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# WEIGHTED INTEGRAL INEQUALITIES FOR MODIFIED INTEGRAL HARDY OPERATORS

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Abstract. In this article, we study the weak and extra-weak type integral inequalities for the modified integral Hardy operators. We provide suitable conditions on the weights $\omega, \rho, \phi$ and $\psi$ to hold the following weak type modular inequality

$$\mathcal{U}^{-1}\Big(\int_{\{|\mathcal{I}f|>\gamma\}} \mathcal{U}\Big(\gamma\omega\Big)\rho\Big) \leq \mathcal{V}^{-1}\Big(\int_0^\infty \mathcal{V}\Big(C|f|\phi\Big)\psi\Big),$$

where $\mathcal{I}$ is the modified integral Hardy operators. We also obtain a necessary and sufficient condition for the following extra-weak type integral inequality

$$\omega\Big(\Big\{|\mathcal{I}f| > \gamma\Big\}\Big) \leq \mathcal{U}\circ\mathcal{V}^{-1}\Big(\int_0^\infty \mathcal{V}\Big(\frac{C|f|\phi}{\gamma}\Big)\psi\Big).$$

Further, we discuss the above two inequalities for the conjugate of the modified integral Hardy operators. It will extend the existing results for the Hardy operator and its integral version.

## 1. Introduction

We define the modified integral Hardy operators, $\mathcal{I}$ for a non-negative measurable function $f$ on the real line as

$$\mathcal{I}f(t) = h(t)\int_0^t K(t,\tau)f(\tau)w(\tau)d\tau, \tag{1}$$

where $h$ and $w$ are two positive measurable functions on $\mathbb{R}$, and let the kernel $K$, defined on $\{(t,\tau) : 0 \leq \tau \leq t\}$ satisfy the following conditions.

(i) $K(t,\tau) \geq 0$ for $0 \leq \tau \leq t$.
(ii) $K(t,\tau)$ is non-decreasing in $t$ and non-increasing in $\tau$.
(iii) For $0 \leq \tau \leq z \leq t$ there exists a constant $M \geq 1$ independent of $t, \tau$ and $z$ such that

$$K(t,\tau) \leq M[K(t,z) + K(z,\tau)]. \tag{2}$$

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The kernel satisfying condition (2) is also known as Oinarov's kernel. Particular examples of this type of kernel are $(t-\tau)^\mu$ and $\log^\mu\left(\frac{t}{\tau}\right)$, where $\mu \geq 0, t > \tau > 0$. For $K = 1$, the integral operator (1) is reduced to the modified Hardy operators defined by

$$\mathcal{H}_h f(t) = h(t)\int_0^t f(y)w(y)dy. \tag{3}$$

For $h = 1 = w$, the operators (1) and (3) are usually termed as the integral Hardy and Hardy operators, respectively. Weighted inequalities for the operators (1) and (3) have been studied substantially [6, 10, 12, 13] due to their considerable influence in spectral problem, fractional regularity, and in the construction of energy estimates for partial differential equations, especially in hyperbolic PDEs, etc. In [12], Martín-Reyes and Salvador characterized the pair of weights $(\rho, \psi)$ such that

$$\left(\int_{\{t\in(0,\infty):\mathcal{H}_h f(t)>\gamma\}} \gamma^q \rho(t)dt\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty f(t)^p\psi(t)dt\right)^{\frac{1}{p}} \tag{4}$$

holds for the range $1 \leq p \leq q < \infty$. Through this article, we examine the estimate (4) in the Orlicz space setting for the operators (1) and (3) and also for their conjugates $\tilde{\mathcal{I}}$ and $\tilde{\mathcal{H}}_h$, respectively, defined by

$$\tilde{\mathcal{I}} f(t) = w(t)\int_t^\infty K(\tau, t)f(\tau)h(\tau)d\tau, \tag{5}$$

$$\tilde{\mathcal{H}}_h f(t) = w(t)\int_t^\infty f(\tau)h(\tau)d\tau. \tag{6}$$

Among the various equivalent generalization of the estimate (4), we will consider the following integral inequality

$$\begin{aligned}\mathcal{U}^{-1}\Big(\int_{\{t\in(0,\infty):\mathcal{T}f(t)>\gamma\}} \mathcal{U}\Big(\gamma\omega(t)\Big)\rho(t)dt\Big) \\ \leq \mathcal{V}^{-1}\Big(\int_0^\infty \mathcal{V}\Big(Cf(t)\phi(t)\Big)\psi(t)dt\Big),\end{aligned} \tag{7}$$

where $\mathcal{U}$ and $\mathcal{V}$ are $N$-functions, defined in Section 2; $\omega, \rho, \phi$ and $\psi$ are positive weights on $\mathbb{R}$; $\mathcal{T} = \mathcal{I}, \tilde{\mathcal{I}}, \mathcal{H}_h, \tilde{\mathcal{H}}_h$.

In particular, if we consider $\mathcal{V}(t) = t^p$ and $\mathcal{U}(t) = t^q$ for $1 \leq p \leq q < \infty$, then the inequality (7) extends the estimate (4). In [3], Bloom and Kerman addressed the estimate (7) for the Hardy operator and its integral version, where the authors used the monotonic condition of the operator. They also proved the following stronger version of (7) in the case $h = 1 = w$.

$$\mathcal{U}^{-1}\Big(\int_0^\infty \mathcal{U}\Big(\mathcal{H}_h f(t)\omega(t)\Big)\rho(t)dt\Big) \leq \mathcal{V}^{-1}\Big(\int_0^\infty \mathcal{V}\Big(Cf(t)\phi(t)\Big)\psi(t)dt\Big). \tag{8}$$

It is again due to the monotonic property of the Hardy operator that the estimates (7) and (8) are equivalent in the case of Hardy operator [3] but yet to be verified for the modified Hardy operators. The techniques of Bloom and Kerman [3] do not work for the operators (1) and (3) as they do not possess the monotone property for non-increasing $h$. Salvador [14] analyzed (7) with two weights that is by considering $\rho = 1 = \phi$. The main objective of this article is to investigate conditions on the four weights $\omega, \rho, \phi$ and $\psi$ such that estimates (7) and (8) hold for a suitable constant $C > 0$. We will also discuss a weaker version of (7) of the form

$$\omega\bigg(\{t \in (0,\infty) : \mathcal{T}f(t) > \gamma\}\bigg) \leq \mathcal{U} \circ \mathcal{V}^{-1}\bigg(\int_0^\infty \mathcal{V}\bigg(\frac{Cf(t)\phi(t)}{\gamma}\bigg)\psi(t)dt\bigg). \tag{9}$$

The inequality (9) is called as the extra-weak type integral inequality as it follows from (7) but does not follow contrarily. One of the prime importance of the extra-weak type mixed integral inequalities is to obtain exquisite bounds for the strong type integral estimates [1,2]. We refer to [4,7–9,15] for a detailed investigation on weighted integral inequalities for the Hardy operators and also for maximal functions.

We organize the article as follows. In Section 2 we give some rudimentary properties of $N$-functions and state the main results of the article. The proofs of the results are given in Sections 3-5.

## 2. Preliminaries

In this section, we discuss some basics associated with the $N$-function [5]. After that, we state the weak and extra weak type results for the modified integral Hardy operators.

An $N$-function $\mathcal{U}$ is continuous and convex on $[0,\infty)$ such that $\mathcal{U}(0) = 0$ and $\frac{\mathcal{U}(t)}{t} \to 0$ (and $\infty$) when $t \to 0$ (and $\infty$). It is always possible to write an $N$-function $\mathcal{U}$ in the integral form as, $\mathcal{U}(t) = \int_0^t u(y)dy$, where $u$ is positive, increasing and right continuous at each point and satisfies $u(0) = 0$, $u(r) > 0$ for $r > 0$ and $u(r) \to \infty$ as $r \to \infty$. The complementary function $\tilde{\mathcal{U}}$ corresponding to a given $N$-function $\mathcal{U}$ is defined by $\tilde{\mathcal{U}}(t) = \sup_{\tau\geq 0}(t\tau - \mathcal{U}(\tau))$ also verifies the properties of $N$-functions. For $t, \tau > 0$, the pair $(\mathcal{U}, \tilde{\mathcal{U}})$ satisfies the following relations [3]:

$$t\tau \leq \mathcal{U}(t) + \tilde{\mathcal{U}}(\tau), \tag{10}$$

$$\mathcal{U}\Big(\frac{\tilde{\mathcal{U}}(t)}{t}\Big) \leq \tilde{\mathcal{U}}(t), \tag{11}$$

$$\mathcal{U}(t) \leq tu(t) \leq \mathcal{U}(2t). \tag{12}$$

Next, we state the main results of the article. We first state the weak type result for the modified integral Hardy operators.

