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RINGS WITH A RIGHT DUO FACTOR RING BY AN IDEAL CONTAINED IN THE CENTER

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Abstract. This article concerns a ring property that arises from combining one-sided duo factor rings and centers. A ring R is called *right CIFD* if R/I is right duo by some proper ideal I of R such that I is contained in the center of R. We first see that this property is seated between right duo and right π-duo, and not left-right symmetric. We prove, for a right CIFD ring R, that $W(R)$ coincides with the set of all nilpotent elements of R; that R/P is a right duo domain for every minimal prime ideal P of R; that $R/W(R)$ is strongly right bounded; and that every prime ideal of R is maximal if and only if $R/W(R)$ is strongly regular, where $W(R)$ is the Wedderburn radical of R. It is also proved that a ring R is commutative if and only if $D_3(R)$ is right CIFD, where $D_3(R)$ is the ring of 3 by 3 upper triangular matrices over R whose diagonals are equal. Furthermore, we show that the right CIFD property does not pass to polynomial rings, and that the polynomial ring over a ring R is right CIFD if and only if R/I is commutative by a proper ideal I of R contained in the center of R.

1. Introduction

Throughout this article, every ring is an associative ring with identity unless otherwise stated. Let R be a ring. We use $N(R)$, $J(R)$, $N_*(R)$, $N^*(R)$, and $W(R)$ to denote the set of all nilpotent elements, Jacobson radical, lower nilradical (i.e., prime radical), upper nilradical (i.e., the sum of all nil ideals), and the Wedderburn radical (i.e., the sum of all nilpotent ideals) of R, respectively. It is well-known that $W(R) \subseteq N_*(R) \subseteq N^*(R) \subseteq N(R)$ and $N^*(R) \subseteq J(R)$. The center and the set of all idempotents of R are denoted by $Z(R)$ and $I(R)$, respectively. The group of all units in R is denoted by $U(R)$. The polynomial ring with an indeterminate x over R is denoted by $R[x]$. $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n). Denote the n by n ($n \geq 2$) full

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(resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $T_n(R)$). Write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$. Use E_{ij} for the matrix with (i, j)-entry 1 and zeros elsewhere. I_n denotes the identity matrix in $Mat_n(R)$. $\prod$ means the direct product. The monoid of all non-zero-divisors in R is denoted by $C(R)$.

Following Feller [6], a ring is called *right duo* if every right ideal is two-sided. Left duo rings are defined similarly. A ring is called *duo* if it is both left and right duo. A ring is called *reduced* if it has no nonzero nilpotents. It is easily checked that a ring R is reduced if and only if $a^2 = 0$ for $a \in R$ implies $a = 0$. A ring is called *Abelian* if every idempotent is central. It is easily checked that reduced rings and one-sided duo rings are Abelian. A ring R is Abelian if and only if $D_n(R)$ is Abelian by [11, Lemma 2]. Following [13], a ring R is called *right π-duo* provided that for any $a \in R$ there is an integer $n \geq 1$ such that $Ra^n \subseteq aR$. Left π-duo rings are defined similarly. A ring is called *π-duo* if it is both left and right π-duo. Right duo rings are clearly right π-duo but not conversely by [13, Theorem 1.7]. Right or left π-duo rings are Abelian by [13, Proposition 1.9(4)]. We will freely use the preceding facts.

According to [12], a ring R is called *CIFC* if R/I is commutative by some proper ideal I of R with $I \subseteq Z(R)$. The concepts of right duo and CIFC are independent of each other by [12, Example 1.6]. In fact, the ring in [12, Example 1.6] is CIFC but neither right nor left duo; and every noncommutative division ring is clearly duo but not CIFC.

In this article, we continue this study in a larger class of rings by considering duoness, and will call a ring R *right* (*resp., left*) *CIFD* if R/I is right (resp., left) duo by some proper ideal I of R with $I \subseteq Z(R)$. A ring is called *CIFD* if it is both right and left CIFD.

In Section 2, we show that right CIFD rings are seated between right duo rings and right π-duo rings, and that the CIFD property is not left-right symmetric. In Section 3, we prove, for a right CIFD ring R, that $W(R) = N(R)$, that for any $0 \neq a \in N(R)$ whose index k of nilpotency is greater than or equal to 3, aR contains a nonzero nilpotent ideal $Rx^{k-1}R$ for some $0 \neq x \in aR$, that R/P is a right duo domain for every minimal prime ideal P of R, and that $R/W(R)$ is strongly right bounded. We also show the equivalence of the following conditions when R is a right (or left) CIFD ring: R is π-regular; every prime ideal of R is maximal; and $R/W(R)$ is strongly regular. In Section 4, it is proved that a ring R is commutative if and only if $D_3(R)$ is right CIFD. Furthermore, we show that the right CIFD property does not pass to polynomial rings, and that the polynomial ring over a ring R is right CIFD if and only if R/I is commutative by a proper ideal I of R contained in the center of R.

2. Relation among related rings

In this section, we study the structure of CIFD rings and the relation between right CIFD rings and related rings.

Lemma 2.1.

(1) *Let R be right (resp., left) CIFD ring. Then the following assertions hold.*
   (i) *$Ra^2 \subseteq aR$ (resp., $a^2R \subseteq Ra$) for each $a \in R$.*
   (ii) *Let P be a prime ideal of R. If $ab \in P$ for $a, b \in R$, then $a \in P$ or $b^2 \in P$.*
   (iii) *R is right (resp., left) π-duo. Especially right or left CIFD rings are Abelian.*

(2) *The class of right (left) duo rings is closed under factor rings and direct products.*

(3) *The class of right (left) CIFD rings is closed under factor rings and direct products.*

(4) *The class of CIFC rings is closed under factor rings and direct products.*

(5) [4, Proposition 1.7] *$Z(D_n(R)) = Z(R)I_n + Z(R)E_{1n}$ for a ring R and $n \geq 2$.*

(6) [3, Lemma 1.1(4)] *If R is a division ring, then $D_2(R)$ is a duo ring; but $D_n(A)$ is neither right nor left duo for all $n \geq 3$ over any ring A.*

Proof. (1) (i) Let R be a right CIFD ring. Then R/I is right duo by some proper ideal I of R with $I \subseteq Z(R)$. Let $a \in R$. Then there exists $s \in R$ such that $ra - as \in I$ for any $r \in R$, since R/I is right duo, so that $ra - as \in Z(R)$. This yields $a(ra - as) = (ra - as)a$ and $ra^2 = a(ra - as + sa) \in aR$ follows. That is $Ra^2 \subseteq aR$. The proof for the left case is similar.

