+l \pmod 2, \tag{3.3}$$

and

$$r_5 \equiv n^2(l^2+1)+1 \equiv n(l+1)+1 \pmod 2. \tag{3.4}$$

By Proposition 3.1, and (3.2), (3.3), and (3.4), we get the result of Theorem 1.4. As an example, $S(4;3,2,3)$ is nonempty by Theorem 1.1. In this case, by Table 1, we know that $q_2 = 7$, $r_2 = 2$, and $q_3 = 24$. Thus, $S(4;3,2,3) = \{a_0^2 + \frac{7a_0+1}{12} \,|\, 7a_0+1 \equiv 0 \pmod{12},\ a_0 \geq 1\}$. Since $a_0 \equiv 5 \pmod{12}$, we have $S(4;3,2,3) = \{28, 299, 858, 1705, 2840, \ldots\}$.

Now, we investigate the continued fraction of $(1+\sqrt{d})/2$. In the case that l'_d is 4 or 5, Tomita [18] determined the forms of the continued fraction of $(1+\sqrt{d})/2$ as follows:

Theorem 3.2. *Let d be a positive square-free integer congruent to 1 modulo 4 and ω_d the continued fraction of $(1+\sqrt{d})/2$. Then we get*

(1) *If l'_d is 4,*

$$\omega_d = \begin{cases} [a/2, \overline{1, l, 1, a-1}] & \text{for an odd integer } l \geq 1 \text{ if } a \text{ is even,} \\ [(a+1)/2, \overline{l, v, l, a}] & \text{for two integers } l, v \geq 1 \text{ if } a \text{ is odd,} \end{cases}$$

(2) *If l'_d is 5,*

$$\omega_d = \begin{cases} [a/2, \overline{1, l, l, 1, a-1}] & \text{for integer } l \geq 0 \text{ if } a \text{ is even,} \\ [(a+1)/2, \overline{l, v, v, l, a}] & \text{for two integers } l \geq 2,\ v \geq 0 \text{ if } a \text{ is odd.} \end{cases}$$

We can ask if there always exists d having given continued fraction. Now we restate the result of Proposition 4.1 in [9].

Proposition 3.3. *Let l' be a positive integer. We define the sequences $\{p'_n\}$, $\{q'_n\}$, and $\{r'_n\}$ which is the same with (2.1) for $(1+\sqrt{d})/2$. Let $p'_{l'_d-1}/q'_{l'_d-1}$ be (l'_d-1)-th convergent of $(1+\sqrt{d})/2$. Then $S'(l'; a'_1, \ldots, a'_{l'-1}) := \{d \in \mathbb{Z} \,|\, d >$*

$0, \frac{1+\sqrt{d}}{2} = [a'_0, \overline{a'_1, \ldots, a'_{l'-1}, 2a'_0 - 1}]$ *where* $a'_0 = \lfloor \frac{1+\sqrt{d}}{2} \rfloor\}$ *is nonempty if and only if one of the following two cases holds*:

(1) $q'_{l'-1}$ *is odd*;
(2) *Both* $q'_{l'-2}$ *and* $r'_{l'-2}$ *are odd, and* $q'_{l'-1}$ *is even.*

Remark 3.4. In [9], Hashimoto gave the expression for $S'(l'; a'_1, \ldots, a'_{l'-1})$ only when d is a prime congruent to 1 modulo 4 and l' is odd. Depending on the parity of l', the explicit expression for $S'(l'; a'_1, \ldots, a'_{l'-1})$ is slightly different, but Proposition 3.3 also holds even if d is a non-square positive integer congruent to 1 modulo 4 and l' is either odd or even. In fact,

$$(3.5) \quad S'(l'; a'_1, \ldots, a'_{l'-1}) = \{4[a'_0(a'_0 - 1) + \frac{(2a'_0 - 1)q'_{l'-2} + r'_{l'-2}}{q'_{l'-1}}] + 1 \mid$$
$$a_0 \text{ is a solution of } (2x-1)q'_{l'-2} + r'_{l'-2} \equiv 0 \pmod{q'_{l'-1}},\ a_0 \geq 1\}.$$

Therefore, $S'(l'; a'_1, \ldots, a'_{l'-1})$ is nonempty if and only if

$$(3.6) \qquad (2x-1)q'_{l'-2} + r'_{l'-2} \equiv 0 \pmod{q'_{l'-1}}$$

is solvable. By using (3.6) and the fact that $q_{l'-1}r_{l'-2} - q^2_{l'-2} = (-1)^{l'-1}$, we can obtain the result of Proposition 3.3.