**Theorem 2.1.** *Let $\mathcal{V}$ be an $N$-function with complementary $N$-function $\tilde{\mathcal{V}}$ and $\mathcal{U}$ be a strictly increasing positive function such that $\mathcal{V} \circ \mathcal{U}^{-1}$ is countably* *subadditive. Suppose that the weights $\omega, \rho, \phi$ and $\psi$ are positive and locally integrable functions on $(0,\infty)$. Let the function $h$ be monotone on $\mathbb{R}$. Then the following conditions are equivalent.*

(a) *There exists $C > 0$ such that*

$$\mathcal{U}^{-1}\Big(\int_{\{t\in(0,\infty):\mathcal{I}f(t)>\gamma\}}\mathcal{U}\Big(\gamma\omega(t)\Big)\rho(t)dt\Big) \leq \mathcal{V}^{-1}\Big(\int_0^\infty \mathcal{V}\Big(Cf(t)\phi(t)\Big)\psi(t)dt\Big) \tag{13}$$

*holds for each $\gamma > 0$ and all non-negative functions $f$.*

(b) *Let $0 < t < \alpha$ and $\gamma > 0$. Then there exists $C > 0$ such that*

$$\int_0^t \tilde{\mathcal{V}}\left[\frac{(\inf_{(t,\alpha)} h)K(t,\tau)w(\tau)\eta(\gamma;t,\alpha)}{C\gamma\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau \leq \eta(\gamma;t,\alpha) \tag{14}$$

*and for $0 < z \leq t < y < \alpha$ the following*

$$\int_0^z \tilde{\mathcal{V}}\left[\frac{(\inf_{y\in(t,\alpha)} h(y)K(y,z))w(\tau)\eta(\gamma;t,\alpha)}{C\gamma\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau \leq \eta(\gamma;t,\alpha) \tag{15}$$

*hold, where*

$$\eta(\gamma;t,\alpha) = \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_t^\alpha \mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big).$$

If we consider $K \equiv 1$, then the conditions (14) and (15) are the same and reduced to the following:

$$\int_0^t \tilde{\mathcal{V}}\left[\frac{(\inf_{(t,\alpha)} h)w(\tau)\eta(\gamma;t,\alpha)}{C\gamma\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau \leq \eta(\gamma;t,\alpha).$$

We now state the weak type estimate for the modified Hardy operators as a corollary of Theorem 2.1.

**Corollary 2.2.** *Let $\mathcal{U}, \mathcal{V}, \tilde{\mathcal{V}}, \mathcal{V}\circ\mathcal{U}^{-1}$ and $h$ satisfy all the assumptions stated in Theorem 2.1. Then the following conditions are equivalent.*

(a) *There exists $C > 0$ such that*

$$\mathcal{U}^{-1}\Big(\int_{\{t\in(0,\infty):\mathcal{H}_h f(t)>\gamma\}}\mathcal{U}\Big(\gamma\omega(t)\Big)\rho(t)dt\Big) \leq \mathcal{V}^{-1}\Big(\int_0^\infty \mathcal{V}\Big(Cf(t)\phi(t)\Big)\psi(t)dt\Big)$$

*holds for each $\gamma > 0$ and all non-negative functions $f$.*

(b) *There exists $C > 0$ such that*

$$\int_0^t \tilde{\mathcal{V}}\left[\frac{(\inf_{(t,\alpha)} h)w(\tau)\eta(\gamma;t,\alpha)}{C\gamma\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau \leq \eta(\gamma;t,\alpha)$$

*holds for each $0 < t < \alpha$ and $\gamma > 0$.*

Next, we state the weak type result for the conjugate of the modified integral operators.

**Theorem 2.3.** *Let $\mathcal{U}, \mathcal{V}, \tilde{\mathcal{V}}$ and $\mathcal{V} \circ \mathcal{U}^{-1}$ satisfy all the assumptions stated in Theorem 2.1. Suppose the function $w$ is monotone on $\mathbb{R}$. Then the following conditions are equivalent.*

(a) *There exists $C > 0$ such that*

$$\begin{aligned} &\mathcal{U}^{-1}\Big(\int_{\{t\in(0,\infty):\tilde{\mathcal{I}}f(t)>\gamma\}} \mathcal{U}\Big(\gamma\omega(t)\Big)\rho(t)dt\Big) \\ &\le \mathcal{V}^{-1}\Big(\int_0^\infty \mathcal{V}\Big(Cf(t)\phi(t)\Big)\psi(t)dt\Big) \end{aligned} \tag{16}$$

*holds for each $\gamma > 0$ and all non-negative functions $f$.*

(b) *Let $0 \le \epsilon < t < \alpha$ and $\gamma > 0$. Then there exists $C > 0$ such that*

$$\int_t^\alpha \tilde{\mathcal{V}}\left[\frac{(\inf_{(0,t)} w)K(\tau,t)h(\tau)\eta(\gamma;\epsilon,t)}{C\gamma\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau \le \eta(\gamma;\epsilon,t) \tag{17}$$

*and for $0 \le \epsilon < t \le z < \alpha$ the following inequality*

$$\int_z^\alpha \tilde{\mathcal{V}}\left[\frac{(\inf_{y\in(\epsilon,t)} w(y)K(z,y))h(\tau)\eta(\gamma;\epsilon,t)}{C\gamma\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau \le \eta(\gamma;\epsilon,t) \tag{18}$$

*hold, where*

$$\eta(\gamma;\epsilon,t) = \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_\epsilon^t \mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big).$$

Again the result for the conjugate of the modified Hardy operators can be obtained from Theorem 2.3 by considering $K \equiv 1$.

**Corollary 2.4.** *Let $\mathcal{U}, \mathcal{V}, \tilde{\mathcal{V}}$ and $\mathcal{V} \circ \mathcal{U}^{-1}$ satisfy all the assumptions stated in Theorem 2.1. Assume that the function $w$ is monotone on $\mathbb{R}$. Then the following conditions are equivalent.*

(a) *There exists $C > 0$ such that*

$$\mathcal{U}^{-1}\Big(\int_{\{t\in(0,\infty):\tilde{\mathcal{H}}_h f(t)>\gamma\}} \mathcal{U}\Big(\gamma\omega(t)\Big)\rho(t)dt\Big) \le \mathcal{V}^{-1}\Big(\int_0^\infty \mathcal{V}\Big(Cf(t)\phi(t)\Big)\psi(t)dt\Big),$$

*holds for each $\gamma > 0$ and all non-negative functions $f$.*

(b) *There exists $C > 0$ such that*

$$\int_t^\alpha \tilde{\mathcal{V}}\left[\frac{(\inf_{(\epsilon,t)} w)h(y)\eta(\gamma;\epsilon,t)}{C\gamma\phi(y)\psi(y)}\right]\psi(y)dy \le \eta(\gamma;\epsilon,t)$$

*holds for each $0 \le \epsilon < t < \alpha$ and $\gamma > 0$.*

Next, we discuss the extra-weak type results.

**Theorem 2.5.** *Let $\mathcal{U}, \mathcal{V}, \tilde{\mathcal{V}}, \mathcal{V} \circ \mathcal{U}^{-1}$ and $h$ satisfy all the assumptions stated in Theorem 2.1. Then the following conditions are equivalent.*

(a) *There exists $C > 0$ such that*

$$\omega\Big(\{t \in (0,\infty) : \mathcal{I}f(t) > \gamma\}\Big) \leq \Big(\mathcal{U} \circ \mathcal{V}^{-1}\Big)\Big(\int_0^\infty \mathcal{V}\Big(\frac{Cf(t)\phi(t)}{\gamma}\Big)\psi(t)dt\Big) \tag{19}$$

*holds for each $\gamma > 0$ and all non-negative functions $f$.*

(b) *Let $0 < t < \alpha$ and $\gamma > 0$. Then there exists $C > 0$ such that*

$$\int_0^t \tilde{\mathcal{V}}\left[\frac{(\inf_{(t,\alpha)} h)K(t,\tau)w(\tau)\theta(t,\alpha)}{C\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau \leq \theta(t,\alpha) \tag{20}$$

*and for $0 < z \leq t < y < \alpha$ the following*

$$\int_0^z \tilde{\mathcal{V}}\left[\frac{(\inf_{y\in(t,\alpha)} h(y)K(y,z))w(\tau)\theta(t,\alpha)}{C\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau \leq \theta(t,\alpha) \tag{21}$$

*hold, where*

$$\theta(t,\alpha) = \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_t^\alpha \omega(\tau)d\tau\Big).$$

Considering $K \equiv 1$, we obtain the following result.

**Corollary 2.6.** *Let $\mathcal{U}, \mathcal{V}, \tilde{\mathcal{V}}, \mathcal{V} \circ \mathcal{U}^{-1}$ and $h$ satisfy all the assumptions stated in Theorem 2.1. Then the following conditions are equivalent.*

(a) *There exists $C > 0$ such that*

$$\omega\Big(\{t \in (0,\infty) : \mathcal{H}_h f(t) > \gamma\}\Big) \leq \Big(\mathcal{U} \circ \mathcal{V}^{-1}\Big)\Big(\int_0^\infty \mathcal{V}\Big(\frac{Cf(t)\phi(t)}{\gamma}\Big)\psi(t)dt\Big)$$

*holds for each $\gamma > 0$ and all non-negative functions $f$.*

(b) *There exists $C > 0$ such that*

$$\int_0^t \tilde{\mathcal{V}}\left[\frac{(\inf_{(t,\alpha)} h)w(\tau)\Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_t^\alpha \omega\Big)}{C\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau \leq \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_t^\alpha \omega(\tau)d\tau\Big)$$

*holds for each $0 < t < \alpha$.*

Next, we state the extra-weak type result for the conjugate of the modified integral operators.