(ii) Let $ab \in P$ for $a, b \in R$. Since R is right CIFD, $aRb^2 \subseteq abR$ by (i). Suppose $a \notin P$. Then $b^2 \in P$, since P is prime and $aRb^2 \subseteq P$.

(iii) Every right (resp., left) CIFD ring is right (resp., left) π-duo by (i). So right or left CIFD rings are Abelian by [13, Proposition 1.9(4)].

(2) This is clear from definition.

(3) Let R be a right CIFD ring. Then R/I is right duo by some proper ideal I of R with $I \subseteq Z(R)$. Let J be a proper ideal of R and consider R/J. If $I \subseteq J$, then R/J is also right duo by (2). So assume $I \nsubseteq J$.

Suppose $I + J = R$. Then $R/J = (I + J)/J \cong I/(I \cap J)$. But $I/(I \cap J)$ is a commutative ring since $I \subseteq Z(R)$, so that R/J is commutative (hence CIFD).

Suppose $I + J = R$. Write $K = I + J$. For $a \in I$ and $b \in J$, $((a+b)+J)(r+J) = (a+J)(r+J) = ar+J = ra+J = (r+J)(a+J) = (r+J)((a+b)+J)$ for all $r \in R$, because $a \in Z(R)$. This implies that K/J is contained in $Z(R/J)$. Note $(R/J)/(K/J) \cong R/K$. Let $r, s \in R$. Since R/I is right duo, $rs = st+u$ for some $t \in R$ and $u \in I$. Then we have $(r+K)(s+K) = rs+K = (st+u)+K = (st+K)+(u+K) = st+K = (s+K)(t+K) \in (s+K)(R/K)$, showing that R/K is right duo. Consequently, $(R/J)/(K/J)$ is right duo by the proper ideal K/J of R/J with $K/J \subseteq Z(R/J)$, and therefore R/J is right CIFD.

Next, let $R = \prod_{i \in \Lambda} R_i$ be the direct product of rings R_i for $i \in \Lambda$. Note $Z(R) = \prod_{i \in \Lambda} Z(R_i)$. Suppose that every R_i is right CIFD. Then R_i/J_i is right duo by some proper ideal J_i of R_i with $J_i \subseteq Z(R_i)$. Set $J = \prod_{i \in \Lambda} J_i$. Then

J is a proper ideal of R such that $J \subseteq Z(R)$. Moreover $R/J = \prod_{i \in \Lambda} R_i/J_i$ is right duo by (2). Thus R is right CIFD. The left cases can be similarly proved.

(4) This is proved by applying the proof of (3). □

Notice that right duo rings are clearly right CIFD, and right CIFD rings are right π-duo rings by Lemma 2.1(1)(i), but each converse does not hold by the following example.

Example 2.2. (1) We use the ring in [12, Example 1.6]. Let K be a field and $A = K\langle x, y\rangle$ be the free algebra generated by noncommuting indeterminates x, y over K. Consider the ideal I of A that is generated by abc and set $R = A/I$, where $a, b, c \in \{f \in A \mid \text{the constant term of } f \text{ is zero}\}$. Then R is CIFC (hence CIFD) by the argument in [12, Example 1.6]. We identity elements in A with their images in R for simplicity. Indeed, $Rx \nsubseteq xR$ and $Rx \nsupseteq xR$ because $yx \notin xR$ and $xy \notin Rx$. Thus R is neither right nor left duo.

(2) Let A be a division ring and $R = D_n(A)$ for $n \geq 3$. Then R is π-duo by [13, Theorem 1.7]. We claim that R is not right CIFD. Indeed, $E_{12}E_{33} = 0$, but $0 \neq E_{13} = E_{12}E_{23}E_{33}^2 \in E_{12}RE_{33}^2$. This, together with Proposition 2.7(1)(ii) to follow, implies that R is not right CIFD .

Next set $n \geq 4$ and let I be a proper ideal of R contained in $Z(R)$. If $I = 0$, then R/I is neither right nor left duo by the preceding argument. So suppose $I \neq 0$. Then $I = Z(A)E_{1n}$ by Lemma 2.1(6). Consider R/I. Write $\bar{R} = R/I$ and $\bar{r} = r + I$ for $r \in R$. Since $\bar{E}_{12}\bar{E}_{23} = \bar{E}_{13} \notin \bar{E}_{23}\bar{R}$ and $\bar{E}_{12}\bar{E}_{23} = \bar{E}_{13} \notin \bar{R}\bar{E}_{12}$, $\bar{R}$ is neither right nor left duo. Thus R is neither right nor left CIFD.

Lemma 2.1(1)(i) need not hold for right π-duo rings. Let A be a division ring and $R = \{(a_{ij}) \in T_4'(A) \mid a_{11} = \cdots = a_{44}\}$, where $T_4'(A)$ is the 4 by 4 lower triangular matrix ring over A. Then R is π-duo by applying the argument in Example 2.2(2). Let $\alpha = E_{21} + E_{32} \in R$. Then $R\alpha$ and $R\alpha^2$ are not contained in αR. Thus R is not right CIFD.

The following diagram shows all implications among the concepts above.

{right duo} ↙

• {Abelian} ⟵ {right π-duo} ⟵ right CIFD

↖ {CIFC}

Following Goodearl [7], a ring R is called *von Neumann regular* (simply, *regular*) if for every $x \in R$, there exists $y \in R$ such that $xyx = x$, and a ring R is called *strongly regular* if for each $x \in R$, there exists $y \in R$ such that $x^2y = x$. A ring is strongly regular if and only if it is Abelian regular by [7, Theorem 3.5]. We have the following equivalences by [7, Theorem 3.2] and Lemma 2.1(1).

Proposition 2.3. *For a regular ring R, the following conditions are equivalent*:

(1) *R is right* (*left*) *CIFD*;

(2) *R is right (left) duo*;
(3) *R is right (left) π-duo*;
(4) *R is Abelian.*

In the following, we check information about Abelian rings.

