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FREE PRODUCTS OF OPERATOR SYSTEMS

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Abstract. In this paper we introduce the notion of universal free product for operator systems and operator spaces, and prove extension results for the operator system lifting property (OSLP) and operator system local lifting property (OSLLP) to the universal free product.

1. Introduction

For C^*-algebras, free products have long been a topic that employed almost exclusively C^*-techniques and methods. The use of operator space methods in relation to free products of C^*-algebras was initiated by Boca in a series of papers [2–4], where he proved that unital, completely positive maps defined on C^*-algebras extend canonically to the full free product of the algebras.

This paper originates in our attempt to investigate the extent to which free products can be defined in the category of operator systems and operator spaces, in a way that is consistent with the well-known properties from the C^*-case.

An immediate problem that requires attention is that, in contrast with the C^*-case, order isomorphic operator systems need not generate isomorphic C^*-algebras, so if free products of systems are to be related in any way to C^*-free products, the C^*-algebras generated by particular representations of operator systems are not helpful. A more subtle issue is that operator systems are not assumed to be uniformly closed while, as we shall see, closure is important when discussing lifting.

Despite these constraints, we will define a notion of free product that is suitable for the category of operator systems, and will prove lifting results for free products of operator systems similar to the results known for free products of C^*-algebras.

The paper is organized as follows. In Section 2 we introduce the universal free product of a family of operator systems, completed with the appropriate counterpart for operator spaces in Section 4. Section 5 is devoted to preliminary results needed in Section 6 to prove the main results: the universal free product

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of a countable family of separable operator systems with the operator system lifting property (OSLP) has the OSLP and the universal free product of an arbitrary family of operator systems with the operator system local lifting property (OSLLP) has the OSLLP.

2. The universal free product of operator systems

In this section we introduce the notion of free product for operator systems. Recall that a concrete operator system is a unital, selfadjoint subspace of $B(H)$. Operator systems are not assumed to be closed in any topology. We should mention that Arveson's extension theorem, originally stated for norm-closed operator systems, also holds in this more general situation, as shown in Theorem 7.5 in [10].

Given operator systems $E \subset B(H)$ and $F \subset B(K)$, it is perhaps tempting to consider the free product of the two C^*-algebras generated by E and, respectively, F. However, this approach turns out to be incorrect, at least for one critical reason. In the C^*-algebra situation, if A_1 and A_2 are isomorphic, then $A_1 * B$ and $A_2 * B$ are isomorphic. But for operator systems, it is possible that $E_1 \subset B(H)$ and $E_2 \subset B(K)$ are completely order isomorphic, yet the C^*-algebras generated by E_1 and E_2 are not isomorphic. This may preclude the isomorphism between $E_1 * F$ and $E_2 * F$, where F is some other operator system. Indeed, the operator system $M_2(\mathbb{C})$ acting on a two-dimensional Hilbert space generates a nuclear C^*-algebra, while the universal C^*-algebra of the system $M_2(\mathbb{C})$ is not even exact ([8]).

To remedy this shortcoming, we will use the universal C^*-algebra of an operator system, introduced by Kirchberg and Wassermann. In [8] they proved that, given an operator system E, there exists a C^*-algebra $C^*_u(E)$, unique up to isomorphism, satisfying

1. There exists a unital completely isometric map $\iota : E \to C^*_u(E)$.
2. $C^*_u(E)$ is the C^*-algebra generated by $\iota(E)$.
3. If $\theta : E \to A$ is a unital completely positive (u.c.p.) map with values in a C^*-algebra A, then there exists a $*$-homomorphism $\pi : C^*_u(E) \to A$ such that $\theta = \pi \circ \iota$.

We note that ι is the direct sum of all u.c.p. maps from E with values in some $M_n(\mathbb{C})$. In the situation when we deal with a family of operator systems $(E_i)_{i\in I}$, in order to avoid any possible confusion we will use the notation $\iota_i(E_i)$ instead of $\iota(E_i)$.

It can be proved ([8]) that, if E and F are completely order isomorphic, then $C^*_u(E)$ and $C^*_u(F)$ are $*$-isomorphic and the $*$-isomorphism between them induces a complete order isomorphism between $\iota(E)$ and $\iota(F)$.