**Theorem 2.7.** *Let $\mathcal{U}, \mathcal{V}, \tilde{\mathcal{V}}$, and $\mathcal{V} \circ \mathcal{U}^{-1}$ satisfy all the assumptions stated in Theorem 2.1. Suppose the function $w$ is monotone on $\mathbb{R}$. Then the following conditions are equivalent.*

(a) *There exists $C > 0$ such that*

$$\omega\Big(\{t \in (0,\infty) : \tilde{\mathcal{I}}f(t) > \gamma\}\Big) \le \Big(\mathcal{U} \circ \mathcal{V}^{-1}\Big)\Big(\int_0^\infty \mathcal{V}\Big(\frac{Cf(t)\phi(t)}{\gamma}\Big)\psi(t)dt\Big) \tag{22}$$

*holds for each $\gamma > 0$ and all non-negative functions $f$.*

(b) *Let $0 \le \epsilon < t < \alpha$ and $\gamma > 0$. Then there exists $C > 0$ such that*

$$\int_t^\alpha \tilde{\mathcal{V}}\left[\frac{(\inf_{(\epsilon,t)} w)K(\tau,t)h(\tau)\theta(0,t)}{C\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau \le \theta(\epsilon,t) \tag{23}$$

*and for $0 \le \epsilon < t \le z < \alpha$ the following*

$$\int_z^\alpha \tilde{\mathcal{V}}\left[\frac{(\inf_{y\in(\epsilon,t)} w(y)K(z,y))h(\tau)\theta(\epsilon,t)}{C\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau \le \theta(\epsilon,t) \tag{24}$$

*hold, where*

$$\theta(\epsilon,t) = \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_\epsilon^t \omega(\tau)d\tau\Big).$$

Similarly, the extra-weak type result for the modified Hardy operators follows from Theorem 2.7 by considering $K \equiv 1$.

**Corollary 2.8.** *Let $\mathcal{U}, \mathcal{V}, \tilde{\mathcal{V}}$, and $\mathcal{V} \circ \mathcal{U}^{-1}$ satisfy all the assumptions stated in Theorem 2.1. Suppose the function $w$ is monotone on $\mathbb{R}$. Then the following conditions are equivalent.*

(a) *There exists $C > 0$ such that*

$$\omega\Big(\{t \in (0,\infty) : \tilde{\mathcal{H}}_h f(t) > \gamma\}\Big) \le \Big(\mathcal{U} \circ \mathcal{V}^{-1}\Big)\Big(\int_0^\infty \mathcal{V}\Big(\frac{Cf(t)\phi(t)}{\gamma}\Big)\psi(t)dt\Big)$$

*holds for each $\gamma > 0$ and all non-negative functions $f$.*

(b) *There exists $C > 0$ such that*

$$\int_t^\alpha \tilde{\mathcal{V}}\left[\frac{(\inf_{(\epsilon,t)} w)h(y)\Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_\epsilon^t \omega(y)dy\Big)}{C\phi(y)\psi(y)}\right]\psi(y)dy \le \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_\epsilon^t \omega(y)dy\Big)$$

*holds for each $0 \le \epsilon < t < \alpha$.*

We prove Theorem 2.1 in Section 3 and then Theorem 2.3 in Section 4. The proofs of the extra-weak type results are almost similar to the corresponding weak type results. We only sketch the proof of Theorem 2.5. For this, we use the same notations and constructions from Section 3. We use the techniques developed in [9, 11, 13, 14].

## 3. Proof of Theorem 2.1

*Proof.* (b) $\Longrightarrow$ (a). Following the argument from [11], we assume that $f$ is non-negative with support $(0, L)$ for some $L > 0$. We define a decreasing sequence $\{\xi_m\}_{m\geq 0}$ with the iteration $\xi_0 = L$ and given $\xi_m$, we consider

$$P(\xi_m) = \int_0^{\xi_m} K(\xi_m, \tau) f(\tau) w(\tau) d\tau = (M+1)^{-m} P(\xi_0). \tag{25}$$

Now, using the inequalities (2) and (25) we have

$$\begin{aligned}
P(\xi_m) &= (M+1)^2 P(\xi_{m+2}) \\
&= (M+1)^2 \int_0^{\xi_{m+2}} K(\xi_{m+2}, \tau) f(\tau) w(\tau) d\tau \\
&= (M+1)^2 \Big[ \int_0^{\xi_{m+3}} + \int_{\xi_{m+3}}^{\xi_{m+2}} \Big] K(\xi_{m+2}, \tau) f(\tau) w(\tau) d\tau \\
&\leq (M+1)^2 \Bigg[ \int_0^{\xi_{m+3}} M \Big\{ K(\xi_{m+2}, \xi_{m+3}) + K(\xi_{m+3}, \tau) \Big\} \\
&\qquad + \int_{\xi_{m+3}}^{\xi_{m+2}} K(\xi_{m+2}, \tau) \Bigg] f(\tau) w(\tau) d\tau \\
&\leq (M+1)^3 \Bigg[ K(\xi_{m+2}, \xi_{m+3}) \int_0^{\xi_{m+3}} f(\tau) w(\tau) d\tau \\
&\qquad + \int_{\xi_{m+3}}^{\xi_{m+2}} K(\xi_{m+2}, \tau) f(\tau) w(\tau) d\tau \Bigg] \\
&\quad + M(M+1)^2 \int_0^{\xi_{m+3}} K(\xi_{m+3}, \tau) f(\tau) w(\tau) d\tau. \qquad (26)
\end{aligned}$$

From the construction of $P(\xi_m)$, we have

$$\begin{aligned}
\int_0^{\xi_{m+3}} K(\xi_{m+3}, \tau) f(\tau) w(\tau) d\tau = P(\xi_{m+3}) &= (M+1)^{-(m+3)} P(\xi_0) \\
&= (M+1)^{-3} P(\xi_m).
\end{aligned}$$

Thus (26) implies that

$$\begin{aligned}
P(\xi_m) \leq (M+1)^4 \Bigg[ & K(\xi_{m+2}, \xi_{m+3}) \int_0^{\xi_{m+3}} f(\tau) w(\tau) d\tau \\
& + \int_{\xi_{m+3}}^{\xi_{m+2}} K(\xi_{m+2}, \tau) f(\tau) w(\tau) d\tau \Bigg]. \qquad (27)
\end{aligned}$$

For $k = 1, 2$, we define $\delta_{k,m} = \inf \Omega_{k,m}$ and $\beta_{k,m} = \sup \Omega_{k,m}$, where

$$\Omega_{1,m} = \left\{ y \in (\xi_{m+1}, \xi_m) : h(y) K(\xi_{m+2}, \xi_{m+3}) \int_0^{\xi_{m+3}} f(\tau) w(\tau) d\tau > \frac{\gamma}{2(M+1)^4} \right\},$$

$$\Omega_{2,m} = \left\{ z \in (\xi_{m+1}, \xi_m) : h(z) \int_{\xi_{m+3}}^{\xi_{m+2}} K(\xi_{m+2}, \tau) f(\tau) w(\tau) d\tau > \frac{\gamma}{2(M+1)^4} \right\}.$$

Thus (25) and (27) gives

$$\begin{aligned} &\Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big( \int_{\{t : \mathcal{I}f(t) > \gamma\}} \mathcal{U}\Big(\gamma \omega(t)\Big) \rho(t) dt \Big) \\ &\leq \sum_{m \geq 0} \Big\{ \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big( \int_{\Omega_{1,m}} \mathcal{U}\Big(\gamma \omega(t)\Big) \rho(t) dt \Big) \\ &\qquad + \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big( \int_{\Omega_{2,m}} \mathcal{U}\Big(\gamma \omega(t)\Big) \rho(t) dt \Big) \Big\}. \end{aligned} \tag{28}$$

Next, we estimate the first sum of (28). For this, we define a cover sequence $\{x_k\}$ for the interval $(0, L)$ with the iteration $x_0 = L$ and

$$\int_0^{x_k} f(\tau) w(\tau) d\tau = 2 \int_0^{x_{k+1}} f(\tau) w(\tau) d\tau.$$

Then $\{x_k\}$ is decresing and verifies

$$\int_0^{x_k} f(\tau) w(\tau) d\tau = 4 \int_{x_{k+2}}^{x_{k+1}} f(\tau) w(\tau) d\tau.$$

Using [9, Lemma 1] we construct a subsequence $\{x'_n\}$ from the sequence $\{x_k\}$ with the iteration $x'_0 = x_0$ and if $x_{k+1} \leq \xi_m < x_k$, then $x'_{n+1} = x_{k+1}$, otherwise we delete the term $x_{k+1}$ and continue the process. Thus, we get a subsequence $\{x'_n\}$ of $\{x_k\}$. Let $\tilde{\delta}_{1,n} = \inf \tilde{\Omega}_{1,n}$ and $\tilde{\beta}_{1,n} = \sup \tilde{\Omega}_{1,n}$, where $\tilde{\Omega}_{1,n} = \cup_{\{m : x'_{n+1} \leq \xi_{m+3} < x'_n\}} \Omega_{1,m}$. Now, if $x_{k+1} = x'_{n+1} \leq \xi_{m+3} < x'_n$, then by construction $\xi_{m+3} \leq x_k$ and $x'_{n+2} \leq x_{k+2}$. Suppose that $\tilde{\Omega}_{1,n} \neq \phi$. Then for $t \in \tilde{\Omega}_{1,n}$, we have

$$\begin{aligned} \frac{\gamma}{2(M+1)^4} &< h(t) K(\xi_{m+2}, \xi_{m+3}) \int_0^{\xi_{m+3}} f(\tau) w(\tau) d(\tau) \\ &\leq h(t) K(t, \xi_{m+3}) \int_0^{x_k} f(\tau) w(\tau) d\tau. \end{aligned} \tag{29}$$

As the estimate (29) holds for each $t \in \tilde{\Omega}_{1,n}$, thus

$$\gamma \leq 8(M+1)^4 \inf_{t \in (\tilde{\delta}_{1,n}, \tilde{\beta}_{1,n})} h(t) K(t, \xi_{m+3}) \int_{x'_{n+2}}^{x'_{n+1}} f(\tau) w(\tau) d(\tau). \tag{30}$$

Let us denote

$$\eta(\gamma;\tilde{\delta}_{1,n},\tilde{\beta}_{1,n}) = \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\delta}_{1,n}}^{\tilde{\beta}_{1,n}}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big).$$