Remark 2.4. (1) Let R be a right or left CIFD ring. Then R is Abelian by Lemma 2.1(1) which is obtained from the fact that right or left π-duo rings are Abelian. Here we provide another proof by using the definition. Since R is right CIFD, R/I is right duo for some proper ideal I of R with $I \subseteq Z(R)$. Then for each $(r,e) \in R \times I(R)$, there exist $s,t \in R$ such that $re - es$, $r(1-e)-(1-e)t \in I \subseteq Z(R)$. So $re - ese = (re - es)e = e(re - es) = e^2(re - es) = e(re - es)e = ere - ese$ and $r(1-e)-(1-e)t(1-e) = (r(1-e)-(1-e)t)(1-e) = (1-e)(r(1-e)-(1-e)t) = (1-e)^2(r(1-e)-(1-e)t) = (1-e)(r(1-e)-(1-e)t)(1-e) = (1-e)r(1-e) - (1-e)t(1-e)$; hence $re = ere$ and $r(1-e) = (1-e)r(1-e)$ (this yields $er = ere$). Thus $re = er$.

(2) Let R be an Abelian ring and suppose that $ae \in N_*(R)$ for some $a, e^2 = e \in R\backslash\{0\}$. Then $aeR \subseteq N_*(R)$, and so we get $aRe \subseteq N_*(R)$ since $e \in Z(R)$. Assuming $a \notin P$ for each prime ideal P of R, we obtain $e \in P$, so that $0 \neq e \in N_*(R)$, a contradiction. Thus there exists a prime ideal Q of R such that $a \in Q$.

The following shows that the CIFD property is not left-right symmetric.

Example 2.5. Consider a skewed trivial extension in [17, Definition 1.3] as follows. Let R be a commutative ring with an endomorphism σ and M be an R-module. For $R \oplus M$, the addition and multiplication are given by $(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$ and $(r_1, m_1)(r_2, m_2) = (r_1 r_2, \sigma(r_1)m_2 + m_1 r_2)$. Then this construction forms a ring. Following the literature, this extension is called the *skew-trivial extension* of R by M, denoted by $R \propto M$. Note that $R \propto R$ is isomorphic to $R[x;\sigma]/(x^2)$ via the corresponding $(r,m) \mapsto r + \bar{x}m$, where $R[x;\sigma]$ is the skew polynomial ring, with the coefficients written on the right, only subject to $ax = x\sigma(a)$ for $a \in R$ and (x^2) is the ideal of $R[x;\sigma]$ generated by x^2.

Now let K be a field with a monomorphism σ and M be a K-module. Suppose that σ is not surjective. Then $R' = K \propto M$ is a right duo ring that is not left duo by [15, Theorem 2.5(1)]. Hence R' is clearly right CIFD.

As in the proof of [15, Theorem 2.5(1)], there exists $s \in K$ with $s \notin \sigma(K)$ (since σ is not surjective) such that for each $0 \neq (0,m) \in R'$, $(0,m)(s,0) = (0, sm) \neq (0, \sigma(t)m) = (t,n)(0,m)$ for any $(t,n) \in R'$, entailing $(0,m)(s,0) \notin R'(0,m)$. This implies $(0,m) \notin Z(R')$ when $m \neq 0$.

Letting $(k,0)(h,n) = (h,n)(k,0)$, we get $\sigma(k)n = kn$ and $\sigma(k) = k$ follows. Thus $Z(R') = F \oplus 0$, where F is the subfield of K left fixed by σ. Note that every nonzero ideal must contain $(0,n)$ for some $0 \neq n \in M$. Thus 0 is the only proper ideal of R' contained in $Z(R')$. This concludes that R' is not left CIFD because R' is not left duo.

We investigate the forms of ideals of matrix rings contained in the center.

Remark 2.6. (1) Let A be a ring and $R = D_n(A)$ for $n \geq 2$. Let I be a nonzero proper ideal of R with $I \subseteq Z(R)$. Then $I \subseteq Z(A)I_n + Z(A)E_{1n}$ by Lemma 2.1(5).

(i) Suppose $n \geq 3$. Let $0 \neq (b_{ij}) \in I$. Assume $b_{ii} \neq 0$, β say. Then $0 \neq \beta E_{12} = (b_{ij})E_{12} \in I$, contrary to $I \subseteq Z(A)I_n + Z(A)E_{1n}$. So $b_{ii} = 0$. This implies $I \subseteq Z(A)E_{1n}$, so that $I = JE_{1n}$ for some nonzero ideal J of A such that $J = \{a_{1n} \mid (a_{ij}) \in I\}$. Note $J \subseteq Z(A)$.

In the preceding argument, if $J \cap U(A) \neq \emptyset$, then $J = A$ and hence A is a commutative ring. Equivalently, if A is noncommutative, then J cannot contain a unit of A.

(ii) Suppose $n = 2$. In this case, the form of I is different to the case of $n \geq 3$. For example, if A is commutative, then $D_2(A)$ is also commutative; hence every ideal of $D_2(A)$ is contained in the center. Consider the case of A being noncommutative. Then the proper ideal I of R with $I \subseteq Z(R)$ is one of the following: $D_2(J)$ or $\{(a_{ij}) \in R \mid a_{ii} \in J_1 \text{ and } a_{12} \in J_2\}$, where J and J_1 are proper ideals of A such that $J, J_1 \subseteq Z(A)$ and J_2 is an ideal of R such that $J_2 \subseteq Z(R)$.

(2) Let A be a ring and $R = Mat_n(A)$ for $n \geq 2$. It is well-known that $Z(R) = Z(A)I_n$. So the zero ideal is the only proper ideal of R contained in $Z(R)$. In fact, letting I be such an ideal of R, we get $IE_{12} \subseteq I \subseteq Z(A)I_n$; but if $IE_{12} \neq 0$ (i.e., $I \neq 0$), then $IE_{12} \nsubseteq Z(R)$, contrary to $I \subseteq Z(R)$. This result also holds for $T_n(A)$ for $n \geq 2$.

Following Faith [5], a ring is called *strongly right* (resp., *left*) *bounded* if every nonzero right (resp., left) ideal contains a nonzero ideal. A ring is called *strongly bounded* if it is both strongly right and left bounded. Right duo rings are clearly strongly right bounded, but there exists a strongly bounded ring that is neither right nor left duo by [12, Example 1.6].

