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CONSTRUCTIONS OF REGULAR SPARSE ANTI-MAGIC SQUARES

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Abstract. For positive integers n and d with $d < n$, an $n \times n$ array A based on $\mathcal{X} = \{0, 1, \ldots, nd\}$ is called *a sparse anti-magic square of order n with density d*, denoted by SAMS(n, d), if each non-zero element of $\mathcal{X}$ occurs exactly once in A, and its row-sums, column-sums and two main diagonal-sums constitute a set of $2n + 2$ consecutive integers. An SAMS(n, d) is called *regular* if there are exactly d non-zero elements in each row, each column and each main diagonal. In this paper, we investigate the existence of regular sparse anti-magic squares of order $n \equiv 1, 5 \pmod 6$, and prove that there exists a regular SAMS(n, d) for any $n \geq 5$, $n \equiv 1, 5 \pmod 6$ and d with $2 \leq d \leq n - 1$.

1. Introduction

Magic square had a long history and had been widely studied, and the interested reader may refer to the two monographs [3] and [5]. *A magic square of order n* is an $n \times n$ array whose entries are an arrangement of the integers $1, 2, \ldots, n^2$, in which all elements in each row, each column and each main diagonal, add to the same sum. As an example, it is easy to verify that the following array A is a magic square of order 3:

$$A = \begin{array}{|c|c|c|} \hline 4 & 9 & 2 \\ \hline 3 & 5 & 7 \\ \hline 8 & 1 & 6 \\ \hline \end{array}\,.$$

There exists a magic square of order n if and only if $n \geq 3$ [19]. Regarding the construction methods and history of magic squares, the interested reader may refer to [2–5, 19].

Sparse semi-magic squares were introduced by Gray and MacDougall in 2006 [13] and applied to construct vertex-magic edge labeling for digraphs. *A sparse*

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semi-magic square is an $n \times n$ array where the numbers $1, 2, \ldots, r$ with $r < n^2$ are placed once each with the remaining entries 0 such that all the rows and columns have a constant sum s. Let

$B =$

			8	13
6		10		5
2	7		12	
4	14			3
9		11	1	

,

where the empty positions of B indicate 0. Then it is easy to verify that B is a sparse semi-magic square.

For positive integers n and d with $d < n$, *a sparse magic square of order n with density* d, denoted by $\mathrm{SMS}(n, d)$, is an $n \times n$ integer array containing the numbers $1, 2, \ldots, nd$ with the remainder of its entries 0's and all the rows, columns and two main diagonals have a constant sum. An $\mathrm{SMS}(n, d)$ is called *regular* if there are exactly d non-zero elements in each row, each column and each main diagonal. The array C below illustrates a regular $\mathrm{SMS}(5, 4)$, where the empty positions of C indicate 0:

$C =$

16		4	15	7
2	13	9	18	
10	20		1	11
	3	12	8	19
14	6	17		5

.

The existence of a regular $\mathrm{SMS}(n, d)$ has been completely solved by Li et al. [20]. They showed that for any positive integers n and d with $d < n$, there exists a regular $\mathrm{SMS}(n, d)$ if and only if $d \geq 3$ when n is odd, or d is even and $d \geq 4$ when n is even.

An anti-magic square of order n is an $n \times n$ array whose entries are an arrangement of the integers $1, 2, \ldots, n^2$, and its row-sums, column-sums and two main diagonal sums constitute a set of $2n + 2$ consecutive integers. We give the following array D as an example of an anti-magic square of order 4, where the last column and the last row indicate its row-sums and column-sums, respectively, and two main diagonal-sums are 29 and 34, respectively:

$D =$

2	15	5	13	35
16	3	7	12	38
9	8	14	1	32
6	4	11	10	31
33	30	37	36	

.

In 1994, Abe [1] proposed an open problem about the existence of an anti-magic square of order n, that is,

Problem 2.23. Find a method of constructing an anti-magic square of every order.

In 2002, Jiang [17] proved that for any even $n \geq 4$, there exists an anti-magic square of order n. Cormie et al. [10] showed that there exists an anti-magic square of order n if and only if $n \geq 4$.

Sparse anti-magic squares can be viewed as a generalization of anti-magic squares. For positive integers n and d with $d < n$, let A be an $n \times n$ array on the elements $1, 2, \ldots, nd$ with the remainder of its entries 0's and let S_A be the set of row-sums, column-sums and two main diagonal sums of A. We call S_A the *sum set* of A. Then A is called a *sparse anti-magic square of order n with density d*, denoted by SAMS(n, d), if each element of $\{1, 2, \ldots, nd\}$ occurs exactly once in A and S_A consists of $2n + 2$ consecutive integers. In [12], an SAMS(n, d) is also called a *sparse totally anti-magic square*. An SAMS(n, d) is called *regular* if there are d non-zero elements in each row, each column and each main diagonal. As an example, a regular SAMS$(5, 2)$ is listed below:

$$E = \begin{array}{|c|c|c|c|c|} \hline 1 & 6 & & & \\ \hline & & 2 & 8 & \\ \hline 5 & & & & 3 \\ \hline & 9 & 7 & & \\ \hline & & & 4 & 10 \\ \hline \end{array}\,,$$

where the empty positions of E indicate 0.

Sparse anti-magic squares and sparse magic squares are useful in graph theory. For instance, they can be used to construct vertex-magic total labeling for bipartite graphs [12, 16, 18], regular graphs [11, 14, 15], trees and cubic graphs [21, 22].

Recently, Chen et al. [6–8] proved the following results.

Lemma 1.1 ([8], Theorem 1.1)**.** *There exists a regular SAMS$(n, n-1)$ if and only if $n \geq 4$.*

Lemma 1.2 ([7], Theorem 1.2)**.** *There exists a regular SAMS$(n, n-2)$ if and only if $n \geq 4$.*

Lemma 1.3 ([6], Theorem 1.3 and Theorem 1.4)**.** (i) *There exists a regular SAMS$(n, 3)$ if and only if $n \geq 4$.* (ii) *There exists a regular SAMS$(n, 5)$ if and only if $n \geq 6$.*

In this paper, we investigate the existence of regular sparse anti-magic squares of order $n \equiv 1,\ 5 \pmod 6$ and obtain the following theorem.

Theorem 1.4. *Suppose that n is a positive integer satisfying $n \geq 5$ and $n \equiv 1$, $5 \pmod 6$, there exists a regular SAMS(n, d) for any d with $2 \leq d \leq n - 1$.*

For convenience, the following notations are used throughout this paper. Let $\mathbb{Z}$ be the set of integers, $I_n = \{1, 2, 3, \ldots, n\}$ and we always use I_m and I_n to label the rows and columns of an $m \times n$ array, respectively. Let $a, b \in \mathbb{Z}$ and $[a, b]$ be the set of integers v such that $a \leq v \leq b$. If A is an array based on $\mathbb{Z}$, let $R(A)$ and $C(A)$ be the set of row-sums and the set of column-sums of

A, respectively. Usually, the main diagonal of a square array from upper left to lower right is called *the left diagonal*, the other is called *the right diagonal*. Let $l(A)$ and $r(A)$ be the sum of the elements in the left diagonal and the right diagonal of A, respectively. Then $S_A = R(A) \cup C(A) \cup \{l(A), r(A)\}$. Suppose that a and n are positive integers, and define $\langle a \rangle_n$ as follows.

$$\langle a \rangle_n = \begin{cases} r, & \text{if} \quad n \nmid a,\ a = mn + r \ \text{ and } \ 0 < r < n, \\ n, & \text{if} \quad n \mid a. \end{cases}$$

Clearly, $1 \le \langle a \rangle_n \le n$.

The rest of this paper is organized as follows. In Section 2, we shall define some terminologies such as Latin square, symmetric diagonal Kotzig arrays and symmetric forward diagonal arrays, and state some results on them for later use. We show that there exists a regular SAMS$(n, 2)$ and a regular SAMS$(n, 4)$ for $n \equiv 1, 5 \pmod 6$ and $n \ge 5$ via direct construction in Section 3 and Section 4, respectively. In Section 5, we prove that there exists a regular SAMS(n, d) for $n \equiv 1, 5 \pmod 6$ and $d \in [6, n-3]$. In Section 6, the proof of Theorem 1.4 is presented, and we give some concluding remarks for future work.

2. Preliminaries

In this section, we shall give some definitions such as Latin square, symmetric diagonal Kotzig arrays and symmetric forward diagonal arrays etc. and state some new results on them for later use.

Definition 2.1. A *Latin square* of order n is an $n \times n$ array in which each cell contains a single symbol from an n-set S, such that each symbol occurs exactly once in each row and exactly once in each column.

A *transversal* in a Latin square of order n is a set of n cells, one from each row and column, containing each of the n symbols exactly once. *A diagonal Latin square of order* n is a Latin square of order n whose left diagonal and right diagonal are both transversals. *A forward diagonal* of an array $B = (b_{i,j})$ of order $m \times n$ is the set $\{b_{i,\langle j+i \rangle_n} \mid i \in I_m\}$ for each $j \in I_n$.

Definition 2.2. Suppose n and d are positive integers with $d \le n$. A $d \times n$ rectangular array $A = (a_{i,j})$, $i \in I_d$, $j \in I_n$, is a *symmetric diagonal Kotzig array* if it has the following properties:

1. Each row is a permutation of the set $I_n = \{1, 2, \ldots, n\}$.
2. All columns have the same sum.
3. All forward diagonals have the same sum.
4. $a_{i,j} + a_{d+1-i,n+1-j} = n + 1$ for each $(i, j) \in I_d \times I_n$.

Three-row arrays satisfying the first two conditions of Definition 2.2 were used by Kotzig [18] to construct edge-magic labelings and there is an account of this in [23,24] where they are called *Kotzig arrays*. Gray and MacDougall have constructed a d-row generalization of these Kotzig arrays and they have been used to construct vertex-magic labelings for complete bipartite graphs [16]. The arrays satisfying the first three conditions of Definition 2.2 were used by Gray and MacDougall [13] to construct sparse semi-magic square and vertex-magic labelings, and they are called *diagonal Kotzig arrays.* Our constructions of squares require diagonal Kotzig arrays with the additional diagonal condition stated as property 4 above.

Definition 2.3. Suppose n and t are positive integers and $t \leq n$. A $t \times n$ array $A = (a_{i,j})$, $i \in I_t, j \in I_n$, whose positions are an arrangement of the integers $1, 2, \ldots, nt$, is a *symmetric forward diagonals array*, denoted by SFD(t, n) for short, if it satisfies the following properties:

1. All columns have the same sum.
2. All forward diagonals have the same sum.
3. $a_{i,j} + a_{t+1-i,n+1-j}$ is a constant for any $(i, j) \in I_t \times I_n$.

