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STRONG CLASSIFICATION OF EXTENSIONS OF CLASSIFIABLE C^*-ALGEBRAS

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Abstract. We show that certain extensions of classifiable C^*-algebras are strongly classified by the associated six-term exact sequence in K-theory together with the positive cone of K_0-groups of the ideal and quotient. We use our results to completely classify all unital graph C^*-algebras with exactly one non-trivial ideal.

1. Introduction

The classification program for C^*-algebras has for the most part progressed independently for the classes of infinite and finite C^*-algebras, and great strides have been made in this program for each of these classes. In the finite case, Elliott's theorem classifies all AF-algebras up to stable isomorphism by the ordered K_0-group. In the infinite case, there are a number of results for purely infinite C^*-algebras. The Kirchberg-Phillips theorem classifies certain simple purely infinite C^*-algebras up to stable isomorphism by the K_0-group together with the K_1-group. For non-simple purely infinite C^*-algebras many partial results have been obtained: Rørdam has shown that certain purely infinite C^*-algebras with exactly one non-trivial ideal are classified up to stable isomorphism by the associated six-term exact sequence of K-groups [56], the second named author has shown that non-simple Cuntz-Krieger algebras satisfying Condition (II) are classified up to stable isomorphism by their filtered K-theory [51, Theorem 4.2], Meyer and Nest have shown that all separable, nuclear, purely infinite C^*-algebras in the bootstrap category that have finite linear ideal lattices are classified up to stable isomorphism by their filtrated K-theory [48, Theorem 4.14], and Bentmann and Köhler showed that this can be extended to algebras with the primitive ideal space an accordion space [5].

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However, in all of these situations the non-simple C^*-algebras that are classified have the property that they are either AF-algebras or purely infinite, and consequently all of their ideals and quotients are of the same type.

The authors have provided a framework for classifying non-simple C^*-algebras that are not necessarily AF-algebras or purely infinite C^*-algebras. In particular, the authors have shown in [22] that certain extensions of classifiable C^*-algebras may be classified up to stable isomorphism by their associated six-term exact sequence in K-theory. This has allowed for the classification of certain non-simple C^*-algebras in which there are ideals and quotients of mixed type (some finite and some infinite). The results in [22] were then used by the first named author and Tomforde in [29] to show that all graph C^*-algebras with exactly one non-trivial ideal are classified up to stable isomorphism by their associated six-term exact sequence in K-theory.

In this paper, we prove that the classification up to stable isomorphism of certain graph C^*-algebras with exactly one non-trivial ideal is *strong* in the sense that the given isomorphism of the invariant lifts to an isomorphism of the C^*-algebras studied. We then proceed to use that result to establish exact classification in that same class, and stable classification in some cases with more than one non-trivial ideal. Our methods differ rather dramatically from the methods in [29] and [26] — in particular, we use the traditional methods of classification via existence and uniqueness theorems.

This paper has existed in preprint form for close to a decade. A main motivation for it was the extension problem for simple graph C^*-algebras, which we essentially solve in the unital case here. However, due to repercussions from a mistake in the literature (not related to the present work, see Remark 5.4), this project ground to a halt and was only recently completed in [17]. Also, we have recently completed, with Sørensen, the project of classifying non-simple unital graph C^*-algebras ([28]) for which the first step was taken in Theorem 5.3 of the present paper. We are expecting methods developed here to be useful in the general, non-unital case, and all of this motivates us to update the paper and seek permanent publication.

The preprint version of the paper has been cited by the authors and others over the years, and unfortunately some inconsistencies have resulted from the fact that several versions have been circulated. We have chosen to enumerate the present version with all items adjusted to match the arxiv version of the paper, since the only works in print which substantially has references to a different version are coauthored by one or more of us. We indicate how to resolve these references in Table 1.

We briefly indicate the organization of the paper. After establishing preliminaries in Section 2, we show in Section 3 a “meta-theorem” which in very general terms explains how a strong and stable classification result yields an exact classification result at the expense of keeping track of the class of the unit (or scale) in the relevant K_0-groups. In Section 4 we establish that various known classification results are in fact strong. This is rather technical and in most cases requires a completely new proof. In Section 5 we then collect applications.

Table 1.

[20]	p. 316	Uniqueness follows from [ERR, Corollary 4.22]	Corollary 4.20
[25]	p. 114	By Corollary 3.10 and Theorem 3.13 of [17]...	(2): Corollary 4.8 (3): Corollary 4.17
	p. 115	In [17], we prove 1 and 2	Lemma 5.5
	p. 116	By Theorem 3.9 of [17]...	Theorem 4.7
	p. 116	A proof is given in [17]	Theorem 5.7
	p. 117	By Lemma 4.5 of [17]...	Lemma 5.5
[35]	p. 196	Theorem 3.2 of [16]	Theorem 3.3

2. Preliminaries

In the following, X will denote a finite topological space satisfying the T_0-separation theorem. Many of the definitions and some of the results work more generally, but this is not relevant to the main results.

2.1. C^*-algebras over topological spaces

Let X be a topological space and let $\mathbb{O}(X)$ be the set of open subsets of X, partially ordered by set inclusion $\subseteq$. A subset Y of X is called *locally closed* if $Y = U \setminus V$ where $U, V \in \mathbb{O}(X)$ and $V \subseteq U$. The set of all locally closed subsets of X will be denoted by $\mathbb{LC}(X)$. The set of all connected, non-empty, locally closed subsets of X will be denoted by $\mathbb{LC}(X)^*$.

The partially ordered set $(\mathbb{O}(X), \subseteq)$ is a *complete lattice*, that is, any subset S of $\mathbb{O}(X)$ has both an infimum $\bigwedge S$ and a supremum $\bigvee S$. More precisely, for any subset S of $\mathbb{O}(X)$,

$$\bigwedge_{U \in S} U = \left(\bigcap_{U \in S} U \right)^\circ \quad \text{and} \quad \bigvee_{U \in S} U = \bigcup_{U \in S} U.$$

For a C^*-algebra $\mathfrak{A}$, let $\mathbb{I}(\mathfrak{A})$ be the set of closed ideals of $\mathfrak{A}$, partially ordered by $\subseteq$. The partially ordered set $(\mathbb{I}(\mathfrak{A}), \subseteq)$ is a complete lattice. More precisely, for any subset S of $\mathbb{I}(\mathfrak{A})$,

$$\bigwedge_{\mathfrak{I} \in S} \mathfrak{I} = \bigcap_{\mathfrak{I} \in S} \mathfrak{I} \quad \text{and} \quad \bigvee_{\mathfrak{I} \in S} \mathfrak{I} = \overline{\sum_{\mathfrak{I} \in S} \mathfrak{I}}.$$

Definition 2.1. Let $\mathfrak{A}$ be a C^*-algebra. Let $\mathrm{Prim}(\mathfrak{A})$ denote the *primitive ideal space* of $\mathfrak{A}$, equipped with the usual hull-kernel topology, also called the Jacobson topology.