Proposition 2.7.

(1) *Let R be a right (resp., left) CIFD ring. Then the following assertions hold*:
  (i) *For any $a \in R$ with $a^2 \neq 0$, aR (resp., Ra) contains a nonzero ideal of R.*
  (ii) *If $ab = 0$ for $a, b \in R$, then $aRb^2 = 0$ (resp., $a^2Rb = 0$).*
(2) *For a simple ring R, the following conditions are equivalent*:
  (i) *R is right (resp., left) CIFD*;
  (ii) *R is right (resp., left) duo*;
  (iii) *R is a division ring.*
(3) *Reduced right (resp., left) CIFD rings are strongly right (resp., left) bounded.*

(4) *Let R be a ring and I be a nonzero semiprime ideal of R such that $I \subseteq Z(R)$ and R/I is right* (*resp., left*) *duo. Then R is strongly right* (*resp., left*) *bounded.*

Proof. (1) (i) Let $a \in R$ with $a^2 \neq 0$. Since R is right CIFD, $Ra^2 \subseteq aR$ by Lemma 2.1(1)(i), so that $0 \neq Ra^2R \subseteq aR$.

(ii) Let $ab = 0$ for $a, b \in R$. Then $aRb^2 \subseteq abR = 0$ by Lemma 2.1(1)(i). The left case can be proved similarly.

(2) (i) $\Rightarrow$ (ii). Let R be right CIFD. Then since R is simple, $R \cong R/0$ is right duo.

(ii) $\Rightarrow$ (iii). Let R be right duo and $0 \neq a \in R$. Then $RaR = aR$. But since R is simple, we get $aR = R$, entailing $a \in U(R)$.

(iii) $\Rightarrow$ (i) is obvious. The left case can be proved similarly.

(3) This is immediately obtained from (1)(i).

(4) Let K be a nonzero right ideal of R. If $K \cap I \neq 0$, then $RaR = aR \subseteq K$ for every $0 \neq a \in K \cap I$. Next, suppose $K \cap I = 0$. Since R/I is right duo, $(K+I)/I$ is an ideal of R/I. This yields $RK \subseteq K+I$. So we can obtain that K contains the nonzero ideal RK^2 of R, by applying the proof [12, Theorem 1.5]. Thus R is strongly right bounded. The left case can be proved similarly. □

Indeed, the ring in Proposition 2.7(4) is right CIFD. We see examples that elaborate upon Proposition 2.7 in the following.

Example 2.8. (1) There exists a non-reduced right CIFD ring that is not strongly right bounded, which shows that the condition "reduced" in Proposition 2.7(3) is not superfluous. Let A be a commutative ring and $R = D_3(A)$. Then R is CIFD by Theorem 4.1 to follow. But R is neither right nor left strongly bounded. To see that, take $E_{23} \in R$. Then $E_{23}R = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & A \\ 0 & 0 & 0 \end{smallmatrix}\right)$. Next for any aE_{23} with $0 \neq a \in A$, $R(aE_{13})$ contains $\left(\begin{smallmatrix} 0 & 0 & a \\ 0 & 0 & a \\ 0 & 0 & 0 \end{smallmatrix}\right) \notin E_{23}R$. So R cannot be strongly right bounded. Similarly we can show that R is not strongly left bounded by using E_{12}.

(2) Let R be the first Weyl algebra over a field of characteristic zero. Then R is a simple domain that is neither right nor left duo. So R is neither right nor left CIFD by Proposition 2.7(2). Moreover, R is neither right nor left strongly bounded.

Note that the class of right (left) CIFD rings is not closed under subrings. For, let Q be the right quotient division ring of the ring R in Example 2.8(2). Then Q is clearly duo (hence CIFD) but the subring R is neither right nor left CIFD. This fact also shows that the class of right (left) duo rings is not closed under subrings. However, we have the following.

Proposition 2.9. *Let R be a ring and $e \in I(R) \cap Z(R)$ such that $eIe \subsetneq eRe$ for every proper ideal I of R. If R is right* (*resp., left*) *CIFD, then so is eRe.*

Proof. Let R be right CIFD. Then $e \in Z(R)$ (hence $eR = eRe$) by Lemma 2.1(1)(iii). Take $\alpha = ea, \beta = eb \in eR$. Since R is right CIFD, R/I is right duo for some proper ideal I of R such that $I \subseteq Z(R)$. Then $ba - as \in I$ for some $s \in R$, and $eI = eIe$ is a proper ideal of eR by hypothesis. Note $eI \subseteq Z(eR)$. Since $e(ba - as) \in eI$, we see that $\beta\alpha - \alpha\gamma = (eb)(ea) - (ea)(es) = e(ra - as) \in eI$, where $\gamma = es \in eR$. Consequently, eR/eI is right duo, and thus eR is right CIFD. The left case can be proved similarly. □

For a given ring R, recall that R is called *local* if $R/J(R)$ is a division ring; R is called *semilocal* if $R/J(R)$ is semisimple Artinian; and R is called *semiperfect* if R is semilocal and idempotents can be lifted modulo $J(R)$. Local rings are clearly Abelian and semilocal.

Corollary 2.10. *Let R be a ring and suppose that if $e \in I(R) \cap Z(R)$, then $eIe \subsetneq eRe$ for every proper ideal I of R.*

(1) *R is semiperfect right (resp., left) CIFD if and only if R is a finite direct product of local right (resp., left) CIFD rings.*
(2) *Let $e \in I(R) \cap Z(R)$. Then R is right (resp., left) CIFD if and only if both eR and $(1 - e)R$ are right (resp., left) CIFD.*

Proof. (1) Suppose that R is right CIFD and semiperfect. Since R is semiperfect, R has a finite orthogonal set $\{e_1, e_2, \ldots, e_n\}$ of local idempotents whose sum is 1 by [14, Proposition 3.7.2], i.e., each e_iRe_i is a local ring. Since R is right CIFD, $I(R) \subseteq Z(R)$ by Lemma 2.1(1)(iii), whence $e_iR = e_iRe_i$ for each i and this implies $R = \sum_{i=1}^{n} e_iR$. But each e_iR is also a right CIFD ring by Proposition 2.9.