Definition 2.1. If $(E_i)_{i\in I}$ is a family of operator systems, we define the *universal free product* $*^u_{i\in I} E_i$ to be the linear span of the operators of the form

$$x_1 x_2 \cdots x_n,\ n \in \mathbb{N},\ x_j \in \iota_{i_j}(E_{i_j}),\ i_1 \neq i_2 \neq \cdots \neq i_n$$

inside the full free product C^*-algebra $*_{i\in I}\ C_u^*(E_i)$.

Remark 2.2. (1) Note that $\iota_i(1_{E_i}) = \iota_j(1_{E_j}) = 1,\ \forall i, j \in I$.

(2) As a vector space, $*^u_{i\in I}\ E_i$ is the quotient of the vector space which has as basis the set

$$B = \{x_1 x_2 \cdots x_n,\ n \in \mathbb{N},\ x_j \in \iota_{i_j}(E_{i_j}),\ i_1 \neq i_2 \neq \cdots \neq i_n\}$$

by the subspace generated by the relations of the form

$$\begin{aligned} &x_1 \cdots x_{j-1}(\lambda x_j^{(0)} + \mu x_j^{(1)})x_{j+1} \cdots x_n \\ &= \lambda x_1 \cdots x_{j-1} x_j^{(0)} x_{j+1} \cdots x_n + \mu x_1 \cdots x_{j-1} x_j^{(1)} x_{j+1} \cdots x_n\ (\lambda, \mu \in \mathbb{C}) \end{aligned}$$

and

$$x_j = 1 \Rightarrow x_1 \cdots x_n = x_1 \cdots x_{j-1} x_{j+1} \cdots x_n.$$

(3) No assumption is made about the norm closure of the universal free product, as no assumption of this nature is made about the E_n's.

(4) Note that, if $F \subset *^u_{i\in I}\ E_i$ is a finite dimensional subsystem, then there exist a finite subset $J \subset I$ and finite dimensional subsystems $F_j \subset E_j,\ j \in J$, such that $F \subset *_{j\in J}\ \iota_j(F_j)$.

Next, we present the universality property of the universal free product.

Proposition 2.3. *If $(E_i)_{i\in I}$ is a family of operator systems, A is a unital C^*-algebra, and $\varphi_i : E_i \to A$ are u.c.p. maps, then there exists a u.c.p. map $\Phi : *^u_{i\in I} E_i \to A$ such that $\Phi|_{E_i} = \varphi_i$ for all $i \in I$.*

Proof. Let $\pi_i : C_u^*(E_i) \to A$ be the $*$-homomorphisms satisfying $\pi_i \circ \iota_i = \varphi_i$ and consider the $*$-homomorphism $*_{i\in I}\ \pi_i : *_{i\in I}\ C_u^*(E_i) \to A$. The restriction of this $*$-homomorphism to $*_{i\in I}\ \iota_i(E_i) = *^u_{i\in I}\ E_i$ is the desired map. □

Remark 2.4. We must emphasize that the notion of free product defined in this section is exclusively an operator system object and does not extend the classic notion for C^*-algebras. In particular, if A and B are C^*-algebras, the C^*-free product $A * B$ is very different from the universal free product $A *^u B$, when A and B are viewed as operator systems. In spite of this substantial difference, there is a close relationship between the two notions, as the next result illustrates. While it represents no more than a simple remark, its consequences are extremely important.

Proposition 2.5. *If $(E_i)_{i\in I}$ is a family of operator systems, then $\overline{*^u_{i\in I}\ E_i}$, the uniform closure of $*^u_{i\in I}\ E_i$, is equal to the free product C^*-algebra $*_{i\in I}\ C_u^*(E_i)$.*

Proof. It suffices to prove that the uniform closure of $*^u_{i\in I}\ E_i$ contains all operators of the form

$$a_1 a_2 \cdots a_n,\ n \in \mathbb{N},\ a_j \in C_u^*(E_{i_j}),\ a_j \geq 0,\ i_1 \neq i_2 \neq \cdots \neq i_n.$$