Let X be a topological space. A *C^*-algebra over* X is a pair $(\mathfrak{A}, \psi)$ consisting of a C^*-algebra $\mathfrak{A}$ and a continuous map $\psi\colon \mathrm{Prim}(\mathfrak{A}) \to X$. A C^*-algebra over

X, $(\mathfrak{A}, \psi)$, is *separable* if $\mathfrak{A}$ is a separable C^*-algebra. We say that $(\mathfrak{A}, \psi)$ is *tight* if ψ is a homeomorphism.

We always identify $\mathbb{O}(\mathrm{Prim}(\mathfrak{A}))$ and $\mathbb{I}(\mathfrak{A})$ using the lattice isomorphism

$$U \mapsto \bigcap_{\mathfrak{p} \in \mathrm{Prim}(\mathfrak{A}) \setminus U} \mathfrak{p}.$$

Let $(\mathfrak{A}, \psi)$ be a C^*-algebra over X. Then we get a map ψ^* from $\mathbb{O}(X)$ to $\mathbb{O}(\mathrm{Prim}(\mathfrak{A})) \cong \mathbb{I}(\mathfrak{A})$ defined by

$$U \mapsto \{\mathfrak{p} \in \mathrm{Prim}(\mathfrak{A}) \ : \ \psi(\mathfrak{p}) \in U\}.$$

Using the isomorphism $\mathbb{O}(\mathrm{Prim}(\mathfrak{A})) \cong \mathbb{I}(\mathfrak{A})$, we get a map from $\mathbb{O}(X)$ to $\mathbb{I}(\mathfrak{A})$ by

$$U \mapsto \bigcap \{\mathfrak{p} \in \mathrm{Prim}(\mathfrak{A}) \ : \ \psi(\mathfrak{p}) \notin U\}.$$

Denote this ideal by $\mathfrak{A}(U)$. For $Y = U \setminus V \in \mathbb{LC}(X)$, set $\mathfrak{A}(Y) = \mathfrak{A}(U)/\mathfrak{A}(V)$. By [47, Lemma 2.15], $\mathfrak{A}(Y)$ does not depend on U and V.

Example 2.2. For any C^*-algebra $\mathfrak{A}$, the pair $(\mathfrak{A}, \mathrm{id}_{\mathrm{Prim}(\mathfrak{A})})$ is a tight C^*-algebra over $\mathrm{Prim}(\mathfrak{A})$. For each $U \in \mathbb{O}(\mathrm{Prim}(\mathfrak{A}))$, the ideal $\mathfrak{A}(U)$ equals $\bigcap_{\mathfrak{p} \in \mathrm{Prim}(\mathfrak{A}) \setminus U} \mathfrak{p}$.

Example 2.3. Let $X_n = \{1, 2, \dots, n\}$ be partially ordered with $\leq$. Equip X_n with the Alexandrov topology, so the non-empty open subsets are

$$[a, n] = \{x \in X \ : \ a \leq x \leq n\}$$

for all $a \in X_n$; the non-empty closed subsets are $[1, b]$ with $b \in X_n$, and the non-empty locally closed subsets are those of the form $[a, b]$ with $a, b \in X_n$ and $a \leq b$. Let $(\mathfrak{A}, \phi)$ be a C^*-algebra over X_n. We will use the following notation throughout the paper:

$$\mathfrak{A}[k] = \mathfrak{A}(\{k\}), \ \mathfrak{A}[a, b] = \mathfrak{A}([a, b]), \text{ and } \mathfrak{A}(i, j] = \mathfrak{A}[i + 1, j].$$

Using the above notation we have ideals $\mathfrak{A}[a, n]$ such that

$$\{0\} \trianglelefteq \mathfrak{A}[n] \trianglelefteq \mathfrak{A}[n-1, n] \trianglelefteq \cdots \trianglelefteq \mathfrak{A}[2, n] \trianglelefteq \mathfrak{A}[1, n] = \mathfrak{A}.$$

Definition 2.4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras over X. A $*$-homomorphism $\phi \colon \mathfrak{A} \to \mathfrak{B}$ is *X-equivariant* if $\phi(\mathfrak{A}(U)) \subseteq \mathfrak{B}(U)$ for all $U \in \mathbb{O}(X)$. Hence, for every $Y \in \mathbb{LC}(X)$, ϕ induces a $*$-homomorphism $\phi_Y \colon \mathfrak{A}(Y) \to \mathfrak{B}(Y)$. Let $\mathfrak{C}^*\text{-}\mathfrak{alg}(X)$ be the category whose objects are C^*-algebras over X and whose morphisms are X-equivariant homomorphisms.

An X-equivariant homomorphism $\phi \colon \mathfrak{A} \to \mathfrak{B}$ is said to be a *full X-equivariant homomorphism* if for every $a \in \mathfrak{A}$, we have that $a \in \mathfrak{A}(U)$ and $\phi_U(a)$ is norm-full in $\mathfrak{B}(U)$, where U is the infimum of the sets O satisfying $a \in \mathfrak{A}(O)$. Note that if $\mathfrak{A}$ is a tight C^*-algebra over X, ϕ is a full X-equivariant homomorphism if and only if for all $a \in \mathfrak{A}$ with $\mathfrak{A}(U)$ equal to the ideal generated by a, we have $\phi_U(a)$ is norm-full in $\mathfrak{B}(U)$. This notion of full X-equivariant

$*$-homomorphism was introduced by Gabe and the third named author in [34, Definition 5.13].

Remark 2.5. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are tight C^*-algebras over X_n. Then it is clear that $\phi\colon \mathfrak{A} \to \mathfrak{B}$ is an isomorphism if and only if ϕ is an X_n-equivariant isomorphism.

Assume that $\mathfrak{A}$ and $\mathfrak{B}$ are tight C^*-algebras over X_2. It is easy to see that $\phi\colon \mathfrak{A} \to \mathfrak{B}$ is a full X_2-equivariant homomorphism if and only if ϕ is an X_2-equivariant homomorphism and $\phi_{\{1\}}$ and $\phi_{\{2\}}$ are injective. Suppose $\phi\colon \mathfrak{A} \to \mathfrak{B}$ is a full X_2-equivariant homomorphism, $\psi\colon \mathfrak{A} \to \mathfrak{B}$ is a $*$-homomorphism, and there exist a projection $p \in \mathfrak{A} \setminus \mathfrak{A}[2]$ and a non-zero projection $q \in \mathfrak{A}[2]$ such that