If l'_d is 4, the continued expansion of $(1+\sqrt{d})/2$ is the form

$$[a'_0, \overline{m, n, m, 2a'_0 - 1}].$$

If n is even, then q'_3 is even. By Proposition 3.3 and Table 1, $S'(4; m, n, m)$ is empty. It means that there does not exist d having the form of the continued expansion $[a'_0, \overline{m, n, m, 2a'_0 - 1}]$. If n is odd and m is even, by case (2) of Proposition 3.3, $S'(4; m, n, m)$ is nonempty. If both m and n is odd, by case (1) of Proposition 3.3, $S'(4; m, n, m)$ is also nonempty. If l'_d is 5, then the continued expansion of $(1+\sqrt{d})/2$ has the form $[a'_0, \overline{m, n, n, m, 2a'_0 - 1}]$. In this case, we can obtain the fact that $S'(5; m, n, n, m)$ is nonempty for all positive integers m and n by a similar method. Therefore, we have the following results.

Theorem 3.5. *Let* l'_d *be* 4. *If* n *is odd, there always exists* d *having the form of the continued expansion* $(1+\sqrt{d})/2 = [a'_0, \overline{m, n, m, 2a'_0 - 1}]$ *for every* m. *If* n *is even, there does not exist* d *having the form of the continued expansion* $(1+\sqrt{d})/2 = [a'_0, \overline{m, n, m, 2a'_0 - 1}]$.

Theorem 3.6. *Let* l'_d *be* 5. *There always exists* d *having the form of the continued expansion* $(1+\sqrt{d})/2 = [a'_0, \overline{m, n, n, m, 2a'_0 - 1}]$ *for all integers* m *and* n.

Through above procedure, we also obtain the similar results for the case that l'_d is 6 or 7. In fact, if l'_d is 6, then $q_5 \equiv l(mn+1) \pmod 2$, $q_4 \equiv nl(m+1)+1 \pmod 2$, and $r_4 \equiv nl \pmod 2$. If mn is even and l is odd, then q_5 is odd. By Proposition 3.3, $S'(6; m, n, l, n, m)$ is nonempty. On the other hand, if mn is even and l is even, then q_5 is even. In this case, one can see that

r_4 is even. By Proposition 3.3, $S'(6; m, n, l, n, m)$ is empty. We can also check in the case that mn is odd. If l'_d is 7, one can use the congruences (3.2), (3.3), and (3.4) to show Theorem 3.8. Therefore, we have the following theorems.

Theorem 3.7. *Let l'_d be 6. If mn is even and l is odd or m, n, and l are odd, there exists d having the form of the continued expansion $(1+\sqrt{d})/2 = [a'_0, \overline{m, n, l, n, m, 2a'_0 - 1}]$. For the other cases, there does not exist such d.*

Theorem 3.8. *Let l'_d be 7. There always exists d having the form of the continued expansion $(1+\sqrt{d})/2 = [a'_0, \overline{m, n, l, l, n, m, 2a'_0 - 1}]$ for all integers m, n, and l.*

Remark 3.9. We can easily check that there always exists d having the form of the continued expansion $(1+\sqrt{d})/2 = [a'_0, \overline{m, m, 2a'_0 - 1}]$ for all integers m. For the case that l'_d is 1, 3, 5, or 7, it will be meaningful to observe that there always exists the continued expansion of $\sqrt{d}$ with given palindromic sequence of positive integers. Generally, what happens if l'_d is odd ?

4. Relationship between fundamental unit of $\mathbb{Q}(\sqrt{d})$ and continued faction of $\sqrt{d}$(or $(1+\sqrt{d})/2$)

For the relation between the continued fraction of $\sqrt{d}$ and the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{d})$, it is well known as the following theorem (cf. [6, 11]).

Theorem 4.1. *Let d be a positive square-free integer and ϵ_d the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{d})$. Let l_d be the length of the period of the continued fraction of $\sqrt{d}$ and p_{l_d-1}/q_{l_d-1} the (l_d-1)-th convergent of it. Then*

$$\epsilon_d = p_{l_d-1} + q_{l_d-1}\sqrt{d}$$

or

$$\epsilon_d^3 = p_{l_d-1} + q_{l_d-1}\sqrt{d}$$

and the latter can only occur if $d \equiv 5 \pmod 8$.

Except for the case that $d \equiv 5 \pmod 8$, the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{d})$ is $\epsilon_d = p_{l_d-1} + q_{l_d-1}\sqrt{d}$. If d is a positive square-free integer congruent to 5 modulo 8, by Theorem 4.1, then $\epsilon_d = p_{l_d-1} + q_{l_d-1}\sqrt{d}$ or $\epsilon_d^3 = p_{l_d-1} + q_{l_d-1}\sqrt{d}$. Suppose d is a positive square-free integer congruent to 2 or 3 modulo 4.