Consider an operator of the form

$$x_1 x_2 \cdots x_n,\ n \in \mathbb{N},\ x_j \in \iota_{i_j}(E_{i_j}),\ i_1 \neq i_2 \neq \cdots \neq i_n$$

and let $b \in \iota_{i_2}(E_{i_2})$ be a nonzero operator. Then $y_k = x_1^*(I + \frac{b}{k})x_1 x_2 \cdots x_n \in *_{i\in I}^u E_i$ and

$$\lim_{k\to\infty} y_k = x_1^* x_1 x_2 \cdots x_n.$$

It follows that $\overline{*_{i\in I}^u E_i}$ contains all operators of the form $a_1 x_2 \cdots x_n,\ a_1 \in C_u^*(E_1),\ x_j \in \iota_{i_j}(E_{i_j}),\ j \geq 2$. The argument continues in a similar fashion, ending up showing that $a_1 a_2 \cdots a_n \in \overline{*_{i\in I}^u E_i}$. The conclusion follows. □

3. Extending completely bounded maps to free products of C^*-algebras

Boca's theorem [2] states that, if $A_i,\ i \in I$, are unital C^*-algebras and $\varphi_i : A_i \to B(H)$ are unital completely positive maps, then there exists a unital completely positive map $\Phi : *_{i\in I} A_i \to B(H)$ such that $\Phi|_{A_i} = \varphi_i$. In this section we obtain a version of Boca's result for completely bounded maps (Proposition 3.3). We must point out that, beyond its existence, we cannot say much more about the extended completely bounded map in Proposition 3.3. This is in stark contrast with the nice multiplicative properties of Boca's extended u.c.p. maps (see [2] for more details).

Let A be a unital C^*-algebra and $\varphi : A \to B(H)$ a complete contraction such that $\varphi(I) = 0$. It is well known [10] that $\varphi(x) = S\pi(x)T$, where $\pi : A \to B(K)$ is a unital $*$-homomorphism and $T : H \to K,\ S : K \to H$ are such that $||S||, ||T|| \leq 1$ and $ST = 0$. Then

$$\begin{pmatrix} \pi(x) & \pi(x) \\ \pi(x) & \pi(x) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \pi(x) & 0 \\ 0 & \pi(x) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and $\begin{pmatrix} S & 0 \\ 0 & T^* \end{pmatrix} \begin{pmatrix} \pi(x) & \pi(x) \\ \pi(x) & \pi(x) \end{pmatrix} \begin{pmatrix} S^* & 0 \\ 0 & T \end{pmatrix} = \begin{pmatrix} S\pi(x)S^* & S\pi(x)T \\ T^*\pi(x)S^* & T^*\pi(x)T \end{pmatrix}$.

Note that for $x = I$ the latter matrix is equal to

$$\begin{pmatrix} SS^* & 0 \\ 0 & T^*T \end{pmatrix}.$$

Since both S and T are contractions, we can add a small completely positive piece to make it a unital completely positive map (u.c.p.). If θ is a state on A, we add

$$\begin{pmatrix} (I - SS^*)\theta(x) & 0 \\ 0 & (I - T^*T)\theta(x) \end{pmatrix}$$

and thus obtain

$$\Psi(x) = \begin{pmatrix} S\pi(x)S^* + (I - SS^*)\theta(x) & S\pi(x)T \\ T^*\pi(x)S^* & T^*\pi(x)T + (I - T^*T)\theta(x) \end{pmatrix}$$

which is a unital c.p. map defined on A and taking values in $M_2(B(H))$.

We just proved:

Lemma 3.1. *If $\varphi : A \to B(H)$ is a completely contractive map such that $\varphi(I) = 0$, then φ is the upper right corner of a u.c.p. map $\Psi : A \to M_2(B(H))$.*

We can now apply Boca's theorem. By taking free products of u.c.p. maps and then compressing to the upper right corner, we obtain:

Lemma 3.2. *If $A_i,\ i \in I$, are unital C^*-algebras and $\varphi_i : A_i \to B(H)$ are complete contractions such that $\varphi_i(I) = 0$, then there exists a complete contraction $\Phi : *_{i\in I}\ A_i \to B(H)$ such that $\Phi|_{A_i} = \varphi_i$ for all $i \in I$.*

We arrived at the main result of this section.