If $A_1 = (a^{(1)}_{i,j})$ is an SFD(t, n) over I_{nt}, let $a^{(2)}_{i,j} = a^{(1)}_{i,j} + l$, where l is a nonnegative integer, then $A_2 = (a^{(2)}_{i,j})$ is also an SFD(t, n) over $[1 + l, nt + l]$.

Construction 2.4. If there exists a symmetric diagonal Kotzig array of order $d \times n$, then there exists an SFD(d, n).

Proof. Let $A = (a_{i,j})$ be a symmetric diagonal Kotzig array of order $d \times n$ and $B = (b_{i,j})$ be the $d \times n$ array with $b_{i,j} = i - 1$, where $i \in I_d$, $n \in I_n$. We shall show that $S = A + nB = (s_{i,j})$ is an SFD(d, n).

Clearly, $\bigcup\limits_{i=1}^{d} \bigcup\limits_{j=1}^{n} \{s_{i,j}\} = I_{dn}$. Note that the columns of A and B have constant sum, respectively, and therefore the columns of $S = A + nB$ will also have a constant sum s. The forward diagonals of A and B have constant sum, respectively, and so the forward diagonals of S will also have a constant sum s. Since $a_{i,j} + a_{d+1-i,n+1-j}$ is a constant, we have

$$\begin{aligned} s_{i,j} + s_{d+1-i,n+1-j} &= (a_{i,j} + nb_{i,j}) + (a_{d+1-i,n+1-j} + nb_{d+1-i,n+1-j}) \\ &= a_{i,j} + a_{d+1-i,n+1-j} + n(d-1) \end{aligned}$$

is a constant. Hence S is an SFD(d, n). □

Now we give the existence of a symmetric diagonal Kotzig array by using direct construction and the recurrence method.

Theorem 2.5. *There exists a symmetric diagonal Kotzig array of order $d \times n$ for any odd integer $n \geq 3$ and integer $d \in [3, n]$.*

Proof. For $i \in I_3$, $j \in I_n$, let $A_3 = (a_{i,j})$, where

$$a_{1,j} = \begin{cases} n - \frac{j-1}{2}, & \text{if } j \text{ is odd}, \\ \frac{n+1-j}{2}, & \text{if } j \text{ is even}, \end{cases} \quad a_{2,j} = j, \quad a_{3,j} = n + 1 - a_{1,n+1-j}.$$

For $i \in I_4$, $j \in I_n$, let $A_4 = (b_{i,j}) = \binom{B_1}{B_2}$, where

$$b_{1,j} = \begin{cases} j, & \text{if} \quad j \leq \frac{n-1}{2}, \\ j+1, & \text{if} \quad \frac{n+1}{2} \leq j \leq n-1, \\ \frac{n+1}{2}, & \text{if} \quad j = n, \end{cases}$$

$$b_{2,j} = \begin{cases} \frac{n+1}{2}, & \text{if} \quad j = 1, \\ n+2-j, & \text{if} \quad 2 \leq j \leq \frac{n+1}{2}, \\ n+1-j, & \text{if} \quad j > \frac{n+1}{2}, \end{cases}$$

$$b_{3,j} = n+1-b_{2,n+1-j}, \quad b_{4,j} = n+1-b_{1,n+1-j}.$$

For $i \in I_5$, $j \in I_n$, let $A_5 = (c_{i,j})$, where

$$c_{1,j} = [\frac{n+1}{2}(j-1)] \pmod n + 1, \quad c_{2,j} = n+1-j, \quad c_{3,j} = j,$$

$$c_{4,j} = n+1-j, \quad c_{5,j} = j + \frac{n+1}{2} - c_{1,j}.$$

It is readily checked that A_3, A_4, A_5 and $A_6 = \binom{A_3}{A_3}$ are symmetric diagonal Kotzig arrays of order $d \times n$ for $d = 3, 4, 5, 6$, respectively.

We write integer $d \geq 7$ as $d = 4k + \alpha$, where $k \geq 1$ and $\alpha \in \{3, 4, 5, 6\}$. Let

$$E = \begin{pmatrix} B_1 \\ \vdots \\ B_1 \\ A_\alpha \\ B_2 \\ \vdots \\ B_2 \end{pmatrix},$$

where B_i occurs k times for $i = 1, 2$. It is clear that E is a symmetric diagonal Kotzig array of order $(4k + \alpha) \times n$. □

Remark 2.6. (i) It is to be pointed out that the array A_4 also has the property that for any $j \in I_n$,

$$b_{1,j} + b_{2,\langle j+1 \rangle_n} = b_{3,j} + b_{4,\langle j+1 \rangle_n} = n+1.$$

(ii) There are many ways to obtain a symmetric diagonal Kotzig array of order $d \times n$ with $d \geq 7$. Let

$$F = \begin{pmatrix} A_3 \\ B_1 \\ \vdots \\ B_1 \\ A_\alpha \\ B_2 \\ \vdots \\ B_2 \\ A_3 \end{pmatrix},$$

where B_i occurs $k-1$ times for $i = 1, 2$. Then it is easy to check that F is also a symmetric diagonal Kotzig array of order $(4k + 2 + \alpha) \times n$.

(iii) When $d = 2e$ and $e \geq 3$, we can get a symmetric diagonal Kotzig array of order $d \times n$ by joining two symmetric diagonal Kotzig arrays of order $e \times n$

coming from Theorem 2.5. They will be used in the proof of the following when the number of the rows of a symmetric diagonal Kotzig array is even $d \geq 6$.

Combining Construction 2.4 and Theorem 2.5, we have the following theorem.

Theorem 2.7. *For any odd $n \geq 3$ and $t \in [3, n]$, there exists an SFD(t, n) over $[1 + l, nt + l]$ for any nonnegative integer l.*

Remark 2.8. By (i) and (iii) of Remark 2.6, for $e \geq 2$ and any nonnegative integer l, there exists an SFD$(2e, n)$, $F = (f_{i,j})$, over $[1 + l, 2en + l]$ by using Construction 2.4 and Theorem 2.5, and it has the additional properties:

$$f_{i,\langle i+x\rangle_n} + f_{2e+1-i,n+1-\langle i+x\rangle_n} = 2en + 1 + 2l,$$

and

$$\sum_{i=1}^{e} f_{i,\langle i+x\rangle_n} + \sum_{i=e+1}^{2e} f_{i,\langle i+x+y\rangle_n} = (2en + 1 + 2l)e$$

for any $x, y \in I_n$.

3. Regular SAMS$(n, 2)$ for $n \geq 5$ and $n \equiv 1, 5 \pmod 6$

In this section, we shall prove that there exists a regular SAMS$(n, 2)$ for $n \geq 5$ and $n \equiv 1, 5 \pmod 6$. The idea of our construction is divided into three steps. Firstly, we give a special array A and a Latin square B. Secondly we shall put the elements of A into the Latin square B to obtain W such that W is a regular SAMS for $n \equiv 1 \pmod 6$ and a near regular SAMS for $n \equiv 5 \pmod 6$, respectively. Furthermore, for $n \equiv 5 \pmod 6$, we can obtain a regular SAMS, W^*, by doing some column permutations to W.

Theorem 3.1. *There exists a regular SAMS$(n, 2)$ for $n \geq 5$ and $n \equiv 1, 5 \pmod 6$.*

Proof. We write $n \geq 5$ and $n \equiv 1, 5 \pmod 6$ as $n = 2m + 1$, where $m > 1$. Construct a special $2 \times n$ array $A = (a_{i,j})$ over $[1, 4m + 2]$, where $i = 1, 2$, $j \in I_n$ and

$$a_{1,j} = \begin{cases} n + j, & j \in [1, m - 1], \\ n + j + 1, & j \in [m, 2m], \\ n, & j = 2m + 1, \end{cases} \qquad a_{2,j} = \begin{cases} j, & j \in [1, m], \\ 3m + 1, & j = m + 1, \\ j - 1, & j \in [m + 2, 2m + 1]. \end{cases}$$

Let R_σ, $\sigma = 1, 2$, be the set of the elements in the σ-th row of A. It is easy to see that

$$\begin{aligned} R_1 &= [n + 1, n + m - 1] \cup [n + m + 1, n + 2m + 1] \cup \{n\} \\ &= [2m + 2, 3m] \cup [3m + 2, 4m + 2] \cup \{2m + 1\} \\ &= [2m + 1, 4m + 2] \setminus \{3m + 1\}, \\ R_2 &= [1, m] \cup \{3m + 1\} \cup [m + 1, 2m] = [1, 2m] \cup \{3m + 1\}. \end{aligned}$$

We have $R_1 \cup R_2 = [1, 4m + 2]$.

Let S_1 and S_2 be the set of column-sums and forward diagonal-sums, respectively. By a simple calculation, we have

$$\begin{aligned}
S_1 &= \bigcup_{j=1}^{n}\{a_{1,j}+a_{2,j}\}\\
&= \bigcup_{j=1}^{m-1}\{a_{1,j}+a_{2,j}\}\cup\{a_{1,m}+a_{2,m},a_{1,m+1}+a_{2,m+1}\}\\
&\quad \cup\left(\bigcup_{j=m+2}^{2m}\{a_{1,j}+a_{2,j}\}\right)\cup\{a_{1,2m+1}+a_{2,2m+1}\}\\
&= \bigcup_{j=1}^{m-1}\{n+2j\}\cup\{2n,3n+1\}\cup\left(\bigcup_{j=m+2}^{2m}\{n+2j\}\right)\cup\{n+2m\}\\
&= \left[\bigcup_{j=1}^{2m}\{n+2j\}\setminus\{2n+1\}\right]\cup\{2n,3n+1\},
\end{aligned}$$

$$\begin{aligned}
S_2 &= \bigcup_{j=1}^{n-1}\{a_{1,j}+a_{2,j+1}\}\cup\{a_{1,2m+1}+a_{2,1}\}\\
&= \bigcup_{j=1}^{m-1}\{a_{1,j}+a_{2,j+1}\}\cup\{a_{1,m}+a_{2,m+1}\}\cup\left(\bigcup_{j=m+1}^{2m}\{a_{1,j}+a_{2,j+1}\}\right)\\
&\quad \cup\{a_{1,2m+1}+a_{2,1}\}\\
&= \bigcup_{j=1}^{m-1}\{n+2j+1\}\cup\{(n+m+1)+(3m+1)\}\\
&\quad \cup\left(\bigcup_{j=m+1}^{2m}\{n+2j+1\}\right)\cup\{n+1\}\\
&= \left[\bigcup_{j=1}^{2m}\{n+2j+1\}\setminus\{2n\}\right]\cup\{3n,n+1\}.
\end{aligned}$$

It follows that $S_1\cup S_2=[n+1,3n+1]\setminus\{2n+1\}$.