$$\|\phi(p) - \psi(p)\| < 1 \quad \text{and} \quad \|\phi(q) - \psi(q)\| < 1.$$

Since $\phi(q), \psi(q), \phi(p), \psi(q)$ are projections, the above inequalities imply that $\phi(p)$ and $\psi(p)$ generate the same ideal (since they are Murray-von Neumann equivalent) and $\phi(q)$ and $\psi(q)$ generate the same ideal (since they are Murray-von Neumann equivalent). Hence, $\psi(q)$ generates $\mathfrak{B}[2]$ and $\psi(p)$ generates $\mathfrak{B}$ since ϕ is a full X_2-equivariant homomorphism. Moreover, since $\mathfrak{A}$ is a tight C^*-algebra over X_2, $\psi(\mathfrak{A}[2]) \subseteq \mathfrak{B}[2]$. Thus, ψ is an X_2-equivariant homomorphism. Let $a \in \mathfrak{A}$. If $a = 0$, then $\psi(a)$ generates the zero ideal. Suppose a is full in $\mathfrak{A}[2]$. Then the ideal generated by $\psi(a)$ contains $\psi(q)$, thus $\psi(a)$ is full in $\mathfrak{B}[2]$. Suppose a is full in $\mathfrak{A}$. Then the ideal generated by $\psi(a)$ contains $\psi(p)$, thus $\psi(a)$ is full in $\mathfrak{B}$. We have just shown that ψ is a full X_2-equivariant homomorphism which in particular implies that $\psi_{\{1\}}$ and $\psi_{\{2\}}$ are injective homomorphism.

Remark 2.6. Let $\mathfrak{e}_i : 0 \to \mathfrak{B}_i \to \mathfrak{E}_i \to \mathfrak{A}_i \to 0$ be an extension for $i = 1, 2$. Note that $\mathfrak{E}_i$ can be considered as a C^*-algebra over $X_2 = \{1, 2\}$ by sending $\emptyset$ to the zero ideal, $\{2\}$ to the image of $\mathfrak{B}_i$ in $\mathfrak{E}_i$, and $\{1, 2\}$ to $\mathfrak{E}_i$. Hence, there exists a one-to-one correspondence between X_2-equivariant homomorphisms $\phi\colon \mathfrak{E}_1 \to \mathfrak{E}_2$ and homomorphisms from $\mathfrak{e}_1$ and $\mathfrak{e}_2$.

2.2. The ideal related K-theory of $\mathfrak{A}$

Definition 2.7. Let X be a topological space and let $\mathfrak{A}$ be a C^*-algebra over X. For open subsets U_1, U_2, U_3 of X with $U_1 \subseteq U_2 \subseteq U_3$, set $Y_1 = U_2 \setminus U_1, Y_2 = U_3 \setminus U_1, Y_3 = U_3 \setminus U_2 \in \mathbb{LC}(X)$. Then the diagram

$$\begin{array}{ccccc} K_0(\mathfrak{A}(Y_1)) & \xrightarrow{\iota_*} & K_0(\mathfrak{A}(Y_2)) & \xrightarrow{\pi_*} & K_0(\mathfrak{A}(Y_3)) \\ {\scriptstyle \partial_*}\uparrow & & & & \downarrow{\scriptstyle \partial_*} \\ K_1(\mathfrak{A}(Y_3)) & \xleftarrow[\pi_*]{} & K_1(\mathfrak{A}(Y_2)) & \xleftarrow[\iota_*]{} & K_1(\mathfrak{A}(Y_1)) \end{array}$$

is an exact sequence. The *ideal related K-theory of* $\mathfrak{A}$, $K_X(\mathfrak{A})$, is the collection of all K-groups thus occurring and the natural transformations $\{\iota_*, \pi_*, \partial_*\}$.

The *ideal related, ordered K-theory of* $\mathfrak{A}$, $K_X^+(\mathfrak{A})$, is $K_X(\mathfrak{A})$ of $\mathfrak{A}$ together with $K_0(\mathfrak{A}(Y))_+$ for all $Y \in \mathbb{LC}(X)$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras over X, we will say that $\alpha\colon K_X(\mathfrak{A}) \to K_X(\mathfrak{B})$ is an *isomorphism* if for all $Y \in \mathbb{LC}(X)$, there exists a graded group isomorphism

$$\alpha_{Y,*}\colon K_*(\mathfrak{A}(Y)) \to K_*(\mathfrak{B}(Y))$$

preserving the natural transformations $\{\iota_*, \pi_*, \partial_*\}$. We say that $\alpha\colon K_X^+(\mathfrak{A}) \to K_X^+(\mathfrak{B})$ is an *isomorphism* if α is an isomorphism between the ideal related K-theory and $\alpha_{Y,0}$ is an order isomorphism for all $Y \in \mathbb{LC}(X)$.

Remark 2.8. Meyer-Nest in [48] defined a similar functor $\mathrm{FK}_X(-)$ which they called filtrated K-theory. For all known cases in which there exists a UCT, the natural transformation from $\mathrm{FK}_X(-)$ to $K_X(-)$ is an equivalence. In particular, this is true for the space X_n.

If $Y \in \mathbb{LC}(X)$ such that $Y = Y_1 \sqcup Y_2$ with two disjoint relatively open subsets $Y_1, Y_2 \in \mathbb{O}(Y) \subseteq \mathbb{LC}(X)$, then $\mathfrak{A}(Y) \cong \mathfrak{A}(Y_1) \oplus \mathfrak{A}(Y_2)$ for any C^*-algebra $\mathfrak{A}$ over X. Moreover, there is a natural isomorphism from $K_*(\mathfrak{A}(Y))$ to $K_*(\mathfrak{A}(Y_1)) \oplus K_*(\mathfrak{A}(Y_2))$ which is an order isomorphism from $K_0(\mathfrak{A}(Y))$ to $K_0(\mathfrak{A}(Y_1)) \oplus K_0(\mathfrak{A}(Y_2))$. If X is finite, then any locally closed subset is a disjoint union of its connected components. Therefore, we lose no information when we replace $\mathbb{LC}(X)$ by the subset $\mathbb{LC}(X)^*$.

Notation 2.9. Let $\mathcal{N}$ be the bootstrap category of Rosenberg and Schochet in [59].

Let $\mathfrak{KK}(X)$ be the category whose objects are separable C^*-algebras over X and the set of morphisms is $KK(X; \mathfrak{A}, \mathfrak{B})$. For a finite topological space X, let $\mathcal{B}(X) \subseteq \mathfrak{KK}(X)$ be the bootstrap category of Meyer and Nest in [47]. By [47, Corollary 4.13], if $\mathfrak{A}$ is a nuclear C^*-algebra over X, then $\mathfrak{A} \in \mathcal{B}(X)$ if and only if $\mathfrak{A}(\{x\}) \in \mathcal{N}$ for all $x \in X$.