We use (10) and (30) to obtain

$$\begin{aligned}2\eta(\gamma;\tilde{\delta}_{1,n},\tilde{\beta}_{1,n}) \le & \int_{x'_{n+2}}^{x'_{n+1}}\Big[16(M+1)^4Cf(\tau)\phi(\tau)\Big] \\ & \Bigg[\frac{\big(\inf_t h(t)K(t,\xi_{m+3})\big)w(\tau)\eta(\gamma;\tilde{\delta}_{1,n},\tilde{\beta}_{1,n})}{C\gamma\phi(\tau)\psi(\tau)}\Bigg]\psi(\tau)d\tau \\ \le & \int_{x'_{n+2}}^{x'_{n+1}}\mathcal{V}\Big(16(M+1)^4Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau \\ & + \int_{x'_{n+2}}^{x'_{n+1}}\tilde{\mathcal{V}}\Bigg[\frac{\big(\inf_t h(t)K(t,\xi_{m+3})\big)w(\tau)\eta(\gamma;\tilde{\delta}_{1,n},\tilde{\beta}_{1,n})}{C\gamma\phi(\tau)\psi(\tau)}\Bigg]\psi(\tau)d\tau. \end{aligned} \tag{31}$$

Since $K(t,\tau)$ is non-increasing in $\tau$, thus from (15) we get

$$\begin{aligned}&\int_{x'_{n+2}}^{x'_{n+1}}\tilde{\mathcal{V}}\Bigg[\frac{\big(\inf_t h(t)K(t,\xi_{m+3})\big)w(\tau)\eta(\gamma;\tilde{\delta}_{1,n},\tilde{\beta}_{1,n})}{C\gamma\phi(\tau)\psi(\tau)}\Bigg]\psi(\tau)d\tau \\ &\le \eta(\gamma;\tilde{\delta}_{1,n},\tilde{\beta}_{1,n}).\end{aligned} \tag{32}$$

Combining (31) and (32), we have

$$\begin{aligned}&\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\Omega}_{1,n}}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big) \\ &\le \int_{x'_{n+2}}^{x'_{n+1}}\mathcal{V}\Big(16(M+1)^4Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau.\end{aligned} \tag{33}$$

Applying sub-additivity of $\mathcal{V}\circ\mathcal{U}^{-1}$, we obtain

$$\begin{aligned}&\sum_{m\ge 0}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\Omega_{1,m}}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big) \\ &\le \sum_{n\ge 0}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\Omega}_{1,n}}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big) \\ &\le \int_0^{\infty}\mathcal{V}\Big(16(M+1)^4Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau.\end{aligned} \tag{34}$$

We now estimate the second part of (28). We denote

$$\eta(\gamma;\delta_{2,m},\beta_{2,m}) = \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\delta_{2,m}}^{\beta_{2,m}}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big).$$

If $t \in \Omega_{2,m}$, then we have

$$\frac{\gamma}{2(M+1)^4} < h(t) \int_{\xi_{m+3}}^{\xi_{m+2}} K(\xi_{m+2}, \tau) f(\tau) w(\tau) d\tau. \tag{35}$$

As the estimate (35) holds for each $t \in \Omega_{2,m}$, thus

$$\gamma \leq 2(M+1)^4 \inf_{t \in (\delta_{2,m}, \beta_{2,m})} h(t) \int_{\xi_{m+3}}^{\xi_{m+2}} K(\xi_{m+2}, \tau) f(\tau) w(\tau) d\tau. \tag{36}$$

Now, applying (10) and (36) we have

$$\begin{aligned} 2\eta(\gamma; \delta_{2,m}, \beta_{2,m}) \leq & \int_{\xi_{m+3}}^{\xi_{m+2}} \Big[ 4(M+1)^4 C f(\tau)\phi(\tau) \Big] \\ & \left[ \frac{\big(\inf h(t)\big) K(\xi_{m+2}, \tau) w(\tau) \eta(\gamma; \delta_{2,m}, \beta_{2,m})}{C\gamma\phi(\tau)\psi(\tau)} \right] \psi(\tau) d\tau \\ \leq & \int_{\xi_{m+3}}^{\xi_{m+2}} \mathcal{V}\Big(4(M+1)^4 C f(\tau)\phi(\tau)\Big) \psi(\tau) d\tau \\ & + \int_{\xi_{m+3}}^{\xi_{m+2}} \tilde{\mathcal{V}} \left[ \frac{\big(\inf h(t)\big) K(\xi_{m+2}, \tau) w(\tau) \eta(\gamma; \delta_{2,m}, \beta_{2,m})}{C\gamma\phi(\tau)\psi(\tau)} \right] \psi(\tau) d\tau. \end{aligned} \tag{37}$$

As $\xi_{m+2} \leq \delta_{2,m} \leq \beta_{2,m}$, thus from the condition (14) we have

$$\begin{aligned} & \int_{\xi_{m+3}}^{\xi_{m+2}} \tilde{\mathcal{V}} \left[ \frac{\big(\inf h(t)\big) K(\xi_{m+2}, \tau) w(\tau) \eta(\gamma; \delta_{2,m}, \beta_{2,m})}{C\gamma\phi(\tau)\psi(\tau)} \right] \psi(\tau) d\tau \\ & \leq \eta(\gamma; \delta_{2,m}, \beta_{2,m}). \end{aligned} \tag{38}$$

Combining (37), (38) and then summing up in $m$, we obtain

$$\begin{aligned} & \sum_{m \geq 0} \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big) \Big( \int_{\Omega_{2,m}} \mathcal{U}\Big(\gamma\omega(\tau)\Big) \rho(\tau) d\tau \Big) \\ & \leq \int_0^\infty \mathcal{V}\Big(4(M+1)^4 C f(\tau)\phi(\tau)\Big) \psi(\tau) d\tau. \end{aligned} \tag{39}$$

Thus from (28), (34) and (39) we obtain (15) with constant $32(M+1)^4 C$.

(a) $\Longrightarrow$ (b). Let $0 < t < \alpha$ and for each $N \in \mathbb{N}$ we consider the set

$$E_N = \{0 < s < t : \frac{1}{N} \leq K(t,s), w(s) \leq N\}.$$

Applying $\tilde{\mathcal{V}}(s) \leq s\tilde{v}(s)$ we obtain

$$\begin{aligned} & \int_{E_N} \tilde{\mathcal{V}}\Big( \frac{\lambda(\inf h) K(t,y) w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)} \Big) \Big( \frac{\psi(y)+1/k}{\lambda} \Big) dy \\ & \leq ltN^2 (\inf h) \tilde{v}(\lambda(\inf h) lkN^2) < \infty. \end{aligned}$$

We choose $\lambda > 0$ such that for each $\mu > 0$ we have

$$\int_{E_N} \tilde{\mathcal{V}}\left(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\left(\frac{\psi(y)+1/k}{\lambda}\right)dy = (1+\mu)C\gamma.$$

We consider the function $f$ as

$$f(y) = \frac{1}{C}\tilde{\mathcal{V}}\left(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\lambda(\inf h)K(t,y)w(y)}\chi_{E_N}(y),$$

where $C$ is the constatnt in the etimate (13). Let $t < \beta < \alpha$. Then

$$\begin{aligned}\mathcal{I}f(\beta) &= h(\beta)\int_0^\beta K(\beta,y)\frac{1}{C}\tilde{\mathcal{V}}\left(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\\ &\qquad\left(\frac{\psi(y)+1/k}{\lambda(\inf h)K(t,y)w(y)}\right)\chi_{E_N}(y)w(y)dy\\ &\geq \int_{E_N}\frac{1}{C}\tilde{\mathcal{V}}\left(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\left(\frac{\psi(y)+1/k}{\lambda}\right)dy\\ &= (1+\mu)\gamma > \gamma.\end{aligned}$$

This implies that

$$(t,\alpha) \subset \{y : \mathcal{I}f(y) > \gamma\}.$$

Thus using (11) and (13) we obtain

$$\begin{aligned}&\eta(\gamma;t,\alpha)\\ &= \left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_t^\alpha \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\right)\\ &\leq \left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\{\mathcal{I}f>\gamma\}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\right)\\ &\leq \int_{E_N}\mathcal{V}\left(\tilde{\mathcal{V}}\left(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\left(\frac{\psi(y)+1/k}{\lambda(\inf h)K(t,y)w(y)}\right)\phi(y)\right)\psi(y)dy\\ &\leq \int_{E_N}\tilde{\mathcal{V}}\left(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\psi(y)dy\\ &= (1+\mu)C\lambda\gamma.\end{aligned}$$

As $\tilde{\mathcal{V}}(s)/s$ increases with $s$, thus we have

$$\begin{aligned}&\int_{E_N}\tilde{\mathcal{V}}\left(\frac{(\inf h)K(t,y)w(y)\eta(\gamma;t,\alpha)}{(1+\mu)C\gamma(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\eta(\gamma;t,\alpha)}dy\\ &\leq \int_{E_N}\tilde{\mathcal{V}}\left(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{(1+\mu)C\lambda\gamma}dy = 1.\end{aligned}$$

By the monotone convergence theorem, we have

$$\int_{E_N}\tilde{\mathcal{V}}\left(\frac{(\inf h)K(t,y)w(y)\eta(\gamma;t,\alpha)}{(1+\mu)C\gamma(\phi(y)+1/l)\psi(y)}\right)\frac{\psi(y)}{\eta(\gamma;t,\alpha)}dy \leq 1.$$

Letting $l, N \to \infty$ and $\mu \to 0^+$, we get

$$\int_0^t \tilde{\mathcal{V}}\left(\frac{(\inf h)K(t,y)w(y)\eta(\gamma;t,\alpha)}{C\gamma\phi(y)\psi(y)}\right)\psi(y)dy \le \eta(\gamma;t,\alpha).$$