Let $B=(b_{i,j})$, where $b_{i,j}=\langle 2i+j-1\rangle_n$, $i,j\in I_n$. It is easy to check that B is a diagonal Latin square of order n over I_n with the property

$$\begin{aligned}
b_{n+1-i,n+1-j} &= \langle 2(n+1-i)+(n+1-j)-1\rangle_n\\
&= \langle 3n+1-(2i+j-1)\rangle_n=(n+1)-b_{i,j},
\end{aligned}$$

i.e.,

$$b_{i,j}+b_{n+1-i,n+1-j}=n+1.$$

For each $j \in I_n$, define

$$f(x,j) = i \ \text{ if } \ b_{i,j} = x, \ \text{that is, } f(b_{i,j}, j) = i$$

and let

$$g(s) = \langle m + 2s - 1\rangle_n, \ s \in I_n.$$

It is easy to see that for each $j \in I_n$, $f(x,j)$ is a bijection function from I_n to I_n since B is a Latin square over I_n, and g is also a bijection function from I_n to I_n.

We put $a_{1,s}$ and $a_{2,s}$, $s \in I_n$, into the positions $(f(m, g(s)), g(s))$ and $(f(m+2, g(s)), g(s))$ of B, respectively, the other positions of B are all filled by 0, denoted by $W = (w_{i,j})$, where $i, j \in I_n$, that is, $w_{f(m,g(s)),g(s)} = a_{1,s}$ and $w_{f(m+2,g(s)),g(s)} = a_{2,s}$.

It is clear that the elements of the s-th column, $s \in I_n$, of A are filled into the $g(s)$-th column of W, so the non-zero elements in the same column of W is just in the same column of A, $C(W) = S_1$. We shall show that $a_{1,s}$ and $a_{2,\langle s+1\rangle_n}$, $s \in I_n$, are in the same row of W. To do this, we need only prove that for any $s \in I_n$, $f(m, g(s)) = f(m+2, g(s+1))$. Without loss of generality, suppose that $f(m, g(s)) = \xi$. We have $b_{\xi,g(s)} = \langle 2\xi + g(s) - 1\rangle_n = m$ by the definition of f, and also have

$$\begin{aligned} b_{\xi,g(s+1)} &= \langle 2\xi + g(s+1) - 1\rangle_n \\ &= \langle 2\xi + g(s) + 2 - 1\rangle_n \\ &= \langle 2\xi + g(s) - 1\rangle_n + 2 = m + 2. \end{aligned}$$

It follows that $f(m+2, g(s+1)) = \xi$. Then the non-zero elements in the same row of W is just in the forward diagonal of A, $R(W) = S_2$. It is clear that

$$C(W) \cup R(W) = S_1 \cup S_2 = [n+1, 3n+1] \backslash \{2n+1\}.$$

Next, we shall consider the sum of the elements in the two main diagonals of W. There are exactly 2 non-zero elements in each main diagonal of W according to the definition of the diagonal Latin square B. It is easy to calculate that

$$\begin{aligned} a_{1,m+2} &= w_{f(m,g(m+2)),g(m+2)} \\ &= w_{m,m+2} \end{aligned}$$

since $g(m+2) = m+2$, $b_{i,j} = \langle 2i + j - 1\rangle_n = \langle 2m + (m+2) - 1\rangle_n = m$ when $i = m, \ j = m+2$, and $f_{m,j} = i$, i.e., $f_{m,m+2} = m$, and

$$\begin{aligned} a_{2,m+1} &= w_{f(m+2,g(m+1)),g(m+1)} \\ &= w_{m+2,m} \end{aligned}$$

since $g(m+1) = m$, $b_{i,j} = \langle 2i + j - 1\rangle_n = \langle 2(m+2) + m - 1\rangle_n = m+2$ when $i = m+2, \ j = m$, and $f_{m+2,j} = i$, i.e., $f_{m+2,m} = m+2$. Hence the sum of the elements in the right diagonal of W is

$$\begin{aligned} w_{m,m+2} + w_{m+2,m} &= a_{1,m+2} + a_{2,m+1} \\ &= (n + m + 2 + 1) + (3m + 1) = 3n + 2. \end{aligned}$$

We shall divide it into two cases to deal with the left diagonal-sum below.

Case 1: We write $n \geq 7$ and $n \equiv 1 \pmod 6$ as $n = 6k+1$, where $k \geq 1$. Clearly, $m = 3k$.

There are exactly two non-zero elements in the left diagonal of W according to the definition of the diagonal Latin square B. By simple calculation we have

$$\begin{aligned} a_{1,k+1} &= w_{f(m,g(k+1)),g(k+1)} \\ &= w_{n-k,n-k} \end{aligned}$$

since $g(k+1) = n-k$, $b_{n-k,n-k} = \langle 2(n-k) + (n-k) - 1\rangle_n = 3k = m$ and $f(m, n-k) = n-k$, and

$$\begin{aligned} a_{2,n+1-k} &= w_{f(m+2,g(n+1-k)),g(n+1-k)} \\ &= w_{k+1,k+1} \end{aligned}$$

since $g(n+1-k) = k+1$, $b_{k+1,k+1} = \langle 2(k+1) + (k+1) - 1\rangle_n = m+2$ and $f(m+2, k+1) = k+1$. Then the sum of the elements in the left diagonal of W is

$$\begin{aligned} w_{k+1,k+1} + w_{n-k,n-k} &= a_{2,n+1-k} + a_{1,k+1} \\ &= (n+1-k-1) + (n+k+1) \\ &= 2n+1. \end{aligned}$$

So, W is a regular SAMS$(n, 2)$.

Case 2: We write $n \geq 5$ and $n \equiv 5 \pmod 6$ as $n = 6k-1$, where $k \geq 1$. Obviously, $m = 3k-1$.

When $k = 1$, a regular SAMS$(5, 2)$ is given as an example in Section 1.

When $k > 1$, we have there are exactly 2 non-zero elements in the left diagonal of W, but their sum is not $2n+1$. In fact,

$$\begin{aligned} a_{1,5k} &= w_{f(m,g(5k)),g(5k)} \\ &= w_{k,k} \end{aligned}$$

since $g(5k) = k$, $b_{k,k} = \langle 2k + k - 1\rangle_n = 3k-1 = m$ and $f(m, k) = k$, and

$$\begin{aligned} a_{2,k+1} &= w_{f(m+2,g(k+1)),g(k+1)} \\ &= w_{n+1-k,n+1-k} \end{aligned}$$

since $g(k+1) = n+1-k$, $b_{n+1-k,n+1-k} = \langle 2(n+1-k) + (n+1-k) - 1\rangle_n = 3k+1 = m+2$ and $f(m+2, n+1-k) = n+1-k$. So the sum of the elements in the left diagonal of W is

$$\begin{aligned} w_{k,k} + w_{n+1-k,n+1-k} &= a_{1,5k} + a_{2,k+1} \\ &= (5k+1+n) + (k+1) \\ &= 2n+3 \neq 2n+1. \end{aligned}$$

The array $W^* = (w^*_{i,j})$, $i, j \in I_n$, is obtained by exchanging column k with column $k+2$ and exchanging column $n+1-k$ with column $n+1-k-2$ of W. We use the notation $\overline{x}$ to represent $n+1-x$ for short and list the elements in the columns $k, k+2, n+1-k, n+1-k-2$ of B, W and W^* in the following tables, respectively.

B

$i\backslash j$	k	$k+1$	$k+2$	$\overline{k+2}$	$\overline{k+1}$	$\overline{k}$
$k-1$	$m-2$	$m-1$	m	$*$	$*$	$*$
k	$\boldsymbol{m}$	$m+1$	$m+2$	$*$	$*$	$*$
$k+1$	$m+2$	$m+3$	$m+4$	$*$	$*$	$*$
$\overline{k+1}$	$*$	$*$	$*$	$m-2$	$m-1$	m
$\overline{k}$	$*$	$*$	$*$	$\boldsymbol{m}$	$m+1$	$m+2$
$\overline{k-1}$	$*$	$*$	$*$	$m+2$	$m+3$	$m+4$

W

$i\backslash j$	k	$k+1$	$k+2$	$\overline{k+2}$	$\overline{k+1}$	$\overline{k}$
$k-1$	0	0	$a_{1,5k+1}$	0	0	0
k	$a_{1,5k}$	0	$a_{2,5k+1}$	0	0	0
$k+1$	$a_{2,5k}$	0	0	0	0	0
$\overline{k+1}$	0	0	0	0	0	$a_{1,k+1}$
$\overline{k}$	0	0	0	$a_{1,k}$	0	$a_{2,k+1}$
$\overline{k-1}$	0	0	0	$a_{2,k}$	0	0

W^*

$i\backslash j$	k	$k+1$	$k+2$	$\overline{k+2}$	$\overline{k+1}$	$\overline{k}$
$k-1$	$a_{1,5k+1}$	0	0	0	0	0
k	$a_{2,5k+1}$	0	$a_{1,5k}$	0	0	0
$k+1$	0	0	$a_{2,5k}$	0	0	0
$\overline{k+1}$	0	0	0	$a_{1,k+1}$	0	0
$\overline{k}$	0	0	0	$a_{2,k+1}$	0	$a_{1,k}$
$\overline{k-1}$	0	0	0	0	0	$a_{2,k}$

Note that $a_{1,5k}$ and $a_{2,k+1}$ lie in the left diagonal of W, and $a_{2,5k+1}$ and $a_{1,k}$ lie in the left diagonal of W^*. Hence

$$\begin{aligned} w^*_{k,k} + w^*_{\overline{k},\overline{k}} &= w_{k,k+2} + w_{\overline{k},\overline{k+2}} \\ &= a_{2,5k+1} + a_{1,k} \\ &= (5k+1-1) + (n+k) = 2n+1. \end{aligned}$$

The set of row-sums, column-sums and the right diagonal-sum of W^* is the same as that of W. It follows that W^* is a regular SAMS$(n,2)$. □

Remark 3.2. For any array $C = (c_{i,j})_{n\times n}$, let $\Omega(C) = \{(i,j) \,|\, c_{i,j} \neq 0,\ i,j \in I_n\}$. In the proof of Theorem 3.1, we have

$$\Omega(W) = \{(i,j) \,|\, b_{i,j} \in \{m, m+2\},\ i,j \in I_n\},$$

and

$$\Omega(W^*) \subset \{(i,j) \,|\, b_{i,j} \in \{m-2, m, m+2, m+4\},\ i,j \in I_n\} \text{ in } \textbf{Case 2}.$$

To illustrate the proof of Theorem 3.1, we give an example in the following.

Example 3.3. There exists a regular SAMS$(7, 2)$.