Conversely assume that R is a finite direct product of local right CIFD rings. Then R is semiperfect since local rings are semiperfect by [14, Corollary 3.7.1], and moreover R is right CIFD by Lemma 2.1(3). The left case can be proved similarly.

(2) This is proved by Lemma 2.1(4) and Proposition 2.9, since $R = eR \oplus (1 - e)R$ for every $e \in I(R) \cap Z(R)$. □

3. Important properties

In this section, we investigate the structure of right CIFD rings in relation to nilradicals, commutativity, right duoness, strongly right boundedness, π-regularity and strong regularity. We first obtain information for given ideals to be contained in the centers, in relation to the commutativity of rings.

Theorem 3.1.

(1) *Let R be a ring which has a nonzero proper ideal I of R with $I \subseteq Z(R)$. If $I \cap C(R) \neq \emptyset$, then R is commutative.*
(2) *Let R be a domain which has a nonzero proper ideal of R with $I \subseteq Z(R)$. Then R is commutative.*

(3) *Let R be a noncommutative domain. Then R is right (resp., left) CIFD if and only if R is right (resp., left) duo.*
(4) *Let R be a noncommutative non-simple ring. Then every nonzero proper ideal of R, which has a regular element, is not contained in $Z(R)$.*
(5) *Let R be a noncommutative non-simple domain. Then every nonzero proper ideal of R is not contained in $Z(R)$.*

Proof. (1) Suppose $I \cap C(R) \neq \emptyset$ and take $a \in I \cap C(R)$. Let $r, s \in R$. Then since $ra \in I \subseteq Z(R)$, we have

$$r(sa) = r(as) = (ra)s = s(ra),$$

so that $(rs - sr)a = 0$. But since a is a non-zero-divisor, we get $rs - sr = 0$. Thus R is commutative.

(2) This is an immediate consequence of (1).

(3) Let R be right CIFD. Then R/I is right duo by some proper ideal I of R such that $I \subseteq Z(R)$. But since R is a noncommutative domain, I must be zero by (2). Thus R is right duo. The converse is obvious and the proof for the left case is similar.

(4) and (5) are obtained from (1) and (2), respectively. □

Let R be a ring. The index of nilpotency of $a \in N(R)$ is the least $n \geq 1$ such that $a^n = 0$. The index of nilpotency of $S \subseteq R$ is the supremum of the indices of nilpotency of all nilpotent elements in S.

Theorem 3.2.

(1) *If R is a right (left) CIFD ring, then $W(R) = N(R)$.*
(2) *Let R be a non-reduced right (resp., left) CIFD ring. Then, for any $0 \neq a \in N(R)$ whose index k of nilpotency is greater than or equal to 3, aR (resp., Ra) contains a nonzero nilpotent ideal $Rx^{k-1}R$ for some $0 \neq x \in aR$.*

Proof. (1) Since R is right CIFD, R/I is right duo by some proper ideal I of R with $I \subseteq Z(R)$. Let $a \in N(R)$ with $a^n = 0$ for $n \geq 2$. Then for each $r \in R$, there exists $s \in R$ such that $ra - as \in I$, and hence $(ra - as)t \in I \subseteq Z(R)$ holds for all $t \in R$. Let $r_i \in R$ for $i = 1, 2, \ldots$.

Suppose $n = 2$. Then

$$0 = a^2(ra - as)r_1 = a(ra - as)r_1a = arar_1a,$$

so that $aRaRa = 0$. This implies $(RaR)^3 = 0$.

Suppose $n = 3$. Then

$$0 = a^3(ra - as)r_1 = a^2(ra - as)r_1a = a^2rar_1a,$$

so that $a^2RaRa = 0$. From this we can obtain

$$0 = a^3(ra - as)r_1r_2 = a^2(ra - as)r_1ar_2 = a(ra - as)r_1ar_2a = arar_1ar_2a,$$

so that $aRaRaRa = 0$. This implies $(RaR)^4 = 0$.

Suppose that $a^n = 0$ for $n \geq 2$ and apply the preceding method. First, we get

$$0 = a^n(ra - as)r_1 = a^{n-1}(ra - as)r_1 a = a^{n-1}rar_1a,$$

so that $a^{n-1}RaRa = 0$. From this we can obtain

$$\begin{aligned} 0 &= a^n(ra - as)r_1r_2 = a^{n-1}(ra - as)r_1ar_2 = a^{n-2}(ra - as)r_1ar_2a \\ &= a^{n-2}rar_1ar_2a, \end{aligned}$$

so that $a^{n-2}RaRaRa = 0$ (hence $a^{n-2}(Ra)^3 = 0$). Now suppose by induction that

$$a^{n-k}(Ra)^{k+1} = 0$$

for $k < n - 1$. Then we get

$$\begin{aligned} 0 &= a^{n-k}(ra - as)r_1ar_2a \cdots r_kar_{k+1} = a^{n-k-1}(ra - as)r_1ar_2 \cdots r_kar_{k+1}a \\ &= a^{n-k-1}rar_1ar_2 \cdots r_kar_{k+1}a, \end{aligned}$$

so that $a^{n-(k+1)}(Ra)^{k+2} = 0$.

Therefore we now have $a^{n-(n-1)}(Ra)^{(n-1)+1} = 0$ and $(RaR)^{n+1} = 0$, entailing $a \in W(R)$. The proof for the left CIFD ring is similar.

(2) Let $0 \neq a \in N(R)$. Then aR is nilpotent by (1). Suppose that the index k of nilpotency is greater than or equal to 3. Take any $x \in aR$ such that $x^k = 0$ and $x^{k-1} \neq 0$.

Consider first the case of $k-1$ being even. Then we have $Rx^{k-1}R \subseteq x^{\frac{k-1}{2}}R$ by Lemma 2.1(1)(i), from which we infer that

$$(Rx^{k-1}R)^2 = (Rx^{k-1})(Rx^{k-1})R \subseteq (Rx^{k-1})x^{\frac{k-1}{2}}R = Rx^{(k-1)+\frac{k-1}{2}}R = 0,$$

since $(k-1) + \frac{k-1}{2} \geq k$. Hence $(Rx^{k-1}R)^2 = 0$.