Thus we obtain (14). Next we prove (15). For $0 < z \le t < \alpha$ and $N \in \mathbb{N}$ let us consider the set $F_N$ as

$$F_N = \{0 < s < z : \frac{1}{N} \le w(s) \le N\}.$$

Next we choose $\pi > 0$ such that for each $\kappa > 0$ the following inequality holds.

$$\int_{F_N} \tilde{\mathcal{V}}\left(\frac{\pi\big(\inf h(y)K(y,z)\big)w(\tau)}{(\phi(\tau)+1/l)(\psi(\tau)+1/k)}\right)\left(\frac{\psi(\tau)+1/k}{\pi}\right)d\tau = (1+\kappa)C\gamma,$$

where $C$ is the constant of (13) and $l, k \in \mathbb{N}$. Let us consider the function $f$ as

$$\begin{aligned} &f(\tau) \\ &= \frac{1}{C}\tilde{\mathcal{V}}\left(\frac{\pi(\inf_{y\in(t,\alpha)} h(y)K(y,z))w(\tau)}{(\phi(\tau)+1/l)(\psi(\tau)+1/k)}\right)\frac{\psi(\tau)+1/k}{\pi(\inf_{y\in(t,\alpha)} h(y)K(y,z))w(\tau)}\chi_{F_N}(\tau). \end{aligned}$$

If $\tau \in F_N$ and $t < \beta < \alpha$, then $K(\beta,\tau) \ge K(\beta,z)$, which implies

$$h(\beta)K(\beta,z) \ge \inf_{y\in(t,\alpha)} h(y)K(y,z).$$

Thus for $t < \beta < \alpha$, we have

$$\begin{aligned} &\mathcal{I}f(\beta) \\ &= h(\beta)\int_0^\beta K(\beta,\tau)f(\tau)w(\tau)d\tau \\ &\ge h(\beta)\int_{F_N} K(\beta,z)\frac{1}{C}\tilde{\mathcal{V}}\left(\frac{\pi(\inf h(y)K(y,z))w(\tau)}{(\phi(\tau)+1/l)(\psi(\tau)+1/k)}\right)\frac{\psi(\tau)+1/k}{\pi(\inf h(y)K(y,z))w(\tau)}w(\tau)d\tau \\ &\ge (1+\kappa)\gamma > \gamma. \end{aligned}$$

This implies that

$$(t,\alpha) \subset \{\tau : \mathcal{I}f(\tau) > \gamma\}.$$

And then the rest of the proof proceeds similarly. Hence the proof is complete. □

## 4. Proof of Theorem 2.3

*Proof.* (b) $\Longrightarrow$ (a). Let $f$ be a non-negative function on $\mathbb{R}$ with support $(0,\infty)$. We define an increasing sequence $\{\xi_m\}_{m\ge 0}$ with the iteration $\xi_0 = 0$ and given $\xi_m$, we consider

$$Q(\xi_m) = \int_{\xi_m}^\infty K(\tau,\xi_m)f(\tau)h(\tau)d\tau = (M+1)^{-m}Q(\xi_0). \tag{40}$$

Now, using (2) and (40) we have

$$Q(\xi_m) = (M+1)^2 Q(\xi_{m+2})$$

$$
\begin{aligned}
&= (M+1)^2 \int_{\xi_{m+2}}^{\infty} K(\tau,\xi_{m+2}) f(\tau)h(\tau)d\tau \\
&= (M+1)^2 \Big[ \int_{\xi_{m+2}}^{\xi_{m+3}} + \int_{\xi_{m+3}}^{\infty} \Big] K(\tau,\xi_{m+2}) f(\tau)h(\tau)d\tau \\
&\leq (M+1)^2 \Bigg[ \int_{\xi_{m+2}}^{\xi_{m+3}} K(\tau,\xi_{m+2}) \\
&\qquad + \int_{\xi_{m+3}}^{\infty} M\Big\{ K(\tau,\xi_{m+3}) + K(\xi_{m+3},\xi_{m+2}) \Big\} \Bigg] f(\tau)h(\tau)d\tau \\
&\leq (M+1)^3 \Bigg[ \int_{\xi_{m+2}}^{\xi_{m+3}} K(\tau,\xi_{m+2}) f(\tau)h(\tau)d\tau \\
&\qquad + K(\xi_{m+3},\xi_{m+2}) \int_{\xi_{m+3}}^{\infty} f(\tau)h(\tau)d\tau \Bigg] \\
&\quad + M(M+1)^2 \int_{\xi_{m+3}}^{\infty} K(\tau,\xi_{m+3}) f(\tau)h(\tau)d\tau. \qquad (41)
\end{aligned}
$$

From the construction of $Q(\xi_m)$

$$
\begin{aligned}
\int_{\xi_{m+3}}^{\infty} K(\tau,\xi_{m+3}) f(\tau)h(\tau)d\tau = Q(\xi_{m+3}) &= (M+1)^{-(m+3)} Q(\xi_0) \\
&= (M+1)^{-3} Q(\xi_m).
\end{aligned}
$$

Thus (41) implies that

$$
\begin{aligned}
Q(\xi_m) \leq (M+1)^4 \Bigg[ &\int_{\xi_{m+2}}^{\xi_{m+3}} K(\tau,\xi_{m+2}) f(\tau)h(\tau)d\tau \\
&+ K(\xi_{m+3},\xi_{m+2}) \int_{\xi_{m+3}}^{\infty} f(\tau)h(\tau)d\tau \Bigg]. \qquad (42)
\end{aligned}
$$

For $k = 1, 2$, we define $\delta_{k,m} = \inf \Omega_{k,m}$ and $\beta_{k,m} = \sup \Omega_{k,m}$, where

$$
\Omega_{1,m} = \left\{ y \in (\xi_m, \xi_{m+1}) : w(y) \int_{\xi_{m+2}}^{\xi_{m+3}} K(\tau,\xi_{m+2}) f(\tau)h(\tau)d\tau > \frac{\gamma}{2(M+1)^4} \right\},
$$

$$
\Omega_{2,m} = \left\{ z \in (\xi_m, \xi_{m+1}) : w(z) K(\xi_{m+3},\xi_{m+2}) \int_{\xi_{m+3}}^{\infty} f(\tau)h(\tau)d\tau > \frac{\gamma}{2(M+1)^4} \right\}.
$$

From the construction of $Q(\xi_m)$ and (42) we have

$$
\begin{aligned}
&\Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big) \Big( \int_{\{t : \tilde{\mathcal{I}} f(t) > \gamma\}} \mathcal{U}\Big(\gamma\omega(t)\Big)\rho(t)dt \Big) \\
&\leq \sum_{m \geq 0} \Bigg\{ \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big) \Big( \int_{\Omega_{1,m}} \mathcal{U}\Big(\gamma\omega(t)\Big)\rho(t)dt \Big)
\end{aligned}
$$

$$+\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\Omega_{2,m}}\mathcal{U}\Big(\gamma\omega(t)\Big)\rho(t)dt\Big)\Big\}. \tag{43}$$

We now estimate the first sum of (43). Now, if $t\in\Omega_{1,m}$, then we have

$$\frac{\gamma}{2(M+1)^4} < w(t)\int_{\xi_{m+2}}^{\xi_{m+3}} K(\tau,\xi_{m+2})f(\tau)h(\tau)d\tau. \tag{44}$$

Since the inequality valids for each $t\in\Omega_{1,m}$, thus

$$\gamma \le 2(M+1)^4\big(\inf_{(\delta_{1,m},\beta_{1,m})} w(t)\big)\int_{\xi_{m+2}}^{\xi_{m+3}} K(\tau,\xi_{m+2})f(\tau)h(\tau)d\tau. \tag{45}$$

Let us denote

$$\eta(\gamma;\delta_{1,m},\beta_{1,m}) = \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\delta_{1,m}}^{\beta_{1,m}}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big).$$

Applying (10) and (45) we obtain

$$\begin{aligned}
2\eta(\gamma;\delta_{1,m},\beta_{1,m}) \le& \int_{\xi_{m+2}}^{\xi_{m+3}}\Big[4(M+1)^4Cf(\tau)\phi(\tau)\Big]\\
&\left[\frac{\big(\inf w(t)\big)K(\tau,\xi_{m+2})h(\tau)\eta(\gamma;\delta_{1,m},\beta_{1,m})}{C\gamma\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau\\
\le& \int_{\xi_{m+2}}^{\xi_{m+3}}\mathcal{V}\Big(4(M+1)^4Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau\\
&+\int_{\xi_{m+2}}^{\xi_{m+3}}\tilde{\mathcal{V}}\left[\frac{\big(\inf w(t)\big)K(\tau,\xi_{m+2})h(\tau)\eta(\gamma;\delta_{1,m},\beta_{1,m})}{C\gamma\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau.
\end{aligned} \tag{46}$$

As $\xi_{m+2}\ge\beta_{1,m}\ge\delta_{1,m}$, thus from (17) we get

$$\begin{aligned}
&\int_{\xi_{m+2}}^{\xi_{m+3}}\tilde{\mathcal{V}}\left[\frac{\big(\inf w(t)\big)K(\tau,\xi_{m+2})h(\tau)\eta(\gamma;\delta_{1,m},\beta_{1,m})}{C\gamma\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau\\
&\le \eta(\gamma;\delta_{1,m},\beta_{1,m}).
\end{aligned} \tag{47}$$