Proof. As in the proof of Theorem 3.1, take $n = 7$, then $m = 3$ and $m + 2 = 5$.

$$A = \begin{pmatrix} 8 & 9 & 11 & 12 & 13 & 14 & 7 \\ 1 & 2 & 3 & 10 & 4 & 5 & 6 \end{pmatrix},$$

$$B = \begin{pmatrix} 2 & \mathbf{3} & 4 & \mathbf{5} & 6 & 7 & 1 \\ 4 & \mathbf{5} & 6 & 7 & 1 & 2 & \mathbf{3} \\ 6 & 7 & 1 & 2 & \mathbf{3} & 4 & \mathbf{5} \\ 1 & 2 & \mathbf{3} & 4 & \mathbf{5} & 6 & 7 \\ \mathbf{3} & 4 & \mathbf{5} & 6 & 7 & 1 & 2 \\ \mathbf{5} & 6 & 7 & 1 & 2 & \mathbf{3} & 4 \\ 7 & 1 & 2 & \mathbf{3} & 4 & \mathbf{5} & 6 \end{pmatrix}.$$

It is readily checked that $g(1) = \langle m + 2 - 1 \rangle_n = m + 1 = 4$, $f(3, 4) = 7$ and $f(5, 4) = 1$ since $b_{7,4} = 3$ and $b_{1,4} = 5$, then $w_{7,4} = a_{1,1} = 8$ and $w_{1,4} = a_{2,1} = 1$, and so on. We have

$$W = \begin{array}{|c|c|c|c|c|c|c|} \hline & 7 & & 1 & & & \\ \hline & 6 & & & & & 14 \\ \hline & & & & 13 & & 5 \\ \hline & & 12 & & 4 & & \\ \hline 11 & & 10 & & & & \\ \hline 3 & & & & & 9 & \\ \hline & & & 8 & & 2 & \\ \hline \end{array},$$

where the empty positions of W indicate 0. Clearly, $G(W) = [1, 14]$ and there are 2 non-zero elements in each row, each column and each main diagonal of W. On the other hand, the set of row-sums $R(W) = \{8, 20, 18, 16, 21, 12, 10\}$, the set of column-sums $C(W) = \{14, 13, 22, 9, 17, 11, 19\}$, $l(W) = 15$ and $r(W) = 23$. It follows that $S_W = R(W) \cup C(W) \cup \{l(W), r(W)\} = [8, 23]$. So, W is a regular SAMS$(7, 2)$. $\square$

The following example is very similar to the above, we only list the arrays A, B, W and W^* by using the proof of Theorem 3.1.

Example 3.4. There exists a regular SAMS$(11, 2)$.

Proof. We have $m = 5$, $m + 2 = 7$ and $k = 2$.

$$A = \begin{pmatrix} 12 & 13 & 14 & 15 & 17 & 18 & 19 & 20 & 21 & 22 & 11 \\ 1 & 2 & 3 & 4 & 5 & 16 & 6 & 7 & 8 & 9 & 10 \end{pmatrix},$$

$$B = \begin{pmatrix}
2 & 3 & 4 & \mathbf{5} & 6 & \mathbf{7} & 8 & 9 & 10 & 11 & 1 \\
4 & \mathbf{5} & 6 & \mathbf{7} & 8 & 9 & 10 & 11 & 1 & 2 & 3 \\
6 & \mathbf{7} & 8 & 9 & 10 & 11 & 1 & 2 & 3 & 4 & \mathbf{5} \\
8 & 9 & 10 & 11 & 1 & 2 & 3 & 4 & \mathbf{5} & 6 & \mathbf{7} \\
10 & 11 & 1 & 2 & 3 & 4 & \mathbf{5} & 6 & \mathbf{7} & 8 & 9 \\
1 & 2 & 3 & 4 & \mathbf{5} & 6 & \mathbf{7} & 8 & 9 & 10 & 11 \\
3 & 4 & \mathbf{5} & 6 & \mathbf{7} & 8 & 9 & 10 & 11 & 1 & 2 \\
\mathbf{5} & 6 & \mathbf{7} & 8 & 9 & 10 & 11 & 1 & 2 & 3 & 4 \\
\mathbf{7} & 8 & 9 & 10 & 11 & 1 & 2 & 3 & 4 & \mathbf{5} & 6 \\
9 & 10 & 11 & 1 & 2 & 3 & 4 & \mathbf{5} & 6 & \mathbf{7} & 8 \\
11 & 1 & 2 & 3 & 4 & \mathbf{5} & 6 & \mathbf{7} & 8 & 9 & 10
\end{pmatrix},$$

$W =$

			11		1					
	22		10							
	9									21
								20		8
						19		7		
				18		6				
		17		16						
15		5								
4									14	
							13		3	
					12		2			

.

We exchange column 2 with column 4 and column 10 with column 8 of W to obtain W^* as follows.

$W^* =$

	11				1					
	10		22							
			9							21
								20		8
						19		7		
				18		6				
		17		16						
15		5								
4							14			
							3		13	
					12				2	

.

Here, the empty positions of W and W^* indicate 0. It is easy to verify that W^* is a regular SAMS$(11, 2)$. $\square$

4. Regular SAMS$(n, 4)$ for $n \geq 5$ and $n \equiv 1, 5 \pmod 6$

In this section, we shall prove that there exists a regular SAMS$(n, 4)$ for $n \geq 5$ and $n \equiv 1, 5 \pmod 6$ by direct construction.

Theorem 4.1. *There exists a regular SAMS$(n, 4)$ for any $n \geq 5$ and $n \equiv 1, 5$ (mod 6).*

Proof. For $n = 5$, there exists a regular SAMS$(5, 4)$ by Lemma 1.1.

We write $n \geq 7$ and $n \equiv 1, 5 \pmod 6$ as $n = 2m + 1$, where $m \geq 3$. We construct a special array $A = (a_{i,j})$, $i \in I_4$, $j \in I_n$, where

$$a_{1,j} = \begin{cases} 2n + \frac{j+1}{2}, & \text{if } j \text{ is odd,} \\ 2n + m + 1 + \frac{j}{2}, & \text{if } j \text{ is even,} \end{cases}$$
$$a_{2,j} = n + 2 - j,$$
$$a_{3,j} = \begin{cases} 1, & \text{if } j = 1, \\ n + \frac{j+1}{2}, & \text{if } j \geq 3 \text{ is odd,} \\ n + m + 1 + \frac{j}{2}, & \text{if } j \text{ is even,} \end{cases}$$
$$a_{4,j} = 4n + 1 - j.$$

It is easy to calculate that

$$A_1 = \bigcup_{j=1}^{n} \{a_{1,j}\} = [2n+1, 3n], \quad A_2 = \bigcup_{j=1}^{n} \{a_{2,j}\} = [2, n+1],$$

$$A_3 = \bigcup_{j=1}^{n} \{a_{3,j}\} = [n+2, 2n] \cup \{1\}, \quad A_4 = \bigcup_{j=1}^{n} \{a_{4,j}\} = [3n+1, 4n].$$

Then the set of the elements of A is

$$G(A) = A_1 \cup A_2 \cup A_3 \cup A_4 = [1, 4n].$$

By a simple calculation, we have

$$\begin{aligned} C(A) &= \bigcup_{j=1}^{n} \{\sum_{i=1}^{4} a_{i,j}\} \\ &= \{\sum_{i=1}^{4} a_{i,1}\} \cup \left(\bigcup_{e=1}^{m} \{\sum_{i=1}^{4} a_{i,2e+1}, \sum_{i=1}^{4} a_{i,2e}\} \right) \\ &= \{7n+3\} \cup \left(\bigcup_{e=1}^{m} \{8n+3-2e, 9n+4-2e\} \right) \\ &= \{7n+3, 8n+1, 9n+2\} \cup \left(\bigcup_{e=1}^{m-1} \{8n+1-2e, 9n+2-2e\} \right), \end{aligned}$$

$$G_3 = \bigcup_{e=1}^{m-1} \{a_{1,2e-1} + a_{2,2e} + a_{3,2e+1} + a_{4,2e+2}\} = \bigcup_{e=1}^{m-1} \{8n+2-2e\},$$
$$G_4 = \bigcup_{e=1}^{m-1} \{a_{1,2e} + a_{2,2e+1} + a_{3,2e+2} + a_{4,2e+3}\} = \bigcup_{e=1}^{m-1} \{9n+1-2e\},$$
$$G_5 = \{a_{1,n-2} + a_{2,n-1} + a_{3,n} + a_{4,1}\} = \{8n+3\},$$
$$G_6 = \{a_{1,n-1} + a_{2,n} + a_{3,1} + a_{4,2}\} = \{7n+2\},$$
$$G_7 = \{a_{1,n} + a_{2,1} + a_{3,2} + a_{4,3}\} = \{9n+1\}.$$

Denote $F(C) = G_3 \cup G_4 \cup G_5 \cup G_6 \cup G_7$. Then $F(C)$ is the set of forward diagonal-sums. Clearly,

$$C(A) \cup G_3 \cup G_4 \cup G_5 \cup G_6 \cup G_7 = [7n+2, 9n+2] \setminus \{8n+2\}.$$

It follows that $C(A) \cup F(C) = [7n+2, 9n+2] \setminus \{8n+2\}$.

Let $B = (b_{i,j}) = (\langle 2i+j-1 \rangle_n)$ be the Latin square of order n over I_n which comes from the proof of Theorem 3.1. Define

$$i = g_r(i', j') = \langle i' - j' - 1 \rangle_n, \quad j = g_c(i', j') = \langle 2j' - 3 \rangle_n, \ i' \in I_4, \ j' \in I_n.$$

We put the element $a_{i',j'}$ of A into the cell (i, j) of B, the other cells of B are all filled by 0. Then the array is denoted by $D = (d_{i,j})$. The elements in the same column of A are also in the same column of D since $j = g_c(i', j') = \langle 2j' - 3 \rangle_n$, and the elements in the same forward diagonal of A are also in the same row of D since

$$\begin{aligned} g_r(1, j') &= g_r(2, \langle j'+1 \rangle_n) \\ &= g_r(3, \langle j'+2 \rangle_n) \\ &= g_r(4, \langle j'+3 \rangle_n) = \langle -j' \rangle_n, \ j' \in I_n. \end{aligned}$$

Then

$$\begin{aligned} G(D) &= [1, 4n] \text{ and} \\ R(D) \cup C(D) = C(A) \cup F(C) &= [7n+2, 9n+2] \setminus \{8n+2\}. \end{aligned}$$

We also have

$$\begin{aligned} b_{i,j} &= \langle 2i + j - 1 \rangle_n \\ &= \langle 2(g_r(i', j') + g_c(i', j') - 1 \rangle_n \\ &= \langle 2(i' - j' - 1) + (2j' - 3) - 1 \rangle_n \\ &= \langle 2i' - 6 \rangle_n, \ i' \in I_4. \end{aligned}$$

It follows that the element $a_{i',j'}$ of A is filled into the cell (i, j) of B with $b_{i,j} = n-4, n-2, n, 2$, respectively. So there are exactly 4 non-zero elements in each row, each column and each main diagonal of D. Clearly,

$$\begin{aligned} g_r(4,1) = 2, \ g_c(4,1) = n-1; \quad g_r(3,2) = n, \ g_c(3,2) = 1; \\ g_r(2,3) = n-2, \ g_c(2,3) = 3; \quad g_r(1,4) = n-4, \ g_c(1,4) = 5. \end{aligned}$$

We have

$$d_{2,n-1} = a_{4,1}, \ d_{n,1} = a_{3,2}, \ d_{n-2,3} = a_{2,3}, \ d_{n-4,5} = a_{1,4}.$$

Then it is easy to calculate that the right diagonal-sum of D is

$$\begin{aligned} r(D) &= d_{2,n-1} + d_{n,1} + d_{n-2,3} + d_{n-4,5} \\ &= a_{4,1} + a_{3,2} + a_{2,3} + a_{1,4} \\ &= 9n + 3. \end{aligned}$$

Case 1. $n \equiv 1 \pmod 6$ and $n \geq 7$.