If $l_d = 4$, the continued fraction of $\sqrt{d}$ has the form $[a_0, \overline{m, n, m, 2a_0}]$, where m is odd or both m and n are even by Theorem 1.1. In this case, for the real quadratic field $\mathbb{Q}(\sqrt{d})$, by Table 1, we get that $t_d = p_3 = (a_0 m+1)(mn+2)-1$, $u_d = q_3 = m(mn+2)$.

If $l_d = 5$, the continued fraction of $\sqrt{d}$ has the form $[a_0, \overline{m, n, n, m, 2a_0}]$, where m is even or both m and n are odd by Theorem 1.2. In this case, for the real quadratic field $\mathbb{Q}(\sqrt{d})$, we get that $t_d = p_4 = a_0(m^2n^2 + 2mn + m^2 + 1) + mn^2 + m + n$, $u_d = q_4 = m^2n^2 + 2mn + m^2 + 1$.

If $l_d = 6$, the continued fraction of $\sqrt{d}$ has the form $[a_0, \overline{m, n, l, n, m, 2a_0}]$, where mn is even and l is odd or mn is odd and l is even by Theorem 1.3. In this case, for the real quadratic field $\mathbb{Q}(\sqrt{d})$, we get that $t_d = p_5 = a_0(m^2n^2l + 2m^2n + 2mnl + 2m + l) + mn^2 + 2mn + nl + l$, $u_d = q_5 = m^2n^2l + 2m^2n + 2mnl + 2m + l$.

If d is a positive square-free integer congruent to 1 modulo 8, the fundamental unit of $\mathbb{Q}(\sqrt{d})$ is $\epsilon_d = 2t_d + 2u_d\sqrt{d}$. For a positive square-free integer d congruent to 5 modulo 8, in order to determine the fundamental unit of $\mathbb{Q}(\sqrt{d})$, the following lemma is useful (cf. [9, 16]).

Lemma 4.2. *Let d be a positive square-free integer such that $d \equiv 1 \pmod 4$. Assume that the continued fraction expansion $(1+\sqrt{d})/2$ is as follows*:

$$(1+\sqrt{d})/2 = [a'_0, a'_1, \ldots] = [a'_0, \overline{a'_1, \ldots, a'_{l'_d}}],$$

where l'_d is the length of the period of the continued fraction expansion of $(1+\sqrt{d})/2$ and $p'_{l'_d-1}/q'_{l'_d-1}$ is (l'_d-1)-th convergent of it.

(1) *All the positive integer solutions of $x^2 - xy - ((d-1)/4)y^2 = \pm 1$ have the form $(x, y) = (p'_{ml'_d-1}, q'_{ml'_d-1})$. Further, it holds that $p'^2_{ml'_d-1} - p'_{ml'_d-1}q'_{ml'_d-1} - \frac{d-1}{4}q'^2_{ml'_d-1} = (-1)^{ml'_d}$.*

(2) *The diophantine equation $x^2 - xy - ((d-1)/4)y^2 = 1$ (resp. -1) is solvable if and only if $x^2 - dy^2 = 4$ (resp. -4).*

We obtain (t_d, u_d) as the least positive integer solution of $x^2 - dy^2 = \pm 4$. By Lemma 4.2, we have $t_d = 2p'_{l'_d-1} - q'_{l'_d-1}$ and $u_d = q'_{l'_d-1}$. For example, let's consider $S'(4; 3, 1, 3)$. By Theorem 3.5, $S'(4; 3, 1, 3)$ is nonempty. In this case, we know that $q'_2 = 4$, $r'_2 = 1$, and $q'_3 = 15$. By (3.5),

$$\begin{aligned} S'(4; 3, 1, 3) &= \{d \mid 4a'_0(a'_0 - 1) + \frac{4(8a'_0 - 3)}{15} + 1 \in \mathbb{Z}, a'_0 \geq 1\} \\ &= \{133, 1725, 5117, 10309, 17301, 26093, \ldots\}. \end{aligned}$$

We can easily check that if $d \in S'(4; 3, 1, 3)$, then $d \equiv 5 \pmod 8$. In order to determine the fundamental unit of $\mathbb{Q}(\sqrt{133})$, we consider the continued fraction of $(1+\sqrt{133})/2$. Noting that $(1+\sqrt{133})/2 = [6, \overline{3, 1, 3, 11}]$, we have Table 3. By Lemma 4.2 and Table 3, we have $(t_{133}, u_{133}) = (2p'_3 - q'_3, q'_3) = (173, 15)$. Thus, the fundamental unit of $\mathbb{Q}(\sqrt{133})$ is $\frac{173+15\sqrt{133}}{2}$. Moreover, if $d \in S'(4; 3, 1, 3)$ and d is square-free, then u_d is always 15.