Combining (46), (47) and then summing up in $m$ we get

$$\begin{aligned}
&\sum_{m\ge0}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\Omega_{1,m}}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big)\\
&\le \int_0^\infty\mathcal{V}\Big(4(M+1)^4Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau.
\end{aligned} \tag{48}$$

For the second sum of the inequality (43), we define a sequence $\{x_k\}$ on $(0,\infty)$ with the iteration $x_0=0$ and given $\{x_k\}$ we define $\{x_{k+1}\}$ as

$$\int_{x_k}^\infty f(\tau)h(\tau)d\tau = 2\int_{x_{k+1}}^\infty f(\tau)h(\tau)d\tau.$$

Then $\{x_k\}$ is increasing and verifies

$$\int_{x_k}^{\infty} f(\tau)h(\tau)d\tau = 4\int_{x_{k+1}}^{x_{k+2}} f(\tau)h(\tau)d\tau.$$

From the sequence $\{x_k\}$ we define a subsequence $\{x'_n\}$ with the iteration as $x'_0 = x_0$ and if $x_k < \xi_m \leq x_{k+1}$, then $x'_{n+1} = x_{k+1}$, otherwise we delete the term $x_{k+1}$ and continue the process. Thus, we get a subsequence $\{x'_n\}$ of $\{x_k\}$. Let $\tilde{\delta}_{2,n} = \inf \tilde{\Omega}_{2,n}$ and $\tilde{\beta}_{2,n} = \sup \tilde{\Omega}_{2,n}$, where $\tilde{\Omega}_{2,n} = \cup_{\{m: x'_n < \xi_{m+3} \leq x'_{n+1}\}} \Omega_{2,m}$. Now, if $x_{k+1} = x'_{n+1} \geq \xi_{m+3} > x'_n$, then by construction $\xi_{m+3} > x_k$ and $x'_{n+2} \geq x_{k+2}$. Let us suppose $\tilde{\Omega}_{2,n} \neq \phi$. Thus for $t \in \tilde{\Omega}_{2,n}$, we have

$$\begin{aligned} \frac{\gamma}{2(M+1)^4} &< w(t)K(\xi_{m+3}, \xi_{m+2}) \int_{\xi_{m+3}}^{\infty} f(\tau)h(\tau)d(\tau) \\ &\leq 4w(t)K(\xi_{m+3}, t) \int_{x_k}^{\infty} f(\tau)h(\tau)d\tau. \end{aligned} \tag{49}$$

As the estimate (49) holds for each $t \in \tilde{\Omega}_{2,n}$, thus

$$\gamma \leq 8(M+1)^4 \inf_{(\tilde{\delta}_{2,n}, \tilde{\beta}_{2,n})} w(t)K(\xi_{m+3}, t) \int_{x'_{n+1}}^{x'_{n+2}} f(\tau)h(\tau)d(\tau). \tag{50}$$

Let us denote

$$\eta(\gamma; \tilde{\delta}_{2,n}, \tilde{\beta}_{2,n}) = \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\delta}_{2,n}}^{\tilde{\beta}_{2,n}} \mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big).$$

Using (10) and (50), we have

$$\begin{aligned} 2\eta(\gamma; \tilde{\delta}_{2,n}, \tilde{\beta}_{2,n}) \leq &\int_{x'_{n+1}}^{x'_{n+2}} \Big[16(M+1)^4 Cf(\tau)\phi(\tau)\Big] \\ &\left[\frac{\big(\inf w(t)K(\xi_{m+3}, t)\big)h(\tau)\eta(\gamma; \tilde{\delta}_{2,n}, \tilde{\beta}_{2,n})}{C\gamma\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau \\ \leq &\int_{x'_{n+1}}^{x'_{n+2}} \mathcal{V}\Big(16(M+1)^4 Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau \\ &+ \int_{x'_{n+1}}^{x'_{n+2}} \tilde{\mathcal{V}}\left[\frac{\big(\inf w(t)K(\xi_{m+3}, t)\big)h(\tau)\eta(\gamma; \tilde{\delta}_{2,n}, \tilde{\beta}_{2,n})}{C\gamma\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau. \end{aligned} \tag{51}$$

Since the kernel $K$ is increasing in the first variable, thus from (18) we have

$$\begin{aligned} &\int_{x'_{n+1}}^{x'_{n+2}} \tilde{\mathcal{V}}\left[\frac{\big(\inf w(t)K(\xi_{m+3}, t)\big)h(\tau)\eta(\gamma; \tilde{\delta}_{2,n}, \tilde{\beta}_{2,n})}{C\gamma\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau \\ &\leq \eta(\gamma; \tilde{\delta}_{2,n}, \tilde{\beta}_{2,n}). \end{aligned} \tag{52}$$

Combining (51) and (52) we obtain

$$\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\Omega}_{2,n}}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big)$$

$$\le \int_{x'_{n+1}}^{x'_{n+2}}\mathcal{V}\Big(16(M+1)^4Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau. \tag{53}$$

Thus, we have

$$\sum_{m\ge 0}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\Omega_{2,m}}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big)$$

$$\le \sum_{n\ge 0}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\Omega}_{2,n}}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big)$$

$$\le \int_{0}^{\infty}\mathcal{V}\Big(16(M+1)^4Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau. \tag{54}$$

Thus from (48) and (54) we obtain (16) with constant $32(M+1)^4C$.

(a) $\Longrightarrow$ (b). For each $m\in\mathbb{N}$, we consider the set

$$E_m=\Big\{x\in(t,\alpha):\frac{1}{m}\le K(x,t),h(x)\le m\Big\}.$$

We have

$$\int_{E_m}\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\Big(\frac{\psi(y)+1/k}{\lambda}\Big)dy$$
$$\le lm^2(\alpha-t)\big(\inf w\big)\tilde{v}(\lambda lkm^2\inf w)<\infty$$

for each $l,k\in\mathbb{N}$ and $\lambda>0$. Thus for each $\mu>0$ we choose $\lambda>0$ suitably such that

$$\int_{E_m}\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\Big(\frac{\psi(y)+1/k}{\lambda}\Big)dy=(1+\mu)C\gamma,$$

where $C$ is the constant of (16). For $0\le\epsilon<t<\alpha$, we consider

$$f(y)=\frac{1}{C}\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\frac{\psi(y)+1/k}{\lambda\big(\inf w\big)K(y,t)h(y)}\chi_{E_m}(y).$$

If $0\le\epsilon<\beta<t$, then

$$\tilde{\mathcal{I}}f(\beta)$$
$$=w(\beta)\int_{E_m}K(y,\beta)\Big[\frac{1}{C}\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\frac{\psi(y)+1/k}{\lambda\big(\inf w\big)K(y,t)h(y)}\Big]h(y)dy$$
$$\ge\int_{E_m}\frac{1}{C}\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\Big(\frac{\psi(y)+1/k}{\lambda}\Big)dy$$
$$=(1+\mu)\gamma>\gamma.$$

This implies that

$$(\epsilon, t) \subset \{y : \tilde{\mathcal{I}}f(y) > \gamma\}.$$

Thus applying (16) we obtain

$$\begin{aligned}
&\eta(\gamma;\epsilon,t)\\
&= \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_\epsilon^t \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big)\\
&\le \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\{\tilde{\mathcal{I}}f>\gamma\}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big)\\
&\le \int_{E_m} \mathcal{V}\left(\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\frac{\psi(y)+1/k}{\lambda\big(\inf w\big)K(y,t)h(y)}\phi(y)\right)\psi(y)dy\\
&\le \int_{E_m} \tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\psi(y)dy\\
&= (1+\mu)C\lambda\gamma.
\end{aligned}$$

Since $\tilde{\mathcal{V}}(s)/s$ increases as $s$ increases, therefore

$$\begin{aligned}
&\int_{E_m} \tilde{\mathcal{V}}\Big(\frac{\big(\inf w\big)K(y,t)h(y)\eta(\gamma;0,t)}{(1+\mu)C\gamma(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\frac{\psi(y)+1/k}{\eta(\gamma;\epsilon,t)}dy\\
&\le \int_{E_m} \tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\frac{\psi(y)+1/k}{(1+\mu)C\lambda\gamma}dy = 1.
\end{aligned}$$

Using the monotone convergence theorem, we obtain

$$\int_{E_m} \tilde{\mathcal{V}}\Big(\frac{\big(\inf w\big)K(y,t)h(y)\eta(\gamma;\epsilon,t)}{(1+\mu)C\gamma(\phi(y)+1/l)\psi(y)}\Big)\frac{\psi(y)}{\eta(\gamma;\epsilon,t)}dy \le 1.$$

Letting $l, m \to \infty$ and $\mu \to 0^+$, we thus obtain

$$\int_t^\alpha \tilde{\mathcal{V}}\left(\frac{\big(\inf w\big)K(y,t)h(y)\eta(\gamma;\epsilon,t)}{C\gamma\phi(y)\psi(y)}\right)\psi(y)dy \le \eta(\gamma;\epsilon,t).$$

Thus we obtain (17). Next we state the proof of (18). Let $0 \le \epsilon < t \le z < \alpha$ and for $m \in \mathbb{N}$ we define the set $F_m$ as

$$F_m = \{s \in (z,\alpha) : \frac{1}{m} \le h(s) \le m\}.$$

Given $\kappa > 0$ we choose $\pi > 0$ such that

$$\int_{F_m} \tilde{\mathcal{V}}\Big(\frac{\pi\big(\inf w(y)K(z,y)\big)h(x)}{(\phi(x)+1/l)(\psi(x)+1/k)}\Big)\Big(\frac{\psi(x)+1/k}{\pi}\Big)dx = (1+\kappa)C\gamma, \tag{55}$$

where $C$ is the constant of (16) and $l, k \in \mathbb{N}$. Let us define the function $f$ as

$$f(x) = \frac{1}{C}\tilde{\mathcal{V}}\Big(\frac{\pi\big(\inf_{(\epsilon,t)} w(y)K(z,y)\big)h(x)}{(\phi(x)+1/l)(\psi(x)+1/k)}\Big)\frac{\psi(x)+1/k}{\pi\big(\inf_{(\epsilon,t)} w(y)K(z,y)\big)h(x)}\chi_{F_m}(x).$$