Proposition 3.3. *If $(A_i)_{i\in I}$ is a family of unital C^*-algebras and $\varphi_i : A_i \to B(H)$ are completely bounded maps such that $\sup\{||\varphi_i||_{cb} : i \in I\} < \infty$ and $\varphi_i(I) = T$ for some $T \in B(H)$ and all $i \in I$, then there exists a completely bounded map $\Phi : *_{i\in I}\ A_i \to B(H)$ such that $\Phi|_{A_i} = \varphi_i$ for all $i \in I$. In addition, $||\Phi||_{cb} \leq 3\sup\{||\varphi_i||_{cb} : i \in I\}$.*

Proof. Denote $\alpha = \sup\{||\varphi_i||_{cb} : i \in I\}$. Take θ_i to be a state on A_i and consider the maps $\psi_i(x) = (2\alpha)^{-1}(\varphi_i(x) - T\theta_i(x))$. These maps are complete contractions with $\psi_i(I) = 0$, so Lemma 3.2 applies to obtain a complete contraction $\Psi : *_{i\in I}\ A_i \to B(H)$ satisfying $\Psi|_{A_i} = \psi_i$. If we consider the free product state $\Theta = *_{i\in I}\ \theta_i$ on $*_{i\in I}\ A_i$ ([1]), then $\Phi = 2\alpha\Psi + T\Theta$ is the desired map. □

4. Universal free products of operator spaces

If $V_i \subset B(H_i),\ i \in I$, are operator spaces, let $E(V_i)$ be the operator systems generated by V_i. Define $*^u_{i\in I}\ V_i$, the *universal free product* of the family $(V_i)_{i\in I}$ to be the operator space $*_{i\in I}\ \iota_i(V_i)$, viewed as a subspace of $*_{i\in I}\ \iota_i(E(V_i)) = *^u_{i\in I}\ E(V_i) \subset *_{i\in I}\ C^*_u(E(V_i))$. It is easy to see that, if the V_i's are operator systems, the previous definition agrees with the definition of the universal free product for operator systems in Section 2. Since in the category of operator spaces the morphisms are completely bounded maps, we will need an extension theorem for completely bounded maps, in the spirit of Proposition 3.3.

This extension theorem, proved below, represents the universality property of the universal free product of operator spaces.

Proposition 4.1. *If $(V_i)_{i\in I}$ is a family of unital operator spaces and $\varphi_i : V_i \to B(H)$ are completely bounded maps such that $\sup\{||\varphi_i||_{cb} : i \in I\} < \infty$ and $\varphi_i(I) = T$ for some $T \in B(H)$ and all $i \in I$, then there exists a completely bounded map $\Phi : *^u_{i\in I}\ V_i \to B(H)$ such that $\Phi|_{V_i} = \varphi_i$ for all $i \in I$. In addition, $||\Phi||_{cb} \leq 3\sup\{||\varphi_i||_{cb} : i \in I\}$.*

Proof. The maps φ_i extend to c.b. maps $\psi_i : C^*_u(E(V_i)) \to B(H)$ with same c.b. norm ([10]8.2). Proposition 3.3 shows that there exists a c.b. map $\Psi : *_{i\in I}\ C^*_u(E(V_i)) \to B(H)$ such that $\Psi|_{C^*_u(E(V_i))} = \varphi_i$. The restriction of Ψ to $*^u_{i\in I}\ V_i$ is the desired map Φ. □

5. Lifting properties of operator systems

Recall that a unital C^*-algebra A has the Lifting Property (LP) if, for every unital completely positive (u.c.p.) map $\varphi : A \to B/J$, there exists a u.c.p. map

$\psi : A \to B$ such that $\varphi = q \circ \psi$, where $q : B \to B/J$ is the quotient map. In [7] Kirchberg introduced the Local Lifting Property (LLP), which requires, for $\varphi : A \to B/J$ and a fixed, finite dimensional operator system $E \subset A$, only the existence of a u.c.p. lifting $\psi : E \to B$ such that $\varphi|_E = q \circ \psi$,

If E is an operator system, E is said to have the Operator System Lifting Property (OSLP) if, for every C^*-algebra A with a closed, two-sided ideal J, and every u.c.p. map $\varphi : E \to A/J$, there exists a u.c.p map $\psi : E \to A$ such that $\varphi = q \circ \psi$, where $q : A \to A/J$ is the quotient map.

If E is an operator system, E is said to have the Operator System Local Lifting Property (OSLLP) if, for every unital C^*-algebra A with a closed, two-sided ideal J, every finite dimensional operator subsystem $E_0 \subset E$, and every u.c.p. map $\varphi : E \to A/J$, there exists a u.c.p map $\psi : E_0 \to A$ such that $\varphi|_{E_0} = q \circ \psi$.