Theorem 2.10 (Bonkat [8] and Meyer-Nest [48])**.** *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be in* $\mathfrak{KK}(X_n)$ *such that* $\mathfrak{A}$ *is in* $\mathcal{B}(X_n)$. *Then there exists an exact sequence*

$$\begin{aligned} 0 &\to \mathrm{Ext}^1_{\mathcal{NT}}(\mathrm{FK}_{X_n}(\mathfrak{A})[1], \mathrm{FK}_{X_n}(\mathfrak{B})) \\ &\xrightarrow{\delta} KK(X_n; \mathfrak{A}, \mathfrak{B}) \xrightarrow{\Gamma} \mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}_{X_n}(\mathfrak{A}), \mathrm{FK}_{X_n}(\mathfrak{B})) \to 0 \end{aligned}$$

that is natural in both variables. Consequently, if $\mathfrak{B}$ *is in* $\mathcal{B}(X_n)$, *then every isomorphism from* $\mathrm{FK}_{X_n}(\mathfrak{A})$ *to* $\mathrm{FK}_{X_n}(\mathfrak{B})$ *lifts to an invertible element in* $KK(X_n; \mathfrak{A}, \mathfrak{B})$.

Corollary 2.11. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be in* $\mathcal{B}(X_n)$. *Then an isomorphism from* $K_{X_n}(\mathfrak{A})$ *to* $K_{X_n}(\mathfrak{B})$ *lifts to an invertible element in* $KK(X_n; \mathfrak{A}, \mathfrak{B})$.

Proof. This follows from Remark 2.8 and Theorem 2.10. □

Remark 2.12. Let $x \in KK(X_n; \mathfrak{A}, \mathfrak{B})$ be an invertible element. Then $K_{X_n}(x)$ will denote the isomorphism from $K_{X_n}(\mathfrak{A})$ to $K_{X_n}(\mathfrak{B})$ given by $\Gamma(x)$ where we have identified $K_{X_n}(\mathfrak{A})$ with $\mathrm{FK}_{X_n}(\mathfrak{A})$ and $K_{X_n}(\mathfrak{B})$ with $\mathrm{FK}_{X_n}(\mathfrak{B})$.

2.3. Functors

We now define some functors that will be used throughout the rest of the paper. Let X and Y be topological spaces. For every continuous function $f\colon X \to Y$ we have a functor

$$f\colon \mathfrak{C}^*\text{-}\mathfrak{alg}(X) \to \mathfrak{C}^*\text{-}\mathfrak{alg}(Y), \quad (\mathfrak{A}, \psi) \mapsto (\mathfrak{A}, f \circ \psi).$$

(1) Define $g_X^1\colon X \to X_1$ by $g_X^1(x) = 1$. Then g_X^1 is continuous. Note that the induced functor $g_X^1\colon \mathfrak{C}^*\text{-}\mathfrak{alg}(X) \to \mathfrak{C}^*\text{-}\mathfrak{alg}(X_1)$ is the forgetful functor.

(2) Let U be an open subset of X. Define $g_{U,X}^2\colon X \to X_2$ by $g_{U,X}^2(x) = 1$ if $x \notin U$ and $g_{U,X}^2(x) = 2$ if $x \in U$. Then $g_{U,X}^2$ is continuous. Thus the induced functor

$$g_{U,X}^2\colon \mathfrak{C}^*\text{-}\mathfrak{alg}(X) \to \mathfrak{C}^*\text{-}\mathfrak{alg}(X_2)$$

is just specifying the extension $0 \to \mathfrak{A}(U) \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{A}(U) \to 0$.

(3) We can generalize (2) to finitely many ideals. Let $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n = X$ be open subsets of X. Define $g^n_{U_1,U_2,\dots,U_{n-1},X}\colon X \to X_n$ by $g^n_{U_1,U_2,\dots,U_{n-1},X}(x) = n-k+1$ if $x \in U_k \setminus U_{k-1}$. Then $g^n_{U_1,U_2,\dots,U_{n-1},X}$ is continuous. Therefore, any C^*-algebra with ideals $0 \trianglelefteq \mathfrak{I}_1 \trianglelefteq \mathfrak{I}_2 \trianglelefteq \cdots \trianglelefteq \mathfrak{I}_n = \mathfrak{A}$ can be made into a C^*-algebra over X_n.

(4) For all $Y \in \mathbb{LC}(X)$, $r_X^Y\colon \mathfrak{C}^*\text{-}\mathfrak{alg}(X) \to \mathfrak{C}^*\text{-}\mathfrak{alg}(Y)$ is the restriction functor defined in [47, Definition 2.19], i.e., $r_X^Y(\mathfrak{A})(C) = \mathfrak{A}(C)$ for all $C \in \mathbb{LC}(Y) \subseteq \mathbb{LC}(X)$.

(5) If $f\colon X \to Y$ is an embedding of a subset with the subspace topology, we write

$$i_X^Y = f\colon \mathfrak{C}^*\text{-}\mathfrak{alg}(X) \to \mathfrak{C}^*\text{-}\mathfrak{alg}(Y).$$

By [47, Proposition 3.4], the functors defined above induce functors from $\mathfrak{KK}(X)$ to $\mathfrak{KK}(Z)$, where $Z = Y, X_1, X_n$.

Throughout the paper, $\mathcal{M}(\mathfrak{A})$ will denote the multiplier algebra of $\mathfrak{A}$ and $\mathcal{Q}(\mathfrak{A})$ will denote the corona algebra $\mathcal{Q}(\mathfrak{A}) = \mathcal{M}(\mathfrak{A})/\mathfrak{A}$. Recall that an extension $0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{A} \to 0$ is called *full* if the image of any non-zero element of $\mathfrak{A}$ under the Busby map defining the extension is a norm-full element of the corona algebra $\mathcal{Q}(\mathfrak{B})$ (i.e., it is not contained in any proper ideal). Note that a full extension is automatically essential. Moreover, we will use the fact that every quotient of a nuclear C^*-algebra is again nuclear as noted in [41, Section 6], a deep result building on an observation by Tomiyama and Connes' famous classification of injective factors.

Proposition 2.13. *For each $i = 1, 2$, let $\mathfrak{A}_i$ be a nuclear, stable, separable C^*-algebra over X_2 such that $0 \to \mathfrak{A}_i[2] \to \mathfrak{A}_i \to \mathfrak{A}_i[1] \to 0$ is a full extension and such that any full extension*

$$0 \to \mathfrak{A}_i[2] \to \mathfrak{E} \to \mathfrak{B} \to 0$$

with $\mathfrak{B}$ a non-unital, separable, nuclear C^*-algebra is an absorbing extension. Suppose x is in $KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ and there exists an isomorphism $\phi_i \colon \mathfrak{A}_1[i] \to \mathfrak{A}_2[i]$ for each $i = 1, 2$ such that $KK(\phi_i) = r_{X_2}^{\{i\}}(x)$. Then there exists an X_2-equivariant isomorphism $\psi \colon \mathfrak{A}_1 \to \mathfrak{A}_2$ such that $KK(\psi_{\{i\}}) = r_{X_2}^{\{i\}}(x)$.