If $x \in F_m$ and $0 \leq \epsilon < \beta < t$, then

$$\inf_{y\in(\epsilon,t)} w(y)K(z,y) \leq w(\beta)K(x,\beta). \tag{56}$$

Now, combining (55) and (56) and for $0 \leq \epsilon < \beta < t$, we have

$$\begin{aligned}\tilde{\mathcal{I}}f(\beta) &= w(\beta)\int_\beta^\infty K(x,\beta)f(x)h(x)dx \\ &\geq (1+\kappa)\gamma > \gamma.\end{aligned}$$

This implies that

$$(\epsilon,t) \subset \{x : \tilde{\mathcal{I}}f(x) > \gamma\}.$$

Thus applying the weak type boundedness of $\tilde{\mathcal{I}}$, we obtain

$$\begin{aligned}&\eta(\gamma;\epsilon,t)\\ &= \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\bigg(\int_\epsilon^t \mathcal{U}\Big(\gamma\omega(x)\Big)\rho(x)dx\bigg)\\ &\quad\leq \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\bigg(\int_{\{\tilde{\mathcal{I}}f>\gamma\}} \mathcal{U}\Big(\gamma\omega(x)\Big)\rho(x)dx\bigg)\\ &\leq \int_{F_m}\mathcal{V}\Bigg(\tilde{\mathcal{V}}\bigg(\frac{\pi\big(\inf w(y)K(z,y)\big)h(x)}{(\phi(x)+1/l)(\psi(x)+1/k)}\bigg)\frac{\psi(x)+1/k}{\pi\big(\inf w(y)K(z,y)\big)h(x)}\phi(x)\Bigg)\psi(x)dx\\ &\leq \int_{F_m}\tilde{\mathcal{V}}\bigg(\frac{\pi\big(\inf w(y)K(z,y)\big)h(x)}{(\phi(x)+1/l)(\psi(x)+1/k)}\bigg)\psi(x)dx\\ &= (1+\kappa)C\pi\gamma.\end{aligned}$$

Since $\tilde{\mathcal{V}}(s)/s$ increases as $s$ increases, therefore

$$\begin{aligned}&\int_{F_m}\tilde{\mathcal{V}}\bigg(\frac{\big(\inf w(y)K(z,y)\big)h(x)\eta(\gamma;\epsilon,t)}{(1+\kappa)C\gamma(\phi(x)+1/l)(\psi(x)+1/k)}\bigg)\frac{\psi(x)+1/k}{\eta(\gamma;\epsilon,t)}dx\\ &\quad\leq \int_{F_m}\tilde{\mathcal{V}}\bigg(\frac{\pi\big(\inf w(y)K(z,y)\big)h(x)}{(\phi(x)+1/l)(\psi(x)+1/k)}\bigg)\frac{\psi(x)+1/k}{(1+\mu)C\pi\gamma}dx = 1.\end{aligned}$$

Letting $l,k,m\to\infty$ and $\kappa\to 0^+$, we thus obtain

$$\int_z^\alpha \tilde{\mathcal{V}}\Bigg(\frac{\big(\inf w(y)K(z,y)\big)h(x)\eta(\gamma;\epsilon,t)}{C\gamma\phi(x)\psi(x)}\Bigg)\psi(x)dx \leq \eta(\gamma;\epsilon,t).$$

Hence the proof is complete. □

## 5. Proof of Theorem 2.5

*Proof.* (b) $\Longrightarrow$ (a). Let $f$ be a non-negative measurable function on $\mathbb{R}$ with support $(0,L)$ for $L>0$. Let $\xi_0 = L$ and we define $P(\xi_m)$ as

$$P(\xi_m) = \int_0^{\xi_m} K(\xi_m,\tau)f(\tau)w(\tau)d\tau = (M+1)^{-m}P(\xi_0). \tag{57}$$

We recall the estimate (27)

(58)

$$P(\xi_m) \leq (M+1)^4 \Big[ K(\xi_{m+2}, \xi_{m+3}) \int_0^{\xi_{m+3}} f(\tau) w(\tau) d\tau + \int_{\xi_{m+3}}^{\xi_{m+2}} K(\xi_{m+2}, \tau) f(\tau) w(\tau) d\tau \Big].$$

Considering $\Omega_{1,m}$ and $\Omega_{2,m}$ be the sets as defined in Section 3, thus we obtain

$$\Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\{t:\mathcal{I}f(t)>\gamma\}} \omega(t)dt\Big) \leq \sum_{m\geq 0} \Big\{ \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\Omega_{1,m}} \omega(t)dt\Big) + \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\Omega_{2,m}} \omega(t)dt\Big) \Big\}. \tag{59}$$

Next, we consider the first sum of (59). Let $x_0 = L$ for each $x_k$ we define $x_{k+1}$ as

$$\int_0^{x_k} f(\tau)w(\tau)d\tau = 2\int_0^{x_{k+1}} f(\tau)w(\tau)d\tau.$$

Thus we obtain a decresing sequence $\{x_k\}$ with the following property.

$$\int_0^{x_k} f(\tau)w(\tau)d\tau = 4\int_{x_{k+2}}^{x_{k+1}} f(\tau)w(\tau)d\tau.$$

Again we consider a subsequence $\{x'_n\}$ from $\{x_k\}$ with the iteration $x'_0 = x_0$ and if $x_{k+1} \leq \xi_m < x_k$, then $x'_{n+1} = x_{k+1}$, otherwise we delete the term $x_{k+1}$ and continue the process. Thus, we get a subsequence $\{x'_n\}$ of $\{x_k\}$. Let $\tilde{\delta}_{1,n} = \inf \tilde{\Omega}_{1,n}$ and $\tilde{\beta}_{1,n} = \sup \tilde{\Omega}_{1,n}$, where $\tilde{\Omega}_{1,n} = \cup_{\{m:x'_{n+1}\leq \xi_{m+3} < x'_n\}} \Omega_{1,m}$. Now, if $x_{k+1} = x'_{n+1} \leq \xi_{m+3} < x'_n$, then by construction $\xi_{m+3} \leq x_k$ and $x'_{n+2} \leq x_{k+2}$. Now, we have

$$\gamma \leq 8(M+1)^4 \inf_{(\tilde{\delta}_{1,n}, \tilde{\beta}_{1,n})} h(t)K(t, \xi_{m+3}) \int_{x'_{n+2}}^{x'_{n+1}} f(\tau)w(\tau)d(\tau). \tag{60}$$

Let us denote

$$\theta(\tilde{\delta}_{1,n}, \tilde{\beta}_{1,n}) = \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\delta}_{1,n}}^{\tilde{\beta}_{1,n}} \omega(\tau)d\tau\Big).$$

Using the estimates (10) and (60) we obtain

$$\begin{aligned} 2\theta(\tilde{\delta}_{1,n}, \tilde{\beta}_{1,n}) \leq & \int_{x'_{n+2}}^{x'_{n+1}} \Big[\frac{16(M+1)^4 C f(\tau)\phi(\tau)}{\gamma}\Big] \\ & \Bigg[\frac{\big(\inf h(t)K(t,\xi_{m+3})\big) w(\tau)\theta(\tilde{\delta}_{1,n}, \tilde{\beta}_{1,n})}{C\phi(\tau)\psi(\tau)}\Bigg] \psi(\tau)d\tau \\ \leq & \int_{x'_{n+2}}^{x'_{n+1}} \mathcal{V}\Big(\frac{16(M+1)^4 C f(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau \end{aligned}$$

$$+\int_{x'_{n+2}}^{x'_{n+1}} \tilde{\mathcal{V}}\left[\frac{\big(\inf h(t)K(t,\xi_{m+3})\big)w(\tau)\theta(\tilde{\delta}_{1,n},\tilde{\beta}_{1,n})}{C\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau. \tag{61}$$

Using non-increasing property of the kernel $K$ with respect to its second variable and thus from (21) we get

$$\int_{x'_{n+2}}^{x'_{n+1}} \tilde{\mathcal{V}}\left[\frac{\big(\inf h(t)K(t,\xi_{m+3})\big)w(\tau)\theta(\tilde{\delta}_{1,n},\tilde{\beta}_{1,n})}{C\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau \le \theta(\tilde{\delta}_{1,n},\tilde{\beta}_{1,n}). \tag{62}$$

Combining (61) and (62), we obtain

$$\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\bigg(\int_{\tilde{\Omega}_{1,n}}\omega(\tau)d\tau\bigg) \le \int_{x'_{n+2}}^{x'_{n+1}} \mathcal{V}\Big(\frac{16(M+1)^4Cf(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau. \tag{63}$$

Using sub-additivity of $\mathcal{V}\circ\mathcal{U}^{-1}$, we obtain

$$\begin{aligned}\sum_{m\ge 0}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\bigg(\int_{\Omega_{1,m}}\omega(\tau)d\tau\bigg) &\le \sum_{n\ge 0}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\bigg(\int_{\tilde{\Omega}_{1,n}}\omega(\tau)d\tau\bigg)\\ &\le \int_0^{\infty}\mathcal{V}\Big(\frac{16(M+1)^4f(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau.\end{aligned} \tag{64}$$