It is easy to see that, if E is separable, the algebras A in the above definitions can be taken to be separable. It is also clear that a C^*-algebra has the LP (or LLP) if and only if it has the OSLP (or OSLLP, respectively) as an operator system.

We continue this section by highlighting the very natural connection between an operator space and its universal C^*-algebra. First, we prove that E has the OSLLP if and only if $C^*_u(E)$ has the LLP. For this purpose we will need two preliminary results.

Lemma 5.1. *Let $E \subset F$ be finite dimensional operator systems and $\varphi : E \to B(\ell^2)$ and $\psi : F \to B(\ell^2)$ be u.c.p. maps such that $(\varphi - \psi)(E) \subset K(\ell^2)$. Then, for any given $\varepsilon > 0$, there exists a u.c.p. map $\psi_0 : F \to B(\ell^2)$ such that $(\varphi - \psi_0)(E) \subset K(\ell^2)$ and $||(\varphi - \psi_0)|_E|| < \varepsilon$.*

Proof. The proof is almost identical to the proof of Lemma 9 in [4]. If (e_n) denotes a quasicentral approximate unit for $K(\ell^2)$, then we can take $\psi_0 = (I-e)^{1/2}\psi(I-e)^{1/2} + e^{1/2}\widetilde{\varphi}e^{1/2}$, where $\widetilde{\varphi}$ is any Arveson extension of φ to F, and e is equal to e_n for some large enough value of n. □

Proposition 5.2. *Let E be a closed, separable operator system and (E_n) an increasing sequence of finite dimensional operator subsystems of E such that $E = \overline{\cup E_n}$. Let $\varphi : E \to B(\ell^2)/K(\ell^2)$ be a u.c.p. map and suppose that there exist u.c.p. maps $\varphi : E_n \to B(\ell^2)$ such that $q \circ \varphi_n = \varphi|_{E_n}$, where q is the quotient map. Then there exists a u.c.p. map $\psi : E \to B(\ell^2)$ satisfying $q \circ \psi = \varphi$.*

Proof. By taking into account Lemma 5.1, we set $\psi_1 = \varphi_1$ and, for every $n \geq 2$, we let $\psi_n : E_n \to B(\ell^2)$ be a u.c.p. map such that $||(\psi_{n+1} - \psi_n)|_{E_n}|| < 2^{-n}$ and $(\psi_{n+1} - \psi_n)(E_n) \subset K(\ell^2)$. It follows that the sequence $\psi_n(x)$ is Cauchy for every $x \in \cup E_n$, so its limit defines a u.c.p. map $\psi : E \to B(\ell^2)$ satisfying $q \circ \psi = \varphi$. □

We conclude with the two results about universal C^*-algebras announced at the beginning of the section. The first one refers to the Local Lifting Property, while the second refers to the Lifting Property.

Proposition 5.3. *An operator system E has the OSLLP if and only if $C_u^*(E)$ has the LLP.*

Proof. Suppose that E has the OSLLP, the other direction being immediate. Fix a finite dimensional subsystem $F \subset C_u^*(E)$. We will assume first that $F \subset C_u^*(E_0)$, where $E_0 \subset E$ is a finite dimensional subsystem, and note that $C^*(E_0)$ is canonically isomorphic to $C_u^*(E_0)$. Let $\varphi : C_u^*(E) \to B/J$ be a u.c.p. map into a quotient C^*-algebra where the quotient map is $q : B \to B/J$. Consider also a $*$-epimorphism $\rho : C^*(\mathbb{F}_I) \to C_u^*(E)$. Since E has the OSLLP, there exists a u.c.p. map $\alpha : E_0 \to C^*(\mathbb{F}_I)$ such that $\rho \circ \alpha = id|_{E_0}$. By universality, the map α extends to a $*$-homomorphism $\theta : C_u^*(E_0) \to C^*(\mathbb{F}_I)$ such that $\theta(x) = \alpha(x)$ for all $x \in E_0$. It follows that the $*$-homomorphism $\rho \circ \theta$ will be the identity map on E_0. Since E_0 generates $C_u^*(E_0)$ as a C^*-algebra, we get that $\rho \circ \theta$ is the identity map on $C_u^*(E_0)$. Since $C^*(\mathbb{F}_I)$ has the LLP, there exists a u.c.p. map $\beta : \theta(F) \to B$ such that $q \circ \beta = \varphi \circ \rho|_{\theta(F)}$. Then $(\beta \circ \theta)|_F$ is the desired local lifting of φ.