We write $n = 6k + 1$, where $k \geq 1$. There are exactly 4 non-zero elements $a_{1,1}$, $a_{2,2+4k}$, $a_{3,2+2k}$, $a_{4,2}$ in the left diagonal of D because

$$g_r(1,1) = g_c(1,1) = n - 1, \quad g_r(2, 2 + 4k) = g_c(2, 2 + 4k) = 2k,$$
$$g_r(3, 2 + 2k) = g_c(3, 2 + 2k) = 4k + 1, \quad g_r(4, 2) = g_c(4, 2) = 1.$$

By a simple calculation,

$$\begin{aligned} l(D) &= d_{n-1,n-1} + d_{2k,2k} + d_{4k+1,4k+1} + d_{1,1} \\ &= a_{1,1} + a_{2,2+4k} + a_{3,2+2k} + a_{4,2} \\ &= 8n + 2. \end{aligned}$$

So D is a regular SAMS$(n, 4)$.

Case 2. $n \equiv 5 \pmod 6$ and $n \geq 5$.

We write $n = 6k + 5$, where $k \geq 1$. There are exactly 4 non-zero elements $a_{1,1}, a_{2,2k+3}, a_{3,4k+5}, a_{4,2}$ in the left diagonal of D because

$$g_r(1,1) = g_c(1,1) = n - 1, \quad g_r(2, 2k + 3) = g_c(2, 2k + 3) = 4k + 3;$$
$$g_r(3, 4k + 5) = g_c(3, 4k + 5) = 2k + 2, \quad g_r(4, 2) = g_c(4, 2) = 1.$$

By a simple computation,

$$\begin{aligned} l(D) &= d_{n-1,n-1} + d_{4k+3,4k+3} + d_{2k+2,2k+2} + d_{1,1} \\ &= a_{1,1} + a_{2,2k+3} + a_{3,4k+5} + a_{4,2} \\ &= 8n + 2. \end{aligned}$$

So D is a regular SAMS$(n, 4)$. □

5. Regular SAMS(n, d) for $n \equiv 1, 5 \pmod 6$ and $d \in [6, n - 3]$

In this section, we shall prove that there exists a regular SAMS(n, d) for any $n \equiv 1, 5 \pmod 6$ and $d \in [6, n - 3]$ by using the arrays B, W and W^* in the proof of Theorem 3.1 and the existence of an SFD(d, n) from Theorem 2.7. To do this, we also introduce a new concept and some very simple and useful results in the following.

Definition 5.1. Two $m \times n$ arrays $M = (m_{i,j})$ and $N = (n_{i,j})$ are *compatible* if $\Omega(M) \cap \Omega(N) = \emptyset$, where $\Omega(M) = \{(i, j) \,|\, m_{i,j} \neq 0,\ i \in I_m, j \in I_n\}$ and $\Omega(N) = \{(i, j) \,|\, n_{i,j} \neq 0,\ i \in I_m, j \in I_n\}$.

Lemma 5.2. *If there exist a regular SMS(n, d_1) and an SAMS(n, d_2), and they are compatible, then there exists an SAMS$(n, d_1 + d_2)$.*

Proof. Let $M = (m_{i,j})$ be a regular SMS(n, d_1) over $\{0, 1, 2, \ldots, nd_1\}$, and $N = (n_{i,j})$ be an SAMS(n, d_2) over $\{0, 1, 2, \ldots, nd_2\}$. Let $M' = (m'_{i,j})$, where

$$m'_{i,j} = \begin{cases} m_{i,j} + nd_2, & \text{if} \quad m_{i,j} \neq 0, \\ 0, & \text{if} \quad m_{i,j} = 0. \end{cases}$$

It is readily checked that $M' + N$ is an SAMS$(n, d_1 + d_2)$ over $\{0, 1, 2, \ldots, n(d_1 + d_2)\}$. □

Lemma 5.3. *If there exist an SMS(n, d_1) and a regular SAMS(n, d_2), and they are compatible, then there exists an SAMS$(n, d_1 + d_2)$.*

Proof. Let $M = (m_{i,j})$ be an SMS(n, d_1) over $\{0, 1, 2, \ldots, nd_1\}$, and $N = (n_{i,j})$ be a regular SAMS(n, d_2) over $\{0, 1, 2, \ldots, nd_2\}$. Let $N' = (n'_{i,j})$, where

$$n'_{i,j} = \begin{cases} n_{i,j} + nd_1, & \text{if } \ n_{i,j} \neq 0, \\ 0, & \text{if } \ n_{i,j} = 0. \end{cases}$$

It is readily checked that $M + N'$ is an SAMS$(n, d_1 + d_2)$ over $\{0, 1, 2, \ldots, n(d_1 + d_2)\}$. $\square$

Theorem 5.4. *If there exist a regular SMS(n, d_1) and a regular SAMS(n, d_2), and they are compatible, then there exists a regular SAMS$(n, d_1 + d_2)$.*

Theorem 5.5. *If $n \geq 7$ and $n \equiv 1, 5 \pmod 6$, then there exists a regular SAMS(n, d) for $d \equiv 0 \pmod 2$ and $d \in [6, n-3]$.*

Proof. For $n = 7$, there exists a regular SAMS$(7, 6)$ by Lemma 1.1.

For $n \geq 11$ and $n \equiv 1, 5 \pmod 6$, let $n = 2m + 1$, namely, $m = \frac{n-1}{2}$. We write $d \equiv 0 \pmod 2$ and $d \in [6, n-3]$ as $d = 2e + 2$, where $2e \in [4, n-5]$. By Theorem 3.1 there exists a regular SAMS$(n, 2)$, W or W^*. By Theorem 5.4, to show the conclusion, we need only construct a regular SMS$(n, 2e)$ which is compatible with the regular SAMS$(n, 2)$.

By Theorem 2.7 there exists an SFD$(2e, n)$ over $[2n+1, 2n+2en]$, denoted by $C = (c_{i,j})$, where $i \in I_{2e}$, $j \in I_n$. Let Latin square B and f be from the proof of Theorem 3.1. We put $c_{i,j}$, $i \in I_{2e}$, $j \in I_n$, into the cell $(f(\langle 2i' - 2e - 1 + m\rangle_n, \langle 2j + m\rangle_n), \langle 2j + m\rangle_n)$ of B, where

$$i' = \begin{cases} i, & \text{if } \ i \in [1, e], \\ i + 1, & \text{if } \ i \in [e+1, 2e]. \end{cases}$$

the other cells of B are all filled by 0, we obtained an array D. We shall prove that D is the desired regular SMS$(n, 2e)$.

Note that we put the elements in the j-th column of C into the $\langle 2j + m\rangle_n$-th column of D, where $j \in I_n$ and $\{\langle 2j + m\rangle_n \mid j \in I_n\} = I_n$, there are $2e$ non-zero elements in each column of D and the columns of D will have a constant sum $\frac{2e(4n+1+2en)}{2}$ since the columns of C have a constant sum $\frac{2e(4n+1+2en)}{2}$.

For each $j_1, j_2 \in I_n$, the elements in the set $\mathcal{A}_1 = \{c_{i,\langle i+j_1\rangle_n} \mid i \in I_e\}$ are filled into the same row $i_1(j_1)$ of D and the elements in the set $\mathcal{A}_2 = \{c_{i,\langle i+j_2\rangle_n} \mid i \in I_{2e} \setminus I_e\}$ are also filled into the same row $i_2(j_2)$ of D, and it is clear that $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$ and $|\mathcal{A}_1 \cup \mathcal{A}_2| = 2e$. In fact, the element $c_{i,\langle i+j_1\rangle_n}$ of $\mathcal{A}_1$ is filled into the $f(\langle (m+1) - 2(e+1) + 2i\rangle_n, \langle 2\langle i + j_1\rangle_n + m\rangle_n)$-th row of D, the element $c_{i,\langle i+j_2\rangle_n}$ of $\mathcal{A}_2$ is filled into the $f(\langle (m+1) - 2(e+1) + 2(i+1)\rangle_n, \langle 2\langle i+j_2\rangle_n + m\rangle_n)$-th row of D. Let $f(\langle (m+1) - 2(e+1) + 2i\rangle_n, \langle 2\langle i + j_1\rangle_n + m\rangle_n) = \alpha_1$ and $f(\langle (m+1) - 2(e+1) + 2(r+1)\rangle_n, \langle 2\langle i + j_2\rangle_n + m\rangle_n) = \alpha_2$, by the definition of f from the proof of Theorem 3.1, we have

$$b_{\alpha_1, \langle 2\langle i+j_1\rangle_n + m\rangle_n} = \langle (m+1) - 2(e+1) + 2i\rangle_n$$

$$\begin{aligned}
&= \langle 2\alpha_1 + 2(i+j_1) + m - 1\rangle_n,\\
b_{\alpha_2,\langle 2\langle i+j_2\rangle_n+m\rangle_n} &= \langle (m+1) - 2(e+1) + 2(i+1)\rangle_n\\
&= \langle 2\alpha_2 + 2(i+j_2) + m - 1\rangle_n.
\end{aligned}$$

It follows that

$$\alpha_1 = \langle -j_1 - e\rangle_n \text{ and } \{\langle -j_1 - e\rangle_n \,|\, j_1 \in I_n\} = I_n \text{ when } i \in I_e,$$

and

$$\alpha_2 = \langle -j_2 - e + 1\rangle_n \text{ and } \{\langle -j_2 - e + 1\rangle_n \,|\, j_2 \in I_n\} = I_n \text{ when } i \in I_n \setminus I_e,$$

which are both independent of the parameter i. Note that $j_2 = j_1 + 1$ when $\alpha_1 = \alpha_2$, that is, the elements in the same forward diagonal of C aren't filled into the same row of D. But this implies that the elements in the set $\mathcal{A}_1$ and $\mathcal{A}_2$ lie in the same row $i_1(j_1)$ and $i_2(j_2)$ of D, respectively, where $i_1(j_1)$ and $i_2(j_2)$ are both bijections over I_n since $\{\langle 2\langle i + j_\sigma\rangle_n + m\rangle_n \,|\, j_\sigma \in I_n\} = I_n$, $\sigma = 1, 2$, and B is a Latin square. Then the rows of D will also have a constant sum $\frac{2e(4n+1+2en)}{2}$ by the property

$$\sum_{i=1}^{e} c_{i,\langle i+j_1\rangle_n} + \sum_{i=e+1}^{2e} c_{i,\langle i+j_2\rangle_n} = \frac{2e(4n+1+2en)}{2}$$

mentioned in Remark 2.8.