5. Mordell conjecture

There exist two famous conjectures related to the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{p})$ with a prime p. One is the Ankeny-Artin-Chowla conjecture [1], which says that for any prime p congruent to 1 modulo 4, $u_p \not\equiv$

TABLE 3.

k	-1	0	1	2	3	4
a'_k		6	3	1	3	11
p'_k	1	6	19	25	94	
q'_k	0	1	3	4	15	
r'_k	1	0	1	1	4	

0 (mod p). The other one is the Mordell conjecture [13], which says that for any prime p congruent to 3 modulo 4, $u_p \not\equiv 0 \pmod{p}$. The Ankeny-Artin-Chowla conjecture was numerically verified for all primes $p < 2\times 10^{11}$ in [19,20]. Furthermore, Mordell [12] proved the Ankeny-Artin-Chowla conjecture for any regular prime p, i.e., when p does not divide the class number of $\mathbb{Q}(e^{2\pi i/p})$. On the other hand, the Mordell conjecture has also been checked for primes not exceeding 10^7 in [3]. In [7], authors provided an equivalent criterion for the Mordell conjecture by using central term in continued fraction of $\sqrt{p}$. In [6] and [9], it is proved that two conjectures hold for some families as follows:

Theorem 5.1. (1) *For any odd positive integer l' and palindromic sequence of positive integers $a'_1, \ldots, a'_{l'-1}$, it holds that $u_p < p$ for all primes $p \in S'(l'; a'_1, \ldots, a'_{l'-1})$ with one possible exception. If the minimum of $S'(l'; a'_1, \ldots, a'_{l'-1})$ is not prime, the Ankeny-Artin-Chowla conjecture is true for all the primes p belonging to $S'(l'; a'_1, \ldots, a'_{l'})$.*

(2) *For any even positive integer l and palindromic sequence of positive integers $a_1, \ldots, a_{l-1}$, it holds that $u_p < p$ for all primes $p \in S(l; a_1, \ldots, a_{l-1})$ with one possible exception. If the minimum of $S(l; a_1, \ldots, a_{l-1})$ is not prime, the Mordell conjecture is true for all the primes p belonging to $S(l; a_1, \ldots, a_{l-1})$.*

Let p be a prime congruent to 3 modulo 4. In order to have simpler proof of (2) of Theorem 5.1 than [6], we give a different expression of $\sqrt{d}$ for $d := p$ from being done in Section 2. Noting that l_p is even, we can divide (2.9) into the following two equations:

$$p - a_0^2 = -r_{l_p-2}^2 + q_{l_p-2}s,$$
$$2a_0 = -q_{l_p-2}r_{l_p-2} + q_{l_p-1}s$$

for some $s \in \mathbb{Z}$. It implies that

$$p = \left(\frac{-q_{l_p-2}r_{l_p-2} + q_{l_p-1}s}{2}\right)^2 + q_{l_p-2}s - r_{l_p-2}^2 \tag{5.1}$$

for some integer s satisfying $s > \frac{q_{l_p-2}r_{l_p-2}}{q_{l_p-1}}$. Suppose that $p \in S(l; a_1, \ldots, a_{l-1})$ and $p \neq \min S(l; a_1, \ldots, a_{l-1})$. By (5.1), we have

$$s > \frac{q_{l_p-2}r_{l_p-2}}{q_{l_p-1}} + 1$$

and

$$p > \frac{1}{4}q_{l_p-1}^2 + q_{l_p-2} + \frac{r_{l_p-2}(q_{l_p-2}^2 - r_{l_p-2}q_{l_p-1})}{q_{l_p-1}}$$
$$= \frac{1}{4}q_{l_p-1}^2 + q_{l_p-2} + \frac{r_{l_p-2}}{q_{l_p-1}}.$$

It means that $p > q_{l_p-1}$ for $q_{l_p-1} \geq 4$. If q_{l_p-1} is even, we can see $u_p = q_{l_p-1} \not\equiv 0 \pmod{p}$, which implies that the Mordell conjecture holds for this case. It remains only for the case that q_{l_p-1} is 1 or 3. For the case, we can easily check that $p > u_p$. Therefore, in order to prove the Mordell conjecture, it is enough to consider the case that q_{l_p-1} is odd and the minimum of $S(l; a_1, \ldots, a_{l-1})$ is prime.