In general, F is contained in the uniform closure of an increasing sequence of finite dimensional subsystems (F_n) of E with the property that each F_n is contained in some $C^*(E_n)$, for some finite dimensional subsystem $E_n \subset E$.

Let $\varphi : C_u^*(E) \to B(\ell^2)/K(\ell^2)$ be a u.c.p. map with values in the Calkin algebra. By the first part of the proof, each F_n has a lifting $\psi_n : F_n \to B(\ell^2)$ such that $q \circ \psi_n = \varphi|_{F_n}$. We apply Proposition 5.2 to obtain a u.c.p. map $\psi : \overline{\cup F_n} \to B(\ell^2)$ such that $q \circ \psi = \varphi|_{\overline{\cup F_n}}$. The restriction of ψ to F is the desired lifting.

We proved that $\varphi : C_u^*(E) \to B(\ell^2)/K(\ell^2)$ lifts locally. By a theorem of Ozawa ([9], see also [13]) it follows that $C_u^*(E)$ has the LLP. □

We take a moment to note, in retrospect, that in the second part of the proof we had to narrow our argument to maps with values in the Calkin algebra and then rely on Ozawa's theorem because in Lemma 5.1 we needed to apply Arveson's extension theorem which, in turn, required an injective range, like $B(\ell^2)$.

The following is the analogue result for the OSLP and LP.

Proposition 5.4. *A separable operator system E has the OSLP if and only if $C_u^*(E)$ has the LP.*

Proof. Suppose that E (therefore $\iota(E)$) has the OSLP. Since E (and thus $C_u^*(E)$) is separable, let $\pi : C^*(\mathbb{F}_\infty) \to C_u^*(E)$ be a $*$-epimorphism. By hypothesis, there exists a u.c.p. map $\alpha : \iota(E) \to C^*(\mathbb{F}_\infty)$ such that $\pi \circ \alpha$ is the identity map on $\iota(E)$ and, by universality, α extends to a $*$-homomorphism $\beta : C_u^*(E) \to C^*(\mathbb{F}_\infty)$. It follows that $\pi \circ \beta$ is a $*$-homomorphism of $C_u^*(E)$

which acts identically on $\iota(E)$, a generating set, therefore $\pi \circ \beta(x) = x$ for all $x \in C_u^*(E)$. Finally, let $\varphi : C_u^*(E) \to B/J$ be a u.c.p. map. Since $C^*(\mathbb{F}_\infty)$ has the LP, there exists a u.c.p. map $\rho : C^*(\mathbb{F}_\infty) \to B$ such that $q \circ \rho = \varphi \circ \pi$, where $q : B \to B/J$ is the quotient map. Then $\rho \circ \beta$ is the desired lifting of φ.

Conversely, suppose that $C_u^*(E)$ has the LP and let $\varphi : \iota(E) \to B/J$ be a u.c.p. map. By universality, φ extends to a $*$-homomorphism $\theta : C_u^*(E) \to B/J$. By hypothesis, there exists a u.c.p. map $\alpha : C_u^*(E) \to B$ satisfying $q \circ \alpha = \theta$. Then the restriction of α to $\iota(E)$ is the desired lifting of φ. □

We continue with some remarks on operator systems and their norm closures, from the viewpoint of the OSLP.

Proposition 5.5. *An operator system $E \subset B(H)$ has the OSLP if and only if $\overline{E}$, the uniform closure of E, has the OSLP.*

Proof. Suppose that E has the OSLP. If $\varphi : \overline{E} \to A/J$ is a u.c.p. map, denote $\varphi_0 = \varphi|_E$. By hypothesis, there exists a u.c.p. map $\alpha_0 : E \to A$ such that $q \circ \alpha_0 = \varphi_0$, where $q : A \to A/J$ is the quotient map. If α denote the extension by continuity of α_0 to $\overline{E}$, it is easy to see that $q \circ \alpha = \varphi$.