Consider next the case of $k-1$ being odd. Note that $k \geq 4$ and $k-2$ is even. Then we have $Rx^{k-2}R \subseteq x^{\frac{k-2}{2}}R$ by Lemma 2.1(1)(i), from which we infer that

$$(Rx^{k-1}R)^2 \subseteq (Rx^{k-1})(Rx^{k-2})R \subseteq (Rx^{k-1})x^{\frac{k-2}{2}}R = Rx^{(k-1)+\frac{k-2}{2}}R = 0$$

since $(k-1) + \frac{k-2}{2} \geq k$. Hence $(Rx^{k-1}R)^2 = 0$.

Therefore aR contains the nonzero nilpotent ideal $Rx^{k-1}R$ in any case. The proof for the left case is analogous. □

In Theorem 3.2(2), aR is nilpotent by Theorem 3.2(1). Meanwhile, if the index of nilpotency of a is 2, then we also see that aR contains a nonzero nilpotent right ideal of R that is either aR itself or UaR for some $U \subseteq aR$, by the proof of [8, Lemma 1.1] that is a variation of the argument given by Levitzki. Moreover, Theorem 3.2(2) is not valid for elements whose indices of nilpotency are 2. In fact, if A is a commutative ring, then $R = D_3(A)$ is CIFD by Theorem 4.1 to follow. Consider $E_{23}, E_{13} \in R$. Then $E_{23}R = AE_{23}$ and $RE_{13} = AE_{13}$ are both contained in $N(R)$, but these cannot contain any nonzero ideal of R by the argument in Example 2.8(1).

Recall that Köthe's conjecture means "*the sum of two nil left ideals is nil.*"

Corollary 3.3.

(1) *Let R be a right (resp., left) CIFD ring. Then R/P is a right (resp., left) duo domain for every minimal prime ideal P of R.*
(2) *If R is a right (resp., left) CIFD ring, then $R/W(R)$ is strongly right (resp., left) bounded.*
(3) *Köthe's conjecture holds for right or left CIFD rings.*
(4) *Let R be a right (or left) CIFD ring. Then R is semiprime if and only if R is reduced.*
(5) *Let R be a right (resp., left) CIFD ring. If for any $0 \neq a \in N(R)$ whose index of nilpotency is 2, aR (resp., Ra) contains a nonzero nilpotent ideal of R, then R is strongly right (resp., left) bounded.*

Proof. (1) Let R be a right CIFD ring and P be a minimal prime ideal of R. Then $R/N_*(R)$ is a reduced ring by Theorem 3.2(1), noting $W(R) = N_*(R) = N(R)$. So R/P is a domain by [18, Proposition 1.11]. Furthermore since R/P is right CIFD by Lemma 2.1(3), it follows that R/P is right duo by Theorem 3.1(3). The proof for the left case is similar.

(2) Let R be a right CIFD ring. Then, by Theorem 3.2(1), $R/W(R)$ is reduced. Moreover $R/W(R)$ is right CIFD by Lemma 2.1(3). Thus the result follows from Proposition 2.7(3). The proof for the left case is analogous.

(3) and (4) follow from Theorem 3.2(1).

(5) This is proved by Theorem 3.2(2) and Proposition 2.7(1)(i). □

McCoy [16] called a ring R *π-regular* if for each $a \in R$, there exists a positive integer n, depending on a, and $b \in R$ such that $a^n = a^n b a^n$. Regular rings are clearly π-regular, but not conversely by considering $T_n(R)$ and $D_n(R)$ for $n \geq 2$ over a division ring R. It is easy to check whether the Jacobson radicals of π-regular rings are nil. Note that if a ring R is Abelian π-regular, then $N(R) = N^*(R) = J(R)$ by [1, Theorem 2]. Abelian π-regular rings are π-duo by [13, Lemma 1.4(2)]. But there exist Abelian π-regular rings which are neither right nor left CIFD; for example, $D_3(R)$ over a noncommutative division ring (see Theorem 4.1 to follow), and $D_n(R)$ for $n \geq 4$ over a division ring R (see Example 2.2(2)).

Theorem 3.4. *For a right (or left) CIFD ring R, the following conditions are equivalent*:

(1) *R is π-regular*;
(2) *Every prime ideal of R is maximal*;
(3) *$R/W(R)$ is strongly regular.*

Proof. Since R is right CIFD, we have that $W(R) = N_*(R) = N(R)$ (hence every minimal prime ideal of R is completely prime by [18, Proposition 1.11] and $R/N_*(R)$ is reduced) by Theorem 3.2(1) and that $R/N_*(R)$ is right CIFD by Lemma 2.1(3). Note that every prime factor ring of R coincides with one of $R/N_*(R)$. We use these freely.

Since R is right π-duo by Lemma 2.1(1)(iii), we obtain the equivalence of the conditions (1) and (2) by [13, Theorem 1.13], noting that $W(R) = N_*(R) = J(R) = N(R)$ when R is π-regular.

(3) $\Rightarrow$ (2) is proved by [7, Theorem 3.2].

(2) $\Rightarrow$ (3). Suppose that every prime ideal of R is maximal. Then every prime ideal of R is minimal, and hence R/P is a simple domain for every prime ideal P of R. So we can say that every prime ideal of R is completely prime. But R/P is also right CIFD by Lemma 2.1(3), so R/P is a division ring (hence regular) by Proposition 2.7(2). Now since $R/N_*(R)$ is reduced, $R/N_*(R)$ is regular by [7, Theorem 1.21], noting $R/P \cong (R/N_*(R))/(P/N_*(R))$. Therefore $R/W(R) = R/N_*(R)$ is strongly regular by [7, Theorem 3.5].

The proof for left CIFD rings is almost same. □

4. Some matrix rings and polynomial rings

In this section we study the relation among right CIFD rings, right duo rings and commutative rings, which may provide useful information to the studies of matrix rings and polynomial rings. Note that, by Example 2.2(2), $D_n(R)$ is neither right nor left CIFD for $n \geq 4$ over any ring R. We see an equivalent condition for $D_3(R)$ to be right CIFD in the theorem below. Furthermore the following provides a kind of right CIFD rings which are not right duo, noting that $D_n(R)$ is neither right nor left duo for all $n \geq 3$ over any ring R by Lemma 2.1(6).