Proof. Let $\mathfrak{e}_i$ denote the canonical extensions $0 \to \mathfrak{A}_i[2] \to \mathfrak{A}_i \to \mathfrak{A}_i[1] \to 0$, for $i = 1, 2$, let $\mathfrak{f}_1$ be the extension obtained as the push-out of the extension $\mathfrak{e}_1$ via the isomorphism ϕ_2, and let $\mathfrak{f}_2$ be the extension obtained by pulling-back the extension $\mathfrak{e}_2$ via the isomorphism ϕ_1. Let $\widetilde{\mathfrak{A}}_1$ and $\widetilde{\mathfrak{A}}_2$ be the X_2-equivariant C^*-algebras induced by $\mathfrak{f}_1$ and $\mathfrak{f}_2$, respectively. Let $\tau_{\mathfrak{e}_i}$ be the Busby invariant for the extension $\mathfrak{e}_i$ and let $\tau_{\mathfrak{f}_i}$ be the Busby invariant for the extension $\mathfrak{f}_i$. We claim that $[\tau_{\mathfrak{f}_1}] = [\tau_{\mathfrak{f}_2}]$.

By the universal property of the push-out and the pull-back constructions, there exist X_2-equivariant isomorphisms $\alpha \colon \mathfrak{A}_1 \to \widetilde{\mathfrak{A}}_1$ and $\beta \colon \widetilde{\mathfrak{A}}_2 \to \mathfrak{A}_2$ such that $\alpha_{\{2\}} = \phi_2$, $\alpha_{\{1\}} = \mathrm{id}_{\mathfrak{A}_1[1]}$, $\beta_{\{1\}} = \phi_1$, and $\beta_{\{2\}} = \mathrm{id}_{\mathfrak{A}_2[2]}$. Since α and β are X_2-equivariant isomorphisms, by [24, Theorem 3.3],

$$[\tau_{\mathfrak{e}_1}] \times r_{X_2}^{\{2\}}(KK(\alpha)) = r_{X_2}^{\{1\}}(KK(\alpha)) \times [\tau_{\mathfrak{f}_1}]$$

and

$$[\tau_{\mathfrak{f}_2}] \times r_{X_2}^{\{2\}}(KK(\beta)) = r_{X_2}^{\{1\}}(KK(\beta)) \times [\tau_{\mathfrak{e}_2}].$$

Note that we have that

$$r_{X_2}^{\{1\}}(KK(\alpha)) = KK(\alpha_{\{1\}}) = KK(\mathrm{id}_{\mathfrak{A}_1[1]}),$$
$$r_{X_2}^{\{2\}}(KK(\alpha)) = KK(\alpha_{\{2\}}) = KK(\phi_2) = r_{X_2}^{\{2\}}(x),$$
$$r_{X_2}^{\{1\}}(KK(\beta)) = KK(\beta_{\{1\}}) = KK(\phi_1) = r_{X_2}^{\{1\}}(x),$$
$$r_{X_2}^{\{2\}}(KK(\beta)) = KK(\beta_{\{2\}}) = KK(\mathrm{id}_{\mathfrak{A}_2[2]}).$$

Hence, the above equations reduce to

$$[\tau_{\mathfrak{f}_1}] = [\tau_{\mathfrak{e}_1}] \times r_{X_2}^{\{2\}}(x) \quad \text{and} \quad [\tau_{\mathfrak{f}_2}] = r_{X_2}^{\{1\}}(x) \times [\tau_{\mathfrak{e}_2}].$$

By [24, Theorem 3.3], $r_{X_2}^{\{1\}}(x) \times [\tau_{\mathfrak{e}_2}] = [\tau_{\mathfrak{e}_1}] \times r_{X_2}^{\{2\}}(x)$ in $KK^1(\mathfrak{A}_1[1], \mathfrak{A}_2[2])$. Using this and the above equations, we get $[\tau_{\mathfrak{f}_1}] = [\tau_{\mathfrak{f}_2}]$ in $KK^1(\mathfrak{A}_1[1], \mathfrak{A}_2[2])$.

Since $\mathfrak{A}_i[j]$ is a nuclear, separable C^*-algebra, there are trivial extensions $\sigma_1, \sigma_2 \colon \mathfrak{A}_1[1] \to \mathcal{Q}(\mathfrak{A}_2[2])$ and there exists a unitary $v \in \mathcal{M}(\mathfrak{A}_2[2])$ such that $\mathrm{Ad}(\pi(v))\,(\tau_{\mathfrak{f}_1} \oplus \sigma_1) = \tau_{\mathfrak{f}_2} \oplus \sigma_2$, where π is the canonical surjective $*$-homomorphism from $\mathcal{M}(\mathfrak{A}_2[2])$ onto $\mathcal{Q}(\mathfrak{A}_2[2])$. By assumption, there exists a unitary $v_i \in \mathcal{M}(\mathfrak{A}_2[2])$ such that $\mathrm{Ad}(\pi(v_i)) \circ (\tau_{\mathfrak{f}_i} \oplus \sigma_i) = \tau_{\mathfrak{f}_i}$. Set $U = v_2 v v_1^*$. A computation shows that $\mathrm{Ad}(\pi(U)) \circ \tau_{\mathfrak{f}_1} = \tau_{\mathfrak{f}_2}$. Therefore, there exists an X_2-equivariant isomorphism $\gamma \colon \widetilde{\mathfrak{A}}_1 \to \widetilde{\mathfrak{A}}_2$ such that $\gamma_{\{2\}} = \mathrm{Ad}(U)|_{\mathfrak{A}_2[2]}$ and $\gamma_{\{1\}} = \mathrm{id}_{\mathfrak{A}_1[1]}$.

Set $\psi = \beta \circ \gamma \circ \alpha$. Then

$$KK(\psi_{\{1\}}) = KK(\alpha_{\{1\}}) \times KK(\gamma_{\{1\}}) \times KK(\beta_{\{1\}}) = r_{X_2}^{\{1\}}(x)$$

and

$$KK(\psi_{\{2\}}) = KK(\alpha_{\{2\}}) \times KK(\gamma_{\{2\}}) \times KK(\beta_{\{2\}}) = r_{X_2}^{\{2\}}(x). \qquad \square$$

Proposition 2.14. *Let $\mathfrak{I}$ be a stable, nuclear, separable C^*-algebra satisfying the corona factorization property. Then any full extension*

$$0 \to \mathfrak{I} \to \mathfrak{E} \to \mathfrak{A} \to 0$$

with $\mathfrak{A}$ a non-unital, nuclear, separable C^*-algebra is an absorbing extension.

Proof. Since the extension is full and $\mathfrak{I}$ has the corona factorization property, we have that extension is purely large. By [31], the extension is an absorbing extension. $\square$

2.4. Graph C^*-algebras

A *graph* (E^0, E^1, r, s) consists of a countable set E^0 of vertices, a countable set E^1 of edges, and maps $r : E^1 \to E^0$ and $s : E^1 \to E^0$ identifying the range and source of each edge. If E is a graph, the *graph C^*-algebra* $C^*(E)$ is the universal C^*-algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges satisfying

(1) $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$,
(2) $s_e s_e^* \leq p_{s(e)}$ for all $e \in E^1$, and
(3) $p_v = \sum_{\{e \in E^1 : s(e)=v\}} s_e s_e^*$ for all v with $0 < |s^{-1}(v)| < \infty$.