Conversely, if $\overline{E}$ has the OSLP and $\varphi : E \to A/J$ is a u.c.p. map, let $\Phi : \overline{E} \to A/J$ be the extension of φ by continuity. By hypothesis, there exists a u.c.p. map $\alpha : \overline{E} \to A$ such that $q \circ \alpha = \Phi$. If α_0 is the restriction of α to E, then $q \circ \alpha_0 = \Phi|_E = \varphi$. □

At this stage we need to recall some basic facts about tensor products of operator systems. We refer the reader to [5,6,12] for more details on this topic.

(1) *The minimal tensor product* min. If $S \subset B(H)$ and $T \subset B(K)$ are operator systems acting on the Hilbert spaces H, respectively K, then $S \otimes_{\min} T$ is the operator system arising from the natural inclusion of $S \otimes T$ into $B(H \otimes K)$.

(2) *The maximal tensor product* max [5] is the operator system structure on $S \otimes T$ obtained from the Archimedeanization of the matrix order given by positive cones

$$D_n = \{A^*(P \otimes Q)A : \ A \in M_{n,km}(\mathbb{C}), P \in M_k(S)_+, Q \in M_m(T)_+\}.$$

(3) *The commuting tensor product* "c" was introduced in [10] (where it was referred to as max). If $\theta_i : S_i \to B(H)$ are u.c.p. maps with commuting ranges, we have a well-defined map $\theta_1 \otimes \theta_2 : S_1 \otimes S_2 \to B(H)$. For $(x_{ij}) \in M_n(S_1 \otimes S_2)$, we set

$$||(x_{ij})|| = \sup\{||(\theta_1 \otimes \theta_2)(x_{ij})|| : \theta_k : S_k \to B(H), \ k = 1, 2\}$$

where θ_1 and θ_2 are u.c.p. maps with commuting ranges and H is an arbitrary Hilbert space.

Two important facts were proved in [5]: first, $S \otimes_c T \subset C_u^*(S) \otimes_{\max} C_u^*(T)$, which shows in particular that if A is a C^*-algebra, then in the definition of $S \otimes_c A$ the u.c.p. maps θ_2 pertaining to A can be chosen to be $*$-monomorphisms (we identified S and $\iota(S)$). The second result is that if A is a C^*-algebra, then

$S \otimes_c A = S \otimes_{\max} A$. We will, therefore, use the notation "max" when one of the operator systems is a C^*-algebra.

Our next goal is to prove an analogue of Proposition 5.5 for the OSLLP. We will need the following technical result.

Proposition 5.6. *If E is an operator system and A is a C^*-algebra, then $E \otimes_{\min} A = E \otimes_{\max} A$ if and only if $\overline{E} \otimes_{\min} A = \overline{E} \otimes_{\max} A$.*

Proof. ($\Rightarrow$) Fix $E_i \in \overline{E}$ and $a_i \in A$, $1 \le i \le n$, and $\varepsilon > 0$. Choose $e_i \in E$ to satisfy $||e_i - E_i|| < \varepsilon/n \max ||a_i||$. There exist a Hilbert space H, a u.c.p. map $\varphi : \overline{E} \to B(H)$ and a $*$-monomorphism $\pi : A \to B(H)$ such that the ranges of φ and π commute and $||\sum E_i \otimes a_i||_{\max} < ||\sum \varphi(E_i)\pi(a_i)|| + \varepsilon$. We have

$$\begin{aligned} ||\sum E_i \otimes a_i||_{\max} &< ||\sum \varphi(E_i)\pi(a_i)|| + \varepsilon \\ &\le ||\sum \varphi(e_i)\pi(a_i)|| + 2\varepsilon \\ &\le ||\sum e_i \otimes a_i||_{\max} + 2\varepsilon = ||\sum e_i \otimes a_i||_{\min} + 2\varepsilon \\ &\le ||\sum E_i \otimes a_i||_{\min} + 3\varepsilon. \end{aligned}$$

($\Leftarrow$) There exist a Hilbert space H, a u.c.p. map $\varphi : E \to B(H)$, and a $*$-homomorphism $\pi : A \to B(H)$ such that the ranges of φ and π commute and

$$||\sum e_i \otimes a_i||_{\max} < ||\sum \varphi(e_i)\pi(a_i)|| + \varepsilon.$$