Theorem 4.1. *For a ring R, the following conditions are equivalent*:

(1) *R is commutative*;

(2) *$D_3(R)$ is CIFC*;

(3) *$D_3(R)$ is right CIFD*;

(4) *$D_3(R)$ is left CIFD.*

Proof. (1) $\Rightarrow$ (2). Let R be commutative. Consider the proper ideal $I = RE_{13}$ of $D_3(R)$. Then $I \subseteq Z(D_3(R))$ by Lemma 2.1(5). R/I is isomorphic to the ring

$$R_1 = \left\{ \left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \begin{pmatrix} a & c \\ 0 & a \end{pmatrix} \right) \in D_2(R) \times D_2(R) \,\middle|\, a, b, c \in R \right\}$$

by the proof of [3, Theorem 1.2(2)]. Moreover since $D_2(R)$ is commutative, R_1 is also commutative. Thus $D_3(R)$ is CIFC.

(3) $\Rightarrow$ (1). Suppose that R is noncommutative (i.e., $Z(R) \subsetneq R$). Let I be a proper ideal of $D_3(R)$ such that $I \subseteq Z(D_3(R))$. Then, by Remark 2.6(1),

$$I = \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, c \in J \text{ for some ideal } J \text{ of } R \text{ such that } J \subseteq Z(R) \right\}.$$

Here if $J = R$, then $R = Z(R)$, contradicting that R is noncommutative. Thus J is a proper ideal of R.

Now consider

$$D_3(R)/I = \left\{ \begin{pmatrix} a & b & c' \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, d \in R \text{ and } c' \in R/J \right\}$$

and write $\bar{\alpha} = \alpha + I$ for $\alpha \in D_3(R)$. Then $\bar{E}_{12}\bar{E}_{23} = \bar{E}_{13} \notin \bar{E}_{23}(D_3(R)/I)$ and hence $D_3(R)/I$ is not right duo. Thus $D_3(R)$ is not right CIFD.

In this situation, we also have $\bar{E}_{12}\bar{E}_{23} = \bar{E}_{13} \notin (D_3(R)/I)\bar{E}_{12}$ and hence $D_3(R)/I$ is not left duo. Thus $D_3(R)$ is not left CIFD, proving (3) $\Rightarrow$ (1).

(2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are obvious. The proof of (4) $\Rightarrow$ (1) is similar to one of (3) $\Rightarrow$ (1). □

We provide an example demonstrating that there exists a right duo ring R over which $D_2(R)$ is not right CIFD (hence not right duo). Note, by [10, Theorem 1.2(1)], that if R is a ring such that $D_2(R)$ is right duo, then $ab-ba \in b^2R$ for any $a, b \in R$ (hence $ab - ba \in a^2R \cap b^2R$).

Example 4.2. We use the right duo ring in [2]. Let F be a field and $F(x)$ the quotient field of $F[x]$, and set $R = F(x) \times F(x)$ as an additive group and define the multiplication by $(f_1(x), g_1(x))(f_2(x), g_2(x)) = (f_1(x)f_2(x), f_1(x^2)g_2(x) + g_1(x)f_2(x))$. Then R is right duo by the argument in [2]. Let

$$A = \begin{pmatrix} (x,0) & 0 \\ 0 & (x,0) \end{pmatrix}, \; B = \begin{pmatrix} (0,1) & 1 \\ 0 & (0,1) \end{pmatrix} \in D_2(R).$$

Then $AB = \left(\begin{smallmatrix} (0,x^2) & (x,0) \\ 0 & (0,x^2) \end{smallmatrix}\right)$. Assume $AB = BC$ for some

$$C = \begin{pmatrix} (f_1(x), g_1(x)) & (f_2(x), g_2(x)) \\ 0 & (f_1(x), g_1(x)) \end{pmatrix} \in D_2(R).$$

Then

$$\begin{pmatrix} (0,x^2) & (x,0) \\ 0 & (0,x^2) \end{pmatrix} = \begin{pmatrix} (0, f_1(x)) & (f_1(x), g_1(x) + f_2(x)) \\ 0 & (0, f_1(x)) \end{pmatrix},$$

so that $f_1(x) = x$ and $f_1(x) = x^2$, which is impossible. Thus there cannot exist such C and hence $D_2(R)$ is not right duo.

We provide another manner to show that $D_2(R)$ is not right duo. Since

$$(x,0)(0,1) - (0,1)(x,0) = (0, x^2 - x) \notin (0,1)^2R = 0,$$

we see that $D_2(R)$ is not right duo also by [10, Theorem 1.2(1)].

Next we show that $D_2(R)$ is not right CIFD. Let $(f(x), g(x)) \in Z(R)$. Then, from $(f(x), f(x^2) + g(x)) = (f(x), g(x))(1,1) = (1,1)(f(x), g(x)) = (f(x), f(x) + 1(x))$, we get $f(x) = f(x^2)$ and hence $f(x) \in F$, $f(x) = a$ say; and from $(ax, xg(x)) = (a, g(x))(x,0) = (x,0)(a, g(x)) = (ax, x^2g(x))$, we get $g(x) = 0$. So $(f(x), g(x)) = (a, 0)$ and hence $Z(R) = F \times 0$, from which we infer that $Z(D_2(R)) = \left\{ \left(\begin{smallmatrix} (a,0) & (b,0) \\ (0,0) & (a,0) \end{smallmatrix}\right) \mid a, b \in F \right\}$ by Lemma 2.1(5). Note that $\left(\begin{smallmatrix} (a,0) & (b,0) \\ (0,0) & (a,0) \end{smallmatrix}\right) \in U(D_2(R))$ when $a \neq 0$ and that $\left(\begin{smallmatrix} (0,0) & (b,0) \\ (0,0) & (0,0) \end{smallmatrix}\right) D_2(R) =$

$\left(\begin{smallmatrix} (0,0) & R \\ (0,0) & (0,0) \end{smallmatrix}\right) \nsubseteq Z(D_2(R))$ when $b \neq 0$. Thus 0 is the only proper ideal of $D_2(R)$ contained in $Z(D_2(R))$. $R \cong R/0$ is not right duo by the argument above, hence R is not right CIFD.

In the result below, we consider a condition of a right duo ring R over which $D_2(R)$ may be right CIFD.

Proposition 4.3.