Let

$$\begin{aligned}
\Delta &= \{2i' - 2e - 1 + m \,|\, i \in I_{2e}\}\\
&= \{2i - 2e - 1 + m \,|\, i \in I_e\} \cup \{2i - 2e + 1 + m \,|\, i \in [e+1, 2e]\}.
\end{aligned}$$

Clearly $|\Delta| = 2e$. It is easy to check that there are exactly $2e$ non-zero elements in each main diagonal of D since B is a diagonal Latin square and for $i_1, j_1 \in I_n$,

$$\begin{cases} d_{i_1,j_1} = 0, & \text{if} \quad b_{i_1,j_1} \notin \Delta,\\ d_{i_1,j_1} \neq 0, & \text{if} \quad b_{i_1,j_1} \in \Delta. \end{cases}$$

Now we compute the main diagonal-sum of D. For $i \in I_e$, $j \in I_n$, the elements $c_{i,j}$ and $c_{2e+1-i,n+1-j}$ are filled into the cells $(f(\langle 2i-2e-1+m\rangle_n, \langle 2j+m\rangle_n), \langle 2j+m\rangle_n)$ and $(f(\langle 2(2e+1-i+1)-2e-1+m\rangle_n, \langle 2(n+1-j)+m\rangle_n), \langle 2(n+1-j)+m\rangle_n)$ of B, respectively. Let $f(\langle 2i-2e-1+m\rangle_n, \langle 2j+m\rangle_n) = \theta$. We have

$$b_{\theta,\langle 2j+m\rangle_n} = \langle 2\theta + (\langle 2j+m\rangle_n) + 1\rangle_n = \langle 2i - 2e - 1 + m\rangle_n.$$

By the property of the Latin square B mentioned in the proof of Theorem 3.1, we obtain

$$\begin{aligned}
b_{n+1-\theta,n+1-\langle 2j+m\rangle_n} &= (n+1) - b_{\theta,\langle 2j+m\rangle_n}\\
&= (n+1) - \langle (2i - 2e - 1 + m)\rangle_n.
\end{aligned}$$

It is easy to compute that

$$\langle 2(2e+1-i+1) - 2e - 1 + m\rangle_n = \langle (n+1) - (2i - 2e - 1 + m)\rangle_n,$$

$$\langle 2(n+1-j)+m\rangle_n = (n+1) - \langle 2j+m\rangle_n.$$

Therefore,

$$\begin{aligned} &f(\langle 2(2e+1-i+1)-2e-1+m\rangle_n, \langle 2(n+1-j)+m\rangle_n) \\ = &f(\langle (n+1)-(2i-2e-1+m)\rangle_n, (n+1)-\langle (2j+m)\rangle_n) \\ = &(n+1)-\theta. \end{aligned}$$

It follows that for $i \in I_e$, $j \in I_n$, $c_{i,j}$ and $c_{2e+1-i,n+1-j}$ are filled into the cells $(\theta, \langle 2j+m\rangle_n)$ and $(n+1-\theta, n+1-\langle 2j+m\rangle_n)$ of B, respectively. It is easy to see that

$$\begin{aligned} d_{\theta,\langle 2j+m\rangle_n} + d_{n+1-\theta,n+1-\langle 2j+m\rangle_n} &= c_{i,j} + c_{2e+1-i,n+1-j} \\ &= 4n+1+2en. \end{aligned}$$

Then the sum of elements in each diagonal of D is also a constant sum

$$\frac{2e(4n+1+2en)}{2}$$

since there are exactly $2e$ non-zero elements in each diagonals. It follows that D is a regular SMS$(n, 2e)$.

Next, we shall show that D is compatible with the regular SAMS$(n, 2)$ constructed from Theorem 3.1. Denote

$$\begin{aligned} \Omega(D) &= \{(i,j) \,|\, d_{i,j} \neq 0,\ i,j \in I_n\} \\ &= \{(x,y) \,|\, b_{x,y} \in \bigcup_{i=1}^{2e} \{\langle 2i'-2e-1+m\rangle_n\},\ x,y \in I_n\}. \end{aligned}$$

Clearly $e \leq m-2$ since $d = 2e+2 \leq n-3 = (2m+1)-3$. So it is easy to verify that

$$\{m-2, m, m+2, m+4\} \cap \{\langle 2i'-2e-1+m\rangle_n \,|\, i \in I_{2e}\} = \emptyset.$$

It follows that

$$\{(x,y) \,|\, b_{x,y} \in \{m-2, m, m+2, m+4\}\} \cap \Omega(D) = \emptyset.$$

Let W and W^* be from the proof of Theorem 3.1, that is, W is an SAMS$(n, 2)$ for $n \equiv 1 \pmod 6$ and W^* is an SAMS$(n, 2)$ for $n \equiv 5 \pmod 6$. By Remark 3.2,

$$\Omega(W) = \{(i,j) \,|\, b_{i,j} \in \{m, m+2\},\ i,j \in I_n\}$$

and

$$\Omega(W^*) \subset \{(i,j) \,|\, b_{i,j} \in \{m-2, m, m+2, m+4\},\ i,j \in I_n\}.$$

We have $\Omega(W) \cap \Omega(D) = \emptyset$ and $\Omega(W^*) \cap \Omega(D) = \emptyset$. It follows that W and D are compatible, and W^* and D are compatible. So $D+W$ and $D+W^*$ are the desired regular SAMS(n, d)s by Theorem 5.4. □

To illustrate the proof of Theorem 5.5, we give an example in the following.

Example 5.6. There exists a regular SAMS$(11, 8)$.

Proof. Let $n = 2m + 1 = 11$ and $d = 2e + 2 = 8$. Clearly $m = 5$ and $2e = 6$. We construct an SFD$(6, 11)$ over $[1, 66]$ as follows.

$$
\begin{aligned}
C' &= \begin{pmatrix}
11 & 5 & 10 & 4 & 9 & 3 & 8 & 2 & 7 & 1 & 6 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
6 & 11 & 5 & 10 & 4 & 9 & 3 & 8 & 2 & 7 & 1 \\
11 & 5 & 10 & 4 & 9 & 3 & 8 & 2 & 7 & 1 & 6 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
6 & 11 & 5 & 10 & 4 & 9 & 3 & 8 & 2 & 7 & 1
\end{pmatrix} + 11 \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5
\end{pmatrix} \\
&= \begin{pmatrix}
11 & 5 & 10 & 4 & 9 & 3 & 8 & 2 & 7 & 1 & 6 \\
12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 \\
28 & 33 & 27 & 32 & 26 & 31 & 25 & 30 & 24 & 29 & 23 \\
44 & 38 & 43 & 37 & 42 & 36 & 41 & 35 & 40 & 34 & 39 \\
45 & 46 & 47 & 48 & 49 & 50 & 51 & 52 & 53 & 54 & 55 \\
61 & 66 & 60 & 65 & 59 & 64 & 58 & 63 & 57 & 62 & 56
\end{pmatrix}.
\end{aligned}
$$

Then an SFD$(6, 11)$ over $[23, 88]$, C, is obtained in the following.

$$
\begin{aligned}
C = C' &+ \begin{pmatrix}
22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 \\
22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 \\
22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 \\
22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 \\
22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 \\
22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22
\end{pmatrix} \\
&= \begin{pmatrix}
33 & 27 & 32 & 26 & 31 & 25 & 30 & 24 & 29 & 23 & 28 \\
34 & 35 & 36 & 37 & 38 & 39 & 40 & 41 & 42 & 43 & 44 \\
50 & 55 & 49 & 54 & 48 & 53 & 47 & 52 & 46 & 51 & 45 \\
66 & 60 & 65 & 59 & 64 & 58 & 63 & 57 & 62 & 56 & 61 \\
67 & 68 & 69 & 70 & 71 & 72 & 73 & 74 & 75 & 76 & 77 \\
83 & 88 & 82 & 87 & 81 & 86 & 80 & 85 & 79 & 84 & 78
\end{pmatrix}.
\end{aligned}
$$

The arrays B and D are listed below as in the proof of Theorem 5.5 and W^* comes from Example 3.4.

$$
B = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
\mathbf{2} & 3 & \mathbf{4} & 5 & 6 & 7 & \mathbf{8} & 9 & \mathbf{10} & \mathbf{11} & \mathbf{1} \\ \hline
\mathbf{4} & 5 & 6 & 7 & \mathbf{8} & 9 & \mathbf{10} & \mathbf{11} & \mathbf{1} & 2 & 3 \\ \hline
6 & 7 & \mathbf{8} & 9 & \mathbf{10} & \mathbf{11} & \mathbf{1} & \mathbf{2} & 3 & \mathbf{4} & 5 \\ \hline
\mathbf{8} & 9 & \mathbf{10} & \mathbf{11} & \mathbf{1} & \mathbf{2} & 3 & \mathbf{4} & 5 & 6 & 7 \\ \hline
\mathbf{10} & \mathbf{11} & \mathbf{1} & \mathbf{2} & 3 & \mathbf{4} & 5 & 6 & 7 & \mathbf{8} & 9 \\ \hline
\mathbf{1} & \mathbf{2} & 3 & \mathbf{4} & 5 & 6 & 7 & \mathbf{8} & 9 & \mathbf{10} & \mathbf{11} \\ \hline
3 & \mathbf{4} & 5 & 6 & 7 & \mathbf{8} & 9 & \mathbf{10} & \mathbf{11} & \mathbf{1} & \mathbf{2} \\ \hline
5 & 6 & 7 & \mathbf{8} & 9 & \mathbf{10} & \mathbf{11} & \mathbf{1} & \mathbf{2} & 3 & \mathbf{4} \\ \hline
7 & \mathbf{8} & 9 & \mathbf{10} & \mathbf{11} & \mathbf{1} & \mathbf{2} & 3 & \mathbf{4} & 5 & 6 \\ \hline
9 & \mathbf{10} & \mathbf{11} & \mathbf{1} & \mathbf{2} & 3 & \mathbf{4} & 5 & 6 & 7 & \mathbf{8} \\ \hline
\mathbf{11} & \mathbf{1} & \mathbf{2} & 3 & \mathbf{4} & 5 & 6 & 7 & \mathbf{8} & 9 & \mathbf{10} \\ \hline
\end{array},
$$