2.5. Some prerequisites

The following result should be well-known, but we could not find any reference for it. Thus, we provide a proof here.

Lemma 2.15. *Let $\mathfrak{A}$ be a separable C^*-algebra and let $\mathfrak{B}$ be a full hereditary C^*-subalgebra of $\mathfrak{A}$. Then the embedding of $\mathfrak{B}$ into $\mathfrak{A}$ induces a KK-equivalence.*

Proof. Let $\iota \colon \mathfrak{B} \to \mathfrak{A}$ be the embedding of $\mathfrak{B}$ into $\mathfrak{A}$. Then $KK(\iota)$ is represented by the Kasparov module $(\overline{\mathfrak{B}\mathfrak{A}}, \phi \colon \mathfrak{B} \to \mathbb{K}_{\mathfrak{A}}(\overline{\mathfrak{B}\mathfrak{A}}), 0)$, where $\phi(b)$ is left multiplication by b. Let y be the element in $KK(\mathfrak{A}, \mathfrak{B})$ represented by the Kasparov module $(\overline{\mathfrak{A}\mathfrak{B}}, \psi \colon \mathfrak{A} \to \mathbb{K}_{\mathfrak{B}}(\overline{\mathfrak{A}\mathfrak{B}}), 0)$, where $\psi(a)$ is left multiplication by a. Since $\mathfrak{B}$ is a hereditary C^*-subalgebra of $\mathfrak{A}$, we get that $\overline{\mathfrak{B}\mathfrak{A}\mathfrak{B}} = \mathfrak{B}$ which implies that $KK(\iota) \times y = [(\overline{\mathfrak{B}\mathfrak{A}} \otimes_\psi \overline{\mathfrak{A}\mathfrak{B}}, \phi \otimes_\psi 1, 0)] = [(\mathfrak{B}, \mathrm{id}, 0)]$. Since $\mathfrak{B}$ is full, $\overline{\mathfrak{A}\mathfrak{B}\mathfrak{A}} = \mathfrak{A}$ which implies that $y \times KK(\iota) = [(\overline{\mathfrak{A}\mathfrak{B}} \otimes_\phi \overline{\mathfrak{B}\mathfrak{A}}, \psi \otimes_\phi 1, 0)] = [(\mathfrak{A}, \mathrm{id}, 0)]$. Hence, $KK(\iota)$ is a KK-equivalence. $\square$

Lemma 2.16. *Let $\mathfrak{A}$ be a separable, nuclear C^*-algebra, and let $\mathfrak{B}$ be a separable stable C^*-algebra containing a full properly infinite projection p. Then for every x in $KK(\mathfrak{A}, \mathfrak{B})$ there exists a $*$-homomorphism $\varphi \colon \mathfrak{A} \to \mathfrak{B}$ such that $KK(\varphi) = x$.*

Proof. The algebra $p\mathfrak{B}p$ is clearly a unital, separable, properly infinite C^*-algebra. By Lemma 2.15, the embedding δ of $p\mathfrak{B}p$ into $p\mathfrak{B}p \otimes \mathbb{K}$ sending x to $x \otimes e_{1,1}$ (where $\{e_{i,j}\}$ is a system of matrix units for $\mathbb{K}$) induces a KK-equivalence. Since p is a full projection in B and since B is stable, by [9, Corollary 2.6], there exists an isomorphism $\lambda\colon p\mathfrak{B}p \otimes \mathbb{K} \to \mathfrak{B} \otimes \mathbb{K}$.

Suppose that $\mathfrak{A}$ is unital. Then it follows from [57, Theorem 8.3.3] that there exists a $*$-homomorphism η from $\mathfrak{A}$ to $p\mathfrak{B}p \otimes \mathbb{K}$ such that $\mathrm{KK}(\eta) = x \times \mathrm{KK}(\lambda^{-1}) \times \mathrm{KK}(\delta)^{-1} \times \mathrm{KK}(\delta) = x \times \mathrm{KK}(\lambda^{-1})$. Consequently, $\varphi = \lambda \circ \eta$ satisfies what we want.

Now, suppose that $\mathfrak{A}$ is non-unital, and let $\widetilde{\mathfrak{A}}$ be its unitization, and let ε denote the embedding of $\mathfrak{A}$ into its unitization. Then $\widetilde{\mathfrak{A}}$ is a separable, nuclear unital C^*-algebra, and the split exact sequence $0 \to \mathfrak{A} \to \widetilde{\mathfrak{A}} \to \mathbb{C} \to 0$ induces an exact sequence

$$0 \longrightarrow KK(\mathbb{C}, \mathfrak{B}) \longrightarrow KK(\widetilde{\mathfrak{A}}, \mathfrak{B}) \xrightarrow{\varepsilon^*} KK(\mathfrak{A}, \mathfrak{B}) \longrightarrow 0\,.$$

Thus, $x \in KK(\mathfrak{A}, \mathfrak{B})$ lifts to an element $y \in KK(\widetilde{\mathfrak{A}}, \mathfrak{B})$. The unital case above now implies that y is induced by a $*$-homomorphism. By composition with ε, we get the desired result. □

Lemma 2.17. *Let $\mathfrak{A}$ be a C^*-algebra and let $\mathfrak{I}$ be an ideal of $\mathfrak{A}$ such that $\mathfrak{A}/\mathfrak{I}$ is a finite dimensional C^*-algebra. If $p\mathfrak{I}p$ has an approximate identity consisting of projections for every projection p in $\mathfrak{A}$ and every projection in $\mathfrak{A}/\mathfrak{I}$ lifts to a projection in $\mathfrak{A}$, then there exists a $*$-homomorphism $\phi\colon \mathfrak{A}/\mathfrak{I} \to \mathfrak{A}$ such that $\pi \circ \phi = \mathrm{id}_{\mathfrak{A}/\mathfrak{I}}$, where $\pi\colon \mathfrak{A} \to \mathfrak{A}/\mathfrak{I}$ is the quotient map.*

Let $\mathfrak{A}$ be a C^*-algebra and let $\mathfrak{I}$ be an ideal of $\mathfrak{A}$ such that $\mathfrak{A}/\mathfrak{I}$ is an AF-algebra. If $p\mathfrak{I}p$ has an approximate identity consisting of projections for every projection p in $\mathfrak{A}$ and every projection in $\mathfrak{A}/\mathfrak{I}$ lifts to a projection in $\mathfrak{A}$, then there exists a sequence of finite dimensional sub-C^*-algebras $\{\mathfrak{C}_k\}_{k=1}^{\infty}$ of $\mathfrak{A}$ such $\mathfrak{C}_k \cap \mathfrak{I} = 0$ for all k, $\bigcup_{k=1}^{\infty}(\mathfrak{C}_k + \mathfrak{I})$ is dense in $\mathfrak{A}$, and $\mathfrak{C}_k + \mathfrak{I} \subseteq \mathfrak{C}_{k+1} + \mathfrak{I}$ for all k.