(1) *A ring R is CIFC if and only if so is $D_2(R)$.*

(2) *For a simple ring R, the following conditions are equivalent*:

  (i) *$D_2(R)$ is right* (*resp., left*) *CIFD*;

  (ii) *$D_2(R)$ is right* (*resp., left*) *duo*;

  (iii) *R is right* (*resp., left*) *CIFD*;

  (iv) *R is a division ring.*

Proof. (1) Write $E = D_2(R)$. Let R be CIFC. Then R/I is commutative for some proper ideal I of R with $I \subseteq Z(R)$. Set $J = D_2(I)$. Then J is a proper ideal of E, and $J \subseteq Z(E) = D_2(Z(R))$ by Lemma 2.1(5). Furthermore $E/J \cong D_2(R/I)$ is a commutative ring. Thus $D_2(R)$ is CIFC. The converse is proved by Lemma 2.1(4) through $R \cong D_2(R)/RE_{12}$.

(2) (i) $\Rightarrow$ (iii). This is proved by Lemma 2.1(3) through $R \cong D_2(R)/RE_{12}$.

(iii) $\Rightarrow$ (iv). This is proved by Proposition 2.7(2).

(iv) $\Rightarrow$ (ii). This is obtained from Lemma 2.1(6).

(ii) $\Rightarrow$ (i). Evident.

The proof of the left case is similar. □

In the arguments below, we see useful information for polynomial rings to be right CIFD. Recall that the CIFC property can pass to polynomial rings (see [12, Proposition 1.7]). It is easily checked that $Z(R[x]) = Z(R)[x]$, and we freely use this fact.

Theorem 4.4. *For a noncommutative ring R, the following conditions are equivalent*:

(1) *$R[x]$ is right CIFD*;

(2) *$R[x]$ is CIFC*;

(3) *$R[x]$ is left CIFD*;

(4) *R is CIFC.*

Proof. (1) $\Rightarrow$ (2). Let $R[x]$ be right CIFD. Then $R[x]/J$ is right duo for some proper ideal J of $R[x]$ with $J \subseteq Z(R[x])$. If $J = 0$, then $R[x]$ is right duo, so that R is commutative by [9, Lemma 3]. Thus $J \neq 0$. Let $0 \neq f(x) = \sum_{i=0}^{m} a_i x^i \in J$. Assume $a_j \in U(R)$ for some j. Then since $Rf(x) \subseteq J$,

$$ra_j^{-1}f(x) = ra_j^{-1}a_0 + \cdots + ra_j^{-1}a_{j-1}x^{j-1} + rx_j + ra_j^{-1}a_{j+1}x^{j+1}$$
$$+ \cdots + ra_j^{-1}a_m x^m \in J$$

for all $r \in R$, contradicting that $J \subseteq Z(R)[x]$ and $Z(R) \subsetneq R$. Thus $a_i \notin U(R)$ for all i. This implies $(a + x) + J$ and $(b + x) + J$ are both nonzero in $R[x]/J$ for all $a, b \in R$. Write $\bar{g} = g + J$ for $g \in R[x]$. Since $R[x]/J$ is right duo, $(\bar{a} + \bar{x})(\bar{b} + \bar{x}) = (\bar{b} + \bar{x})(\bar{c} + \bar{d}\bar{x})$ for some $\bar{c} + \bar{d}\bar{x} \in R[x]/J$, comparing the degrees of both sides of the equation. Then $\bar{d} = 1$ and this yields $\bar{c} = \bar{a}$, entailing $\bar{a}\bar{b} = \bar{b}\bar{a}$. Thus $R[x]/J$ is a commutative ring, so that $R[x]$ is CIFC.

The proof of (3) $\Rightarrow$ (2) is similar.

(2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) are obvious.

(2) $\Rightarrow$ (4). This is obtained from Lemma 2.1(4) through $R \cong R[x]/xR[x]$.

(4) $\Rightarrow$ (2). This is proved by [12, Proposition 1.7]. □

But "R is right (left) CIFD" is not equivalent to the conditions of Theorem 4.4 as below.

Corollary 4.5. *The right (left) CIFD property does not pass to polynomial rings.*

Proof. Let R be a noncommutative division ring. Then R is CIFD but not CIFC. Assume that $R[x]$ is right CIFD. Then $R[x]/J$ is right duo for some proper ideal J of $R[x]$ with $J \subseteq Z(R[x])$. But $J = 0$ by the argument in the proof of Theorem 4.4, entailing $R[x]$ is right duo. Then R is commutative by [9, Lemma 3], contrary to R being noncommutative. So $R[x]$ is not right CIFD. The proof of the left case is similar. □

In the remark below, we have information about the ideal J in the preceding arguments when given rings are not CIFC.

Remark 4.6. Let R be a ring and J be a proper ideal of $R[x]$ such that $J \subseteq Z(R[x])$ and $R[x]/J$ is right duo. Suppose that R is not CIFC. Then we claim that J is not of the form $I[x]$ with a proper ideal I of R. Note $Z(R[x]) = Z(R)[x]$. If $J = 0$, then $R[x]$ is right duo by hypothesis and it follows that R is commutative by [9, Lemma 3]. This contradicts that R is not CIFC. Thus $J \neq 0$. Assume on the contrary that $J = I[x]$ for some proper ideal I of R. This yields that $\frac{R}{I}[x]$ $(\cong \frac{R[x]}{I[x]} = \frac{R[x]}{J})$ is right duo. But $\frac{R}{I}[x]$ is commutative by [9, Lemma 3], so that R/I is commutative. Since $J \subseteq Z(R[x])$, $I \subseteq Z(R)$. Thus R is CIFC, contrary again to R being not CIFC. Therefore J is not of the form $I[x]$ with a proper ideal I of R.

In the following, we see a difference between polynomial rings being commutative and CIFC (see Theorem 4.4).

Corollary 4.7.

1. *For a ring R, $D_3(R)[x]$ is right (left) CIFD if and only if $R[x]$ is commutative, if and only if R is commutative.*
2. *From any commutative ring, we can construct a right (left) CIFD (polynomial) ring that is noncommutative and non-reduced.*

Proof. (1) Since $D_3(R)[x] \cong D_3(R[x])$, we obtain the proof by Theorem 4.1.

(2) This is an immediate consequence of (1). $\square$

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