$$
D = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
42 & & 51 & & & & 66 & & 68 & 24 & 82 \\ \hline
46 & & & & 61 & & 67 & 30 & 88 & 41 & \\ \hline
 & & 56 & & 77 & 25 & 83 & 40 & & 52 & \\ \hline
62 & & 76 & 31 & 78 & 39 & & 47 & & & \\ \hline
75 & 26 & 84 & 38 & & 53 & & & & 57 & \\ \hline
79 & 37 & & 48 & & & & 63 & & 74 & 32 \\ \hline
 & 54 & & & & 58 & & 73 & 27 & 85 & 36 \\ \hline
 & & & 64 & & 72 & 33 & 80 & 35 & & 49 \\ \hline
 & 59 & & 71 & 28 & 86 & 34 & & 55 & & \\ \hline
 & 70 & 23 & 81 & 44 & & 50 & & & & 65 \\ \hline
29 & 87 & 43 & & 45 & & & & 60 & & 69 \\ \hline
\end{array},
$$

$$W^* = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline & 11 & & & & 1 & & & & & \\ \hline & 10 & & 22 & & & & & & & \\ \hline & & & 9 & & & & & & & 21 \\ \hline & & & & & & & & 20 & & 8 \\ \hline & & & & & & 19 & & 7 & & \\ \hline & & & & 18 & & 6 & & & & \\ \hline & & 17 & & 16 & & & & & & \\ \hline 15 & & 5 & & & & & & & & \\ \hline 4 & & & & & & & 14 & & & \\ \hline & & & & & & & 3 & & 13 & \\ \hline & & & & & 12 & & & & 2 & \\ \hline \end{array},$$

$$D + W^* = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 42 & 11 & 51 & & & 1 & 66 & & 68 & 24 & 82 \\ \hline 46 & 10 & & 22 & 61 & & 67 & 30 & 88 & 41 & \\ \hline & & 56 & 9 & 77 & 25 & 83 & 40 & & 52 & 21 \\ \hline 62 & & 76 & 31 & 78 & 39 & & 47 & 20 & & 8 \\ \hline 75 & 26 & 84 & 38 & & 53 & 19 & & 7 & 57 & \\ \hline 79 & 37 & & 48 & 18 & & 6 & 63 & & 74 & 32 \\ \hline & 54 & 17 & & 16 & 58 & & 73 & 27 & 85 & 36 \\ \hline 15 & & 5 & 64 & & 72 & 33 & 80 & 35 & & 49 \\ \hline 4 & 59 & & 71 & 28 & 86 & 34 & 14 & 55 & & \\ \hline & 70 & 23 & 81 & 44 & & 50 & 3 & & 13 & 65 \\ \hline 29 & 87 & 43 & & 45 & 12 & & & 60 & 2 & 69 \\ \hline \end{array}.$$

Here, all of above empty positions indicate 0. It is easy to check that $D + W^*$ is a regular SAMS$(11, 8)$. $\square$

Theorem 5.7. *If $n \equiv 1, 5 \pmod 6$ and $n \geq 11$, then there exists a regular SAMS(n, d) for $d \equiv 1 \pmod 2$ and $d \in [6, n-3]$.*

Proof. We write $d \equiv 1 \pmod 2$ and $d \in [6, n-3]$ as $d = (2e+1) + 2$, where $2e+1 \in [4, n-5]$, and $m = \frac{n-1}{2}$. By Theorem 3.1 there exists a regular SAMS$(n, 2)$, W or W^*. By Theorem 5.4, to show the conclusion, we need only construct a regular SMS$(n, 2e+1)$, which is compatible with the regular SAMS$(n, 2)$.

By Theorem 2.7 there exists an SFD$(2e+1, n)$ over $[2n+1, 2n+(2e+1)n]$, denoted by $C = (c_{i,j})$, where $i \in I_{2e+1}$, $j \in I_n$. Let Latin square B and f be from the proof of Theorem 3.1. We put $c_{i,j}$, $i \in I_{2e+1}$, $j \in I_n$, into the cell $(f(\langle 2i-2e-1+m\rangle_n, \langle 2j+m\rangle_n), \langle 2j+m\rangle_n)$ of B, the other cells of B are all filled by 0, we obtained an array D. We shall prove that D is the desired regular SMS$(n, 2e+1)$.

Note that we put the elements in the j-th column of C into the $\langle 2j+m\rangle_n$-th column of D, where $j \in I_n$ and $\{\langle 2j+m\rangle_n \mid j \in I_n\} = I_n$. Therefore there are $2e+1$ non-zero elements in each column of D and the columns of D will have a constant sum $\frac{(2e+1)[4n+1+(2e+1)n]}{2}$ since the columns of C have a constant sum $\frac{(2e+1)[4n+1+(2e+1)n]}{2}$.

For each $j \in I_n$, the elements in the set $\mathcal{A} = \{c_{i,\langle i+j\rangle_n} \mid i \in I_{2e+1}\}$ are filled into the same row of D. It is clear that $|\mathcal{A}| = 2e+1$ and the elements in the set

$\mathcal{A}$ are exactly in the same froward diagonal of C. In fact, the element $c_{i,\langle i+j\rangle_n}$ is filled into the $f(\langle(m+1)-2(e+1)+2i\rangle_n, \langle 2\langle i+j\rangle_n+m\rangle_n)$-th row of D. Let $f(\langle(m+1)-2(e+1)+2i\rangle_n, \langle 2\langle i+j\rangle_n+m\rangle_n)=\alpha$, we have

$$\begin{aligned} b_{\alpha,\langle 2\langle i+j\rangle_n+m\rangle_n} &= \langle(m+1)-2(e+1)+2i\rangle_n \\ &= \langle 2\alpha+2(i+j)+m-1\rangle_n. \end{aligned}$$

It follows that $\alpha=\langle -j-e\rangle_n$ and $\{\langle -j-e\rangle_n \mid j\in I_n\}=I_n$, which is independent of the parameter i. This implies that the elements in the set $\mathcal{A}=\{c_{i,\langle i+j\rangle_n} \mid i\in I_{2e+1}\}$ lie in the same row α of D. So all forward diagonals of C become the rows of D. Then the rows of D will also have a constant sum $\frac{(2e+1)[4n+1+(2e+1)n]}{2}$ since all forward diagonals of C have the same sum $\frac{(2e+1)[4n+1+(2e+1)n]}{2}$.

Let $\Delta=\{2i-2e-1+m \mid i\in I_{2e+1}\}$. Clearly $|\Delta|=2e+1$. It is easy to check that there are exactly $2e+1$ non-zero elements in each main diagonal of D since B is a diagonal Latin square and for $i_1,j_1\in I_n$,

$$\begin{cases} d_{i_1,j_1}=0, & \text{if} \quad b_{i_1,j_1}\notin\Delta, \\ d_{i_1,j_1}\neq 0, & \text{if} \quad b_{i_1,j_1}\in\Delta. \end{cases}$$

Now we compute the main diagonal-sum of D. We have $d_{m+1,m+1}=c_{e+1,m+1}$ $=\frac{1+(2e+1)n}{2}+2n$ because the element $c_{e+1,m+1}$ is filled into the cell

$$\begin{aligned} &(f(\langle 2(e+1)-2e-1+m\rangle_n, \langle 2(m+1)+m\rangle_n), \langle 2(m+1)+m\rangle_n) \\ &=(f(m+1,m+1),m+1) \\ &=(m+1,m+1) \end{aligned}$$

of D. It follows that the sum of elements in each diagonal of D is also a constant sum $\frac{(2e+1)[4n+1+(2e+1)n]}{2}$.

Denote

$$\begin{aligned} \Omega(D) &= \{(i,j) \mid d_{i,j}\neq 0,\ i,j\in I_n\} \\ &= \{(f(\langle 2i-2e-1+m\rangle_n, \langle 2j+m\rangle_n), \langle 2j+m\rangle_n) \mid i\in I_{2e+1},\ j\in I_n\} \\ &= \{(x,y \mid b_{x,y}\in \bigcup_{i=1}^{2e+1}\{\langle 2i-2e-1+m\rangle_n\},\ x,y\in I_n\}. \end{aligned}$$

Clearly $\lfloor\frac{2e+1}{2}\rfloor=e\le m-2$ since $d=(2e+1)+2\le n-3=(2m+1)-3=2m-2$. So it is easy to verify that

$$\{m-2,m,m+2,m+4\}\cap\{\langle 2i-2e-1+m\rangle_n \mid i\in I_{2e+1}\}=\emptyset.$$

It follows that

$$\{(x,y) \mid b_{x,y}\in\{m-2,m,m+2,m+4\}\}\cap\Omega(D)=\emptyset.$$

Let W and W^* from the proof of Theorem 3.1. By Remark 3.2, we have $\Omega(W)=\{(i,j) \mid b_{i,j}\in\{m,m+2\},\ i,j\in I_n\}$ and $\Omega(W^*)\subset\{(i,j) \mid b_{i,j}\in\{m-2,m,m+2,m+4\},\ i,j\in I_n\}$. Then $\Omega(W)\cap\Omega(D)=\emptyset$ and $\Omega(W^*)\cap\Omega(D)=\emptyset$. It follows that W and D are compatible, and W^* and D are compatible. So $D+W$ and $D+W^*$ are the regular SAMS(n,d)s by Theorem 5.4. $\square$

To illustrate the proof of Theorem 5.7, we give an example in the following.

Example 5.8. There exists a regular SAMS$(13,9)$.