Proof. The first part of the lemma is proved in the same way as [16, Lemma 9.8]. The key ingredients of the proof are (1) for each projection $p \in \mathfrak{A}$, the hereditary sub-C^*-algebra $p\mathfrak{I}p$ has an approximate identity consisting of projections and (2) every projection in $\mathfrak{A}/\mathfrak{I}$ lifts to a projection in $\mathfrak{A}$.

We now prove the last part of the lemma. Since $\mathfrak{A}/\mathfrak{I}$ is an AF-algebra, there exists a sequence of finite dimensional sub-C^*-algebras $\{\mathfrak{D}_k\}_{k=1}^{\infty}$ of $\mathfrak{A}/\mathfrak{I}$ such that $\mathfrak{A}/\mathfrak{I} = \overline{\bigcup_{k=1}^{\infty} \mathfrak{D}_k}$ and $\mathfrak{D}_k \subseteq \mathfrak{D}_{k+1}$ for all k. By the first part of the lemma, we have a sequence of $*$-homomorphisms, $\{\phi_k\colon \mathfrak{D}_k \to \mathfrak{A}\}_{k=1}^{\infty}$ such that $\pi \circ \phi_k = \mathrm{id}_{\mathfrak{D}_k}$.

Set $\mathfrak{C}_k = \phi_k(\mathfrak{D}_k)$. Then $\mathfrak{C}_k$ is a finite dimensional sub-C^*-algebra of $\mathfrak{A}$. Since $\pi \circ \phi_k = \mathrm{id}_{\mathfrak{D}_k}$, we have that $\mathfrak{C}_k \cap \mathfrak{I} = 0$. Let $x \in \mathfrak{A}$ and let $\epsilon > 0$. Since $\pi(\bigcup_{k=1}^{\infty} \mathfrak{C}_k) = \bigcup_{k=1}^{\infty} \mathfrak{D}_k$, there exists $y_1 \in \mathfrak{C}_i$ (for some i) such that $\|\pi(x) - \pi(y_1)\| < \epsilon$. Thus, there exists $y_2 \in \mathfrak{I}$ such that $\|x - y_1 - y_2\| < \epsilon$.

Since $y_1 + y_2 \in \bigcup_{k=1}^{\infty}(\mathfrak{C}_k + \mathfrak{I})$, we have just shown that $\bigcup_{k=1}^{\infty}(\mathfrak{C}_k + \mathfrak{I})$ is dense in $\mathfrak{A}$.

Since $\mathfrak{D}_k \subseteq \mathfrak{D}_{k+1}$ for all k, we have that $\mathfrak{C}_k + \mathfrak{I} \subseteq \mathfrak{C}_{k+1} + \mathfrak{I}$. Thus, the lemma holds. □

Lemma 2.18. *Let $\mathfrak{A}_1$ be a C^*-algebra over X such that for each $U \in \mathbb{O}(X)$, we have that $\mathfrak{A}_1(U)$ is generated by projections, and let $\mathfrak{A}_2$ be a C^*-algebra over X. Suppose $\phi_0 \colon \mathfrak{A}_1 \to \mathfrak{A}_2$ is an X-equivariant homomorphism, $\phi_1 \colon \mathfrak{A}_1 \to \mathfrak{A}_2$ is a $*$-homomorphism, and $\psi \colon \mathfrak{A}_1 \to C([0,1], \mathfrak{A}_2)$ is a $*$-homomorphism such that $\psi_0 = \phi_0$ and $\psi_1 = \phi_1$. Then ϕ_1 is an X-equivariant homomorphism, and ψ is an X-equivariant homomorphism, where $C([0,1], \mathfrak{A}_2)$ is considered a C^*-algebra over X via $\mathbb{O}(X) \ni U \mapsto C([0,1], \mathfrak{A}_2(U))$.*

Proof. Let $U \in \mathbb{O}(X)$. Since $\mathfrak{A}_1(U)$ is generated by projections, it is enough to show that for all projections $p \in \mathfrak{A}_1(U)$, $\psi(p) \in C([0,1], \mathfrak{A}_2(U))$. Let p be a projection in $\mathfrak{A}_1(U)$. Choose a $\delta > 0$ such that $\|\psi_t(p) - \psi_s(p)\| < 1$ for all $|t - s| < \delta$. Therefore, if $|t - s| < \delta$, then $\psi_t(p) \in \mathfrak{A}_2(U)$ if and only if $\psi_s(p) \in \mathfrak{A}_2(U)$ since they are Murray-von Neumann equivalent. Let $0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$ be a partition of $[0,1]$ such that $|t_i - t_{i-1}| < \delta$. Since $\psi_0(p) = \phi_0(p) \in \mathfrak{A}_2(U)$, for all $t \in [0, t_1]$, $\psi_t(p) \in \mathfrak{A}_2(U)$. Continuing this process, we get that $\psi_t(p) \in \mathfrak{A}_2(U)$ for all $t \in [0,1]$. Therefore, $\psi(p) \in C([0,1], \mathfrak{A}_2(U))$. Consequently, $\psi_1 = \phi_1$ is an X-equivariant homomorphism. □

3. Meta-theorems

In many cases one can obtain a classification result for a class $\mathcal{C}$ of unital C^*-algebras by obtaining a classification result for the class $\mathcal{C} \otimes \mathbb{K}$, where each object in $\mathcal{C} \otimes \mathbb{K}$ is the stabilization of an object in $\mathcal{C}$. A meta-theorem of this sort was proved by the first and second named authors in [21, Theorem 11]. It was shown there that if $\mathcal{C}$ is a subcategory of the category of C^*-algebras, $\mathfrak{C}^*$-$\mathfrak{alg}$, and if F is a functor from $\mathcal{C}$ to an abelian category such that an isomorphism $F(\mathfrak{A} \otimes \mathbb{K}) \cong F(\mathfrak{B} \otimes \mathbb{K})$ lifts to an isomorphism in $\mathfrak{A} \otimes \mathbb{K} \cong \mathfrak{B} \otimes \mathbb{K}$, then under suitable conditions, we have that $F(\mathfrak{A}) \cong F(\mathfrak{B})$ implies $\mathfrak{A} \cong \mathfrak{B}$. In [52], the second and third named authors improved this result by showing that the isomorphism $F(\mathfrak{A}) \cong F(\mathfrak{B})$ lifts to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

In this section, we improve these results in order to deal with cases when $\mathcal{C}$ is a category (not necessarily a subcategory of $\mathfrak{C}^*$-$\mathfrak{alg}$) where the objects are C^*-algebras and there exists a functor from $\mathcal{C}$ to $\mathfrak{C}^*$-$\mathfrak{alg}$. An example of such a category is the category of C^*-algebras over $\{1,2\}$, where $\{1,2\}$ is given the discrete topology. Then $\mathcal{C}$ is not a subcategory of $\mathfrak{C}^*$-$\mathfrak{alg}$ but the forgetful functor (forgetting the $\{1,2\}$-structure) is a functor from $\mathcal{C}$ to $\mathfrak{C}^*$-$\mathfrak{alg}$. We also replace the condition of properly infinite by the stable weak cancellation property.