Next, we estimate the second part of (59). For this, let us define $\delta_{2,m} = \inf\Omega_{2,m}$ and $\beta_{2,m} = \sup\Omega_{2,m}$ and we denote

$$\theta(\delta_{2,m},\beta_{2,m}) = \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\bigg(\int_{\delta_{2,m}}^{\beta_{2,m}}\omega(\tau)d\tau\bigg).$$

Thus we obtain

$$\gamma \le 2(M+1)^4\inf_{(\delta_{2,m},\beta_{2,m})} h(t)\int_{\xi_{m+3}}^{\xi_{m+2}} K(\xi_{m+2},\tau)f(\tau)w(\tau)d\tau. \tag{65}$$

Now, using the conditions (10) and (65) we have

$$\begin{aligned}2\theta(\delta_{2,m},\beta_{2,m}) \le &\int_{\xi_{m+3}}^{\xi_{m+2}}\Big[\frac{4(M+1)^4Cf(\tau)\phi(\tau)}{\gamma}\Big]\\ &\left[\frac{\big(\inf h(t)\big)K(\xi_{m+2},\tau)w(\tau)\theta(\delta_{2,m},\beta_{2,m})}{C\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau\\ \le &\int_{\xi_{m+3}}^{\xi_{m+2}}\mathcal{V}\Big(\frac{4(M+1)^4Cf(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau\\ &+\int_{\xi_{m+3}}^{\xi_{m+2}}\tilde{\mathcal{V}}\left[\frac{\big(\inf h(t)\big)K(\xi_{m+2},\tau)w(\tau)\theta(\delta_{2,m},\beta_{2,m})}{C\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau.\end{aligned} \tag{66}$$

As $\xi_{m+2} \le \delta_{2,m} \le \beta_{2,m}$, thus from the condition (20) we have

$$\int_{\xi_{m+3}}^{\xi_{m+2}}\tilde{\mathcal{V}}\left[\frac{\big(\inf h(t)\big)K(\xi_{m+2},\tau)w(\tau)\eta(\gamma;\delta_{2,m},\beta_{2,m})}{C\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau$$

$$\leq \theta(\delta_{2,m}, \beta_{2,m}). \tag{67}$$

Combining (66), (67), and then summing up in $m$, we get

$$\sum_{m\geq 0} \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\Omega_{2,m}} \omega(\tau)d\tau\Big)$$
$$\leq \int_0^\infty \mathcal{V}\Big(\frac{4(M+1)^4 C f(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau. \tag{68}$$

Thus, combining the inequalities (59), (64) and (68) we obtain (19) with constant $32(M+1)^4C$.

(a) $\Longrightarrow$ (b). For each $N \in \mathbb{N}$ we consider the set

$$E_N = \{0 < s < t : \frac{1}{N} \leq K(t,y), w(y) \leq N\}.$$

We have

$$\int_{E_N} \tilde{\mathcal{V}}\Big(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\Big(\frac{\psi(y)+1/k}{\lambda}\Big)dy$$
$$\leq ltN^2(\inf h)\tilde{v}(\lambda(\inf h)lkN^2) < \infty$$

for each $l, k \in \mathbb{R}$ and $\lambda > 0$. We choose $\lambda > 0$ such that

$$\int_{E_N} \tilde{\mathcal{V}}\Big(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\Big(\frac{\psi(y)+1/k}{\lambda}\Big)dy = (1+\mu)C,$$

where $C$ is the constant in (19). For each $\gamma > 0$ we consider

$$f(y) = \frac{\gamma}{C}\tilde{\mathcal{V}}\Big(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\frac{\psi(y)+1/k}{\lambda(\inf h)K(t,y)w(y)}\chi_{E_N}(y).$$

If $t < \beta < \alpha$, then

$$\mathcal{I}f(\beta) = h(\beta)\int_0^\beta K(\beta,y)\frac{\gamma}{C}\tilde{\mathcal{V}}\Big(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)$$
$$\Big(\frac{\psi(y)+1/k}{\lambda(\inf h)K(t,y)w(y)}\Big)\chi_{E_N}(y)w(y)dy$$
$$\geq \int_{E_N}\frac{\gamma}{C\lambda}\tilde{\mathcal{V}}\Big(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\Big(\psi(y)+1/k\Big)dy$$
$$= (1+\mu)\gamma > \gamma.$$

This implies that

$$(t,\alpha) \subset \{y : \mathcal{I}f(y) > \gamma\}.$$

Thus using the assumption (19) we obtain

$$\theta(t,\alpha) \leq \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_t^\alpha \omega(y)dy\Big)$$
$$\leq \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\{\mathcal{I}f>\gamma\}} \omega(y)dy\Big)$$

$$\leq \int_{E_N} \mathcal{V}\left(\tilde{\mathcal{V}}\left(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\left(\frac{\psi(y)+1/k}{\lambda(\inf h)K(t,y)w(y)}\right)\phi(y)\right)\psi(y)dy$$
$$\leq \int_{E_N} \tilde{\mathcal{V}}\left(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\psi(y)dy$$
$$\leq (1+\mu)C\lambda.$$

Using the increasing property of $\tilde{\mathcal{V}}(s)/s$ with respect to $s$, we obtain

$$\int_{E_N} \tilde{\mathcal{V}}\left(\frac{(\inf h)K(t,y)w(y)\theta(t,\alpha)}{(1+\mu)C(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\theta(t,\alpha)}dy$$
$$\leq \int_{E_N} \tilde{\mathcal{V}}\left(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{(1+\mu)C\lambda}dy = 1.$$

By the monotone convergence theorem, we get

$$\int_{E_N} \tilde{\mathcal{V}}\left(\frac{(\inf h)K(t,y)w(y)\theta(t,\alpha)}{(1+\mu)C(\phi(y)+1/l)\psi(y)}\right)\frac{\psi(y)}{\theta(t,\alpha)}dy \leq 1.$$

Letting $l, N \to \infty$ and $\mu \to 0^+$, we thus obtain

$$\int_0^t \tilde{\mathcal{V}}\left(\frac{(\inf h)K(t,y)w(y)\big(\mathcal{V}\circ\mathcal{U}^{-1}\big)\left(\int_t^\alpha \omega\right)}{C\phi(y)\psi(y)}\right)\psi(y)dy$$
$$\leq \left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_t^\alpha \omega(y)dy\right).$$

We skip the proof of (21) as it proceeds similarly. Hence the proof is complete. □

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## References

[1] R. J. Bagby, *Weak bounds for the maximal function in weighted Orlicz spaces*, Studia Math. **95** (1990), no. 3, 195–204. https://doi.org/10.4064/sm-95-3-195-204

[2] R. J. Bagby and J. D. Parsons, *Orlicz spaces and rearranged maximal functions*, Math. Nachr. **132** (1987), 15–27. https://doi.org/10.1002/mana.19871320103

[3] S. Bloom and R. Kerman, *Weighted Orlicz space integral inequalities for the Hardy-Littlewood maximal operator*, Studia Math. **110** (1994), no. 2, 149–167. https://doi.org/10.4064/sm-110-2-149-167

[4] A. Gogatishvili and L. Pick, *Weak and extra-weak type inequalities for the maximal operator and the Hilbert transform*, Czechoslovak Math. J. **43(118)** (1993), no. 3, 547–566.

[5] V. Kokilashvili and M. Krbec, *Weighted inequalities in Lorentz and Orlicz spaces*, World Scientific Publishing Co., Inc., River Edge, NJ, 1991. https://doi.org/10.1142/9789814360302

[6] A. Kufner, L.-E. Persson, and N. Samko, *Weighted inequalities of Hardy type*, second edition, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017. https://doi.org/10.1142/10052

[7] Q. Lai, *Weighted integral inequalities for the Hardy type operator and the fractional maximal operator*, J. London Math. Soc. (2) **49** (1994), no. 2, 244–266. https://doi.org/10.1112/jlms/49.2.244

[8] Q. Lai, *Two-weight mixed Φ-inequalities for the one-sided maximal function*, Studia Math. **115** (1995), no. 1, 1–22. https://doi.org/10.4064/sm-115-1-1-22

[9] Q. Lai, *Weighted modular inequalities for Hardy type operators*, Proc. London Math. Soc. (3) **79** (1999), no. 3, 649–672. https://doi.org/10.1112/S0024611599012010

[10] E. Lomakina and V. Stepanov, *On the Hardy-type integral operators in Banach function spaces*, Publ. Mat. **42** (1998), no. 1, 165–194. https://doi.org/10.5565/PUBLMAT_42198_09

[11] F. J. Martín-Reyes, *New proofs of weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Proc. Amer. Math. Soc. **117** (1993), no. 3, 691–698. https://doi.org/10.2307/2159130

[12] F. J. Martín-Reyes and P. Ortega, *On weighted weak type inequalities for modified Hardy operators*, Proc. Amer. Math. Soc. **126** (1998), no. 6, 1739–1746. https://doi.org/10.1090/S0002-9939-98-04247-6

[13] F. J. Martín-Reyes and E. Sawyer, *Weighted inequalities for Riemann-Liouville fractional integrals of order one and greater*, Proc. Amer. Math. Soc. **106** (1989), no. 3, 727–733. https://doi.org/10.2307/2047428

[14] P. Ortega Salvador, *Weighted generalized weak type inequalities for modified Hardy operators*, Collect. Math. **51** (2000), no. 2, 149–155.

[15] P. Ortega Salvador and L. Pick, *Two-weight weak and extra-weak type inequalities for the one-sided maximal operator*, Proc. Roy. Soc. Edinburgh Sect. A **123** (1993), no. 6, 1109–1118. https://doi.org/10.1017/S0308210500029760

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