Proof. Let $n = 2m+1 = 13$ and $d = (2e+1)+2 = 9$. Clearly $m = 6$ and $2e+1 = 7$. We construct an SFD$(7,13)$ over $[1,91]$ as follows.

$$C' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & 10 & 11 & 12 & 13 & 7 \\ 7 & 13 & 12 & 11 & 10 & 9 & 8 & 6 & 5 & 4 & 3 & 2 & 1 \\ 13 & 6 & 12 & 5 & 11 & 4 & 10 & 3 & 9 & 2 & 8 & 1 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 7 & 13 & 6 & 12 & 5 & 11 & 4 & 10 & 3 & 9 & 2 & 8 & 1 \\ 13 & 12 & 11 & 10 & 9 & 8 & 6 & 5 & 4 & 3 & 2 & 1 & 7 \\ 7 & 1 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & 10 & 11 & 12 & 13 \end{pmatrix} + 13 \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 2&2&2&2&2&2&2&2&2&2&2&2&2 \\ 3&3&3&3&3&3&3&3&3&3&3&3&3 \\ 4&4&4&4&4&4&4&4&4&4&4&4&4 \\ 5&5&5&5&5&5&5&5&5&5&5&5&5 \\ 6&6&6&6&6&6&6&6&6&6&6&6&6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & 10 & 11 & 12 & 13 & 7 \\ 20 & 26 & 25 & 24 & 23 & 22 & 21 & 19 & 18 & 17 & 16 & 15 & 14 \\ 39 & 32 & 38 & 31 & 37 & 30 & 36 & 29 & 35 & 28 & 34 & 27 & 33 \\ 40 & 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 & 49 & 50 & 51 & 52 \\ 59 & 65 & 58 & 64 & 57 & 63 & 56 & 62 & 55 & 61 & 54 & 60 & 53 \\ 78 & 77 & 76 & 75 & 74 & 73 & 71 & 70 & 69 & 68 & 67 & 66 & 72 \\ 85 & 79 & 80 & 81 & 82 & 83 & 84 & 86 & 87 & 88 & 89 & 90 & 91 \end{pmatrix}.$$

Then an SFD$(7,13)$ over $[27,117]$, C, is obtained in the following.

$$C = \begin{pmatrix} 27 & 28 & 29 & 30 & 31 & 32 & 34 & 35 & 36 & 37 & 38 & 39 & 33 \\ 46 & 52 & 51 & 50 & 49 & 48 & 47 & 45 & 44 & 43 & 42 & 41 & 40 \\ 65 & 58 & 64 & 57 & 63 & 56 & 62 & 55 & 61 & 54 & 60 & 53 & 59 \\ 66 & 67 & 68 & 69 & 70 & 71 & 72 & 73 & 74 & 75 & 76 & 77 & 78 \\ 85 & 91 & 84 & 90 & 83 & 89 & 82 & 88 & 81 & 87 & 80 & 86 & 79 \\ 104 & 103 & 102 & 101 & 100 & 99 & 97 & 96 & 95 & 94 & 93 & 92 & 98 \\ 111 & 105 & 106 & 107 & 108 & 109 & 110 & 112 & 113 & 114 & 115 & 116 & 117 \end{pmatrix}.$$

The arrays D and W are listed below as in the proof of Theorem 5.7 and Theorem 3.1, respectively.

$D =$

	42		53		78		85		103		106	37
	60		77		79		104		105	36		43
	76		86		98		111	35		44		54
	80		92		117	34		45		61		75
	93		116	32		47		55		74		87
	115	31		48		62		73		81		94
30		49		56		72		88		95		114
50		63		71		82		96		113	29	
57		70		89		97		112	28		51	
69		83		99		110	27		52		64	
90		100		109	33		46		58		68	
101		108	39		40		65		67		84	
107	38		41		59		66		91		102	

,

$W =$

				13		1						
		26		12								
25		11										
10											24	
									23		9	
							22		8			
					21		7					
			20		19							
	18		6									
	5											17
										16		4
								15		3		
						14		2				

,

$$D+W=\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline
 & 42 & & 53 & 13 & 78 & 1 & 85 & & 103 & & 106 & 37\\\hline
 & 60 & 26 & 77 & 12 & 79 & & 104 & & 105 & 36 & & 43\\\hline
25 & 76 & 11 & 86 & & 98 & & 111 & 35 & & 44 & & 54\\\hline
10 & 80 & & 92 & & 117 & 34 & & 45 & & 61 & 24 & 75\\\hline
 & 93 & & 116 & 32 & & 47 & & 55 & 23 & 74 & 9 & 87\\\hline
 & 115 & 31 & & 48 & & 62 & 22 & 73 & 8 & 81 & & 94\\\hline
30 & & 49 & & 56 & 21 & 72 & 7 & 88 & & 95 & & 114\\\hline
50 & & 63 & 20 & 71 & 19 & 82 & & 96 & & 113 & 29 & \\\hline
57 & 18 & 70 & 6 & 89 & & 97 & & 112 & 28 & & 51 & \\\hline
69 & 5 & 83 & & 99 & & 110 & 27 & & 52 & & 64 & 17\\\hline
90 & & 100 & & 109 & 33 & & 46 & & 58 & 16 & 68 & 4\\\hline
101 & & 108 & 39 & & 40 & & 65 & 15 & 67 & 3 & 84 & \\\hline
107 & 38 & & 41 & & 59 & 14 & 66 & 2 & 91 & & 102 & \\\hline
\end{array}\,.$$

It is straightforward to verify that $D+W$ is a regular SAMS$(13,9)$. $\square$

6. Concluding remarks

We first give the proof of Theorem 1.4, restated in Theorem 6.1 for convenience.

Theorem 6.1. *Suppose that n is a positive integer satisfying $n \geq 5$ and $n \equiv 1$, $5 \pmod 6$. Then there exists a regular SAMS(n,d) for any d with $2 \leq d \leq n-1$.*

Proof. For $d = 2, 4$, there exists a regular SAMS(n,d) by Theorem 3.1 and Theorem 4.1, respectively.

For $d \in \{3,5\}$, there exists a regular SAMS(n,d) by Lemma 1.3.

For $d \in [6, n-3]$, there exists a regular SAMS(n,d) by Theorems 5.5-5.7.

For $d \in \{n-1, n-2\}$, there exists a regular SAMS(n,d) by Lemmas 1.1-1.2. $\square$

In the present paper, we proved that there exists a regular SAMS(n,d) for any $n \geq 5$, $n \equiv 1$, $5 \pmod 6$ and d with $2 \leq d \leq n-1$. Quite recently, we obtained the existence of a regular SAMS(n,d) for any $n \geq 5$, $n \equiv 3 \pmod 6$ and d with $2 \leq d \leq n-1$ listed below.

Theorem 6.2 ([9], Theorem 1.6)**.** *Suppose that n is a positive integer satisfying $n \geq 9$ and $n \equiv 3 \pmod 6$. Then there exists a regular SAMS(n,d) for any d with $2 \leq d \leq n-1$.*

However, there are still unsolved cases in the existence question for regular SAMS. For ease of reference, we state this as an open problem.

Problem 6.1. Construct a regular SAMS(n,d) for any $n \geq 4$, $n \equiv 0 \pmod 2$ and d with $2 \leq d \leq n-1$.

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References

[1] G. Abe, *Unsolved problems on magic squares*, Discrete Math. **127** (1994), no. 1-3, 3–13. https://doi.org/10.1016/0012-365X(92)00462-Z

[2] M. Ahmed, *Algebraic combinatorics of magic squares*, Ph.D Dissertation, University of California, Davis, 2004.

[3] W. S. Andrews, *Magic Squares and Cubes*, Dover Publications, Inc., New York, 1960.

[4] W. W. R. Ball and H. S. M. Coxeter, *Mathematical Recreations and Essays*, twelfth edition, University of Toronto Press, Toronto, ON, 1974.

[5] S. S. Block and S. A. Tavares, *Before Sudoku*, Oxford University Press, Oxford, 2009.

[6] G. Chen, H. Chen, K. Chen, and W. Li, *Regular sparse anti-magic squares with small odd densities*, Discrete Math. **339** (2016), no. 1, 138–156. https://doi.org/10.1016/j.disc.2015.08.008

[7] K. Chen, G. Chen, and W. Li, *Regular sparse anti-magic squares with the second maximum density*, Linear Algebra Appl. **457** (2014), 12–28. https://doi.org/10.1016/j.laa.2014.05.008

[8] K. Chen, W. Li, G. Chen, and R. Wei, *Regular sparse anti-magic squares with maximum density*, Ars Combin. **127** (2016), 167–183.

[9] G. Chen, W. Li, M. Zhong, and B. Xin, *On the existence of regular sparse anti-magic squares of odd order*, Graphs Combin. **38** (2022), no. 2, Paper No. 47, 39 pp. https://doi.org/10.1007/s00373-021-02437-z

[10] J. Cormie, V. Linek, S. Jiang, and R. Chen, *Investigating the antimagic square*, J. Combin. Math. Combin. Comput. **43** (2002), 175–197.

[11] I. D. Gray, *Vertex-magic total labelings of regular graphs*, SIAM J. Discrete Math. **21** (2007), no. 1, 170–177. https://doi.org/10.1137/050639594

[12] I. D. Gray and J. A. MacDougall, *Sparse anti-magic squares and vertex-magic labelings of bipartite graphs*, Discrete Math. **306** (2006), no. 22, 2878–2892. https://doi.org/10.1016/j.disc.2006.04.032

[13] I. D. Gray and J. A. MacDougall, *Sparse semi-magic squares and vertex-magic labelings*, Ars Combin. **80** (2006), 225–242.

[14] I. D. Gray and J. A. MacDougall, *Vertex-magic labeling of regular graphs: disjoint unions and assemblages*, Discrete Appl. Math. **160** (2012), no. 7-8, 1114–1125. https://doi.org/10.1016/j.dam.2011.11.025

[15] I. D. Gray and J. A. MacDougall, *Vertex-magic labelings of regular graphs. II*, Discrete Math. **309** (2009), no. 20, 5986–5999. https://doi.org/10.1016/j.disc.2009.04.031

[16] I. D. Gray, J. A. MacDougall, R. J. Simpson, and W. D. Wallis, *Vertex-magic total labelings of complete bipartite graphs*, Ars Combin. **69** (2003), 117–127.

[17] S. Jiang, *Anti-magic squares of even order*, Kodai Math. J. **25** (2002), no. 1, 1–7. https://doi.org/10.2996/kmj/1106171071

[18] A. Kotzig, *On Magic Valuations of Trichromatic Graphs*, Reports of the CRM, CRM-148, December 1971.

[19] J. M. Kudrle and S. B. Menard, *Magic square*, in: The CRC Handbook of Combinatorial Designs, 2nd Edition (Edited by C. J. Colbourn, J. H. Dinitz), CRC Press, Boca Raton, FL, 2006, pp. 524–528.

[20] W. Li, K. J. Chen, and R. W. Su, *Existence of regular sparse magic squares*, Acta Math. Appl. Sin. **34** (2011), no. 6, 1118–1135.

[21] Y.-C. Liang, T.-L. Wong, and X. Zhu, *Anti-magic labeling of trees*, Discrete Math. **331** (2014), 9–14. https://doi.org/10.1016/j.disc.2014.04.021

[22] Y.-C. Liang and X. Zhu, *Antimagic labeling of cubic graphs*, J. Graph Theory **75** (2014), no. 1, 31–36. https://doi.org/10.1002/jgt.21718

[23] W. D. Wallis, *Magic Graphs*, Birkhäuser Boston, Inc., Boston, MA, 2001. https://doi.org/10.1007/978-1-4612-0123-6

[24] W. D. Wallis, *Vertex magic labelings of multiple graphs*, Congr. Numer. **152** (2001), 81–83.

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