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References

[1] N. C. Ankeny, E. Artin, and S. Chowla, *The class-number of real quadratic number fields*, Ann. of Math. (2) **56** (1952), 479–493. https://doi.org/10.2307/1969656

[2] T. Azuhata, *On the fundamental units and the class numbers of real quadratic fields*, Nagoya Math. J. **95** (1984), 125–135. https://doi.org/10.1017/S0027763000021036

[3] B. D. Beach, H. C. Williams, and C. R. Zarnke, *Some computer results on units in quadratic and cubic fields*, in Proceedings of the Twenty-Fifth Summer Meeting of the Canadian Mathematical Congress (Lakehead Univ., Thunder Bay, Ont., 1971), 609–648, Lakehead Univ., Thunder Bay, ON, 1971.

[4] L. Bernstein, *Fundamental units and cycles in the period of real quadratic number fields. I*, Pacific J. Math. **63** (1976), no. 1, 37–61. http://projecteuclid.org/euclid.pjm/1102867566

[5] L. Bernstein, *Fundamental units and cycles in the period of real quadratic number fields. II*, Pacific J. Math. **63** (1976), no. 1, 63–78. http://projecteuclid.org/euclid.pjm/1102867567

[6] D. Byeon and S. Lee, *A note on units of real quadratic fields*, Bull. Korean Math. Soc. **49** (2012), no. 4, 767–774. https://doi.org/10.4134/BKMS.2012.49.4.767

[7] D. Chakraborty and A. Saikia, *On a conjecture of Mordell*, Rocky Mountain J. Math. **49** (2019), no. 8, 2545–2556. https://doi.org/10.1216/RMJ-2019-49-8-2545

[8] C. Friesen, *On continued fractions of given period*, Proc. Amer. Math. Soc. **103** (1988), no. 1, 9–14. https://doi.org/10.2307/2047518

[9] R. Hashimoto, *Ankeny-Artin-Chowla conjecture and continued fraction expansion*, J. Number Theory **90** (2001), no. 1, 143–153. https://doi.org/10.1006/jnth.2001.2652

[10] J. Mc Laughlin, *Multi-variable polynomial solutions to Pell's equation and fundamental units in real quadratic fields*, Pacific J. Math. **210** (2003), no. 2, 335–349. https://doi.org/10.2140/pjm.2003.210.335

[11] R. A. Mollin, *Quadratics*, CRC Press Series on Discrete Mathematics and its Applications, CRC Press, Boca Raton, FL, 1996.

[12] L. J. Mordell, *On a Pellian equation conjecture*, Acta Arith. **6** (1960), 137–144. https://doi.org/10.4064/aa-6-2-137-144

[13] L. J. Mordell, *On a Pellian equation conjecture. II*, J. London Math. Soc. **36** (1961), 282–288. https://doi.org/10.1112/jlms/s1-36.1.282

[14] Ö. Özer and F. K. Telci, *On continued fractions of real quadratic fields with period six*, Int. J. Contemp. Math. Sci. **6** (2011), no. 17-20, 833–840.

[15] Ö. Özer, F. K. Telci, and H. İşcan, *On some real quadratic fields with period 4*, Int. J. Contemp. Math. Sci. **4** (2009), no. 25-28, 1389–1396.

[16] O. Perron, *Die Lehre von den Kettenbrüchen. Bd I. Elementare Kettenbrüche*, B. G. Teubner Verlagsgesellschaft, Stuttgart, 1954.

[17] K. Tomita, *Explicit representation of fundamental units of some quadratic fields*, Proc. Japan Acad. Ser. A Math. Sci. **71** (1995), no. 2, 41–43. http://projecteuclid.org/euclid.pja/1195510812

[18] K. Tomita, *Explicit representation of fundamental units of some real quadratic fields. II*, J. Number Theory **63** (1997), no. 2, 275–285. https://doi.org/10.1006/jnth.1997.2088

[19] A. J. van der Poorten, H. J. J. te Riele, and H. C. Williams, *Computer verification of the Ankeny-Artin-Chowla conjecture for all primes less than* 100 000 000 000, Math. Comp. **70** (2001), no. 235, 1311–1328. https://doi.org/10.1090/S0025-5718-00-01234-5

[20] A. J. van der Poorten, H. J. J. te Riele, and H. C. Williams, *Corrigenda and addition to: "Computer verification of the Ankeny-Artin-Chowla conjecture for all primes less than* 100 000 000 000, Math. Comp. **72** (2003), no. 241, 521–523.

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