Denote by $\widetilde{\varphi}$ the extension by continuity of φ to $\overline{E}$. Clearly the ranges of $\widetilde{\varphi}$ and π commute, so we have

$$\begin{aligned} ||\sum e_i \otimes a_i||_{\max} &< ||\sum \varphi(e_i)\pi(a_i)|| + \varepsilon = ||\sum \widetilde{\varphi}(e_i)\pi(a_i)|| + \varepsilon \\ &\le ||\sum e_i \otimes a_i||_{\overline{E} \otimes_{\max} A} + \varepsilon = ||\sum e_i \otimes a_i||_{\overline{E} \otimes_{\min} A} + \varepsilon \\ &= ||\sum e_i \otimes a_i||_{\min} + \varepsilon. \end{aligned}$$

Since ε was arbitrary, the conclusion follows. □

Corollary 5.7. *An operator system $E \subset B(H)$ has the OSLLP if and only if $\overline{E}$ has the OSLLP.*

Proof. By Proposition 8.5 in [6], E has the OSLLP if and only if $E \otimes_{\min} B(H) = E \otimes_{\max} B(H)$. The conclusion follows if we apply this to $A = B(H)$ in Proposition 5.6. □

6. Lifting properties of universal free products

In [11] Pisier proved that the free product of a family of unital C^*-algebras with the LLP has the LLP as well. While not stated specifically, the corresponding result for LP can be shown to hold true in the case of a countable family of separable C^*-algebras. For completeness, we prove this fact in Proposition

6.1. The analogue results for operator systems will follow as consequences. We recall from Proposition 2.5 that $\overline{*^u_{i\in I}\ E_i} = *_{i\in I}\ C^*_u(E_i)$.

Proposition 6.1. *If $(A_i)_{i\in I}$ is a countable family of separable C^*-algebras with the LP, then $*_{i\in I}A_i$ has the LP.*

Proof. Since the A_i's are all separable, so is $*_{i\in I}A_i$, so there exists a $*$-epimorphism $\pi : C^*(\mathbb{F}_\infty) \to *_{i\in I}A_i$. Since the A_i's have the LP, there exist u.c.p. maps $\varphi_i : A_i \to C^*(\mathbb{F}_\infty)$ such that $\pi \circ \varphi_i = id|_{A_i}$ for all $i \in I$. Boca's theorem ensures the existence of a u.c.p. map $\Phi : *_{i\in I}A_i \to C^*(\mathbb{F}_\infty)$ such that $\pi \circ \Phi = id|_{*_{i\in I}A_i}$. Finally, let $\theta : *_{i\in I}A_i \to B/J$ be a u.c.p. map into a quotient C^*-algebra B/J with quotient map $q : B \to B/J$. Since $C^*(\mathbb{F}_\infty)$ has the LP, there exists a lifting $\alpha : C^*(\mathbb{F}_\infty) \to B$ such that $q \circ \alpha = \theta \circ \pi$. Then $\alpha \circ \Phi$ is the desired lifting of θ. □

Corollary 6.2. *If $(E_i)_{i\in I}$ is a countable family of separable operator systems with the OSLP, then $*^u_{i\in I}E_i$ has the OSLP.*

Proof. If the operator systems (E_i) have the OSLP, then $C^*_u(E_i)$ have the LP by Proposition 5.4, and therefore, by Proposition 6.1, so does $*_{i\in I}\ C^*_u(E_i)$. Proposition 2.5 then shows that $\overline{*^u_{i\in I}\ E_i}$ has the OSLP, therefore $*^u_{i\in I}\ E_i$ has the OSLP by Proposition 5.5. □

We arrived at the other main result of this section.

Proposition 6.3. *If $(E_i)_{i\in I}$ is a family of operator systems with the OSLLP, then $*^u_{i\in I}E_i$ has the OSLLP.*

Proof. If the operator systems E_i have the OSLLP, then the algebras $C^*_u(E_i)$ have the LLP by Proposition 5.3. It follows that $*_{i\in I}\ C^*_u(E_i)$ has the LLP by Theorem 1.11 in [11]. This implies, by Proposition 2.5, that $\overline{*^u_{i\in I}\ E_i}$ has the OSLLP, therefore so does $*^u_{i\in I}\ E_i$ by Corollary 5.7. □

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