We first recall definitions of some cancellation properties.

Definition 3.1. A C^*-algebra $\mathfrak{A}$ is said to have the *weak cancellation property* if for all projections p and q in $\mathfrak{A}$, p is Murray-von Neumann equivalent to q whenever p and q generate the same ideal $\mathfrak{I}$ and $[p] = [q]$ in $K_0(\mathfrak{I})$. A C^*-algebra $\mathfrak{A}$ is said to have the *stable weak cancellation property* if $\mathsf{M}_n(\mathfrak{A})$ has the weak cancellation property for every $n \in \mathbb{N}$.

The following is a generalization of [21, Theorem 11].

Theorem 3.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $\mathfrak{C}^*$-$\mathfrak{alg}$ be the category of C^*-algebras, and let $\mathfrak{Ab}$ be the category of abelian groups. Suppose we have covariant functors $F\colon \mathcal{C} \to \mathfrak{C}^*$-$\mathfrak{alg}$, $G\colon \mathcal{C} \to \mathcal{D}$, and $H\colon \mathcal{D} \to \mathfrak{Ab}$ such that*

(1) *$H \circ G = K_0 \circ F$.*
(2) *For objects $\mathfrak{A}$ in $\mathcal{C}$, there exist an object $\mathfrak{A}_{\mathbb{K}}$ and a morphism $\kappa_{\mathfrak{A}}\colon \mathfrak{A} \to \mathfrak{A}_{\mathbb{K}}$ such that $G(\kappa_{\mathfrak{A}})$ is an isomorphism in $\mathcal{D}$, $F(\mathfrak{A}_{\mathbb{K}}) = F(\mathfrak{A}) \otimes \mathbb{K}$, and $F(\kappa_{\mathfrak{A}}) = \mathrm{id}_{F(\mathfrak{A})} \otimes e_{11}$, where $\{e_{ij}\}_{i,j=1}^{\infty}$ is a system of matrix units of $\mathbb{K}$.*
(3) *For all objects $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathcal{C}$, every isomorphism $G(\mathfrak{A}_{\mathbb{K}})$ to $G(\mathfrak{B}_{\mathbb{K}})$ is induced by an isomorphism from $\mathfrak{A}_{\mathbb{K}}$ to $\mathfrak{B}_{\mathbb{K}}$.*

Let $\mathfrak{A}$ and $\mathfrak{B}$ be given such that $F(\mathfrak{A})$ and $F(\mathfrak{B})$ are unital C^*-algebras. Let $\rho\colon G(\mathfrak{A}) \to G(\mathfrak{B})$ be an isomorphism such that $H(\rho)([1_{F(\mathfrak{A})}]) = [1_{F(\mathfrak{B})}]$. If $F(\mathfrak{A})$ and $F(\mathfrak{B})$ are properly infinite or $F(\mathfrak{B})$ has the stable weak cancellation property, then $F(\mathfrak{A}) \cong F(\mathfrak{B})$.

Proof. Note that $G(\kappa_{\mathfrak{A}})$ and $G(\kappa_{\mathfrak{B}})$ are isomorphisms. Therefore $G(\kappa_{\mathfrak{B}}) \circ \rho \circ G(\kappa_{\mathfrak{A}})^{-1}$ is an isomorphism from $G(\mathfrak{A}_{\mathbb{K}})$ to $G(\mathfrak{B}_{\mathbb{K}})$. Thus, there exists an isomorphism $\phi\colon \mathfrak{A}_{\mathbb{K}} \to \mathfrak{B}_{\mathbb{K}}$ such that $G(\phi) = G(\kappa_{\mathfrak{B}}) \circ \rho \circ G(\kappa_{\mathfrak{A}})^{-1}$.

Set $\psi = F(\phi)$. Then $\psi\colon F(\mathfrak{A}) \otimes \mathbb{K} \to F(\mathfrak{B}) \otimes \mathbb{K}$ is a $*$-isomorphism such that

$$\begin{aligned} K_0(\psi) &= K_0(F(\phi)) \\ &= H(G(\kappa_{\mathfrak{B}}) \circ \rho \circ G(\kappa_{\mathfrak{A}})^{-1}) \\ &= H(G(\kappa_{\mathfrak{B}})) \circ H(\rho) \circ H(G(\kappa_{\mathfrak{A}})^{-1}) \\ &= K_0(F(\kappa_{\mathfrak{B}})) \circ H(\rho) \circ K_0(F(\kappa_{\mathfrak{A}}))^{-1} \\ &= K_0(\mathrm{id}_{F(\mathfrak{B})} \otimes e_{11}) \circ H(\rho) \circ K_0(\mathrm{id}_{F(\mathfrak{A})} \otimes e_{11})^{-1}. \end{aligned}$$

Hence,

$$\begin{aligned} &K_0(\psi)([1_{F(\mathfrak{A})} \otimes e_{11}]) \\ &= K_0(\mathrm{id}_{F(\mathfrak{B})} \otimes e_{11}) \circ H(\rho) \circ K_0(\mathrm{id}_{F(\mathfrak{A})} \otimes e_{11})^{-1}([1_{F(\mathfrak{A})} \otimes e_{11}]) \\ &= K_0(\mathrm{id}_{F(\mathfrak{B})} \otimes e_{11}) \circ H(\rho)([1_{F(\mathfrak{A})}]) \\ &= K_0(\mathrm{id}_{F(\mathfrak{B})} \otimes e_{11})([1_{F(\mathfrak{B})}]) \\ &= [1_{F(\mathfrak{B})} \otimes e_{11}]. \end{aligned}$$

Proper infiniteness or stable weak cancellation implies that there exists $v \in F(\mathfrak{B}) \otimes \mathbb{K}$ such that $v^*v = \psi(1_{F(\mathfrak{A})} \otimes e_{11})$ and $vv^* = 1_{F(\mathfrak{B})} \otimes e_{11}$ since $\psi(1_{F(\mathfrak{A})} \otimes e_{11})$ and $1_{F(\mathfrak{B})} \otimes e_{11}$ are full projections in $F(\mathfrak{B}) \otimes \mathbb{K}$ (cf. also the proof of [21, Theorem 11] for the properly infinite case). Set $\gamma(x) = v\psi(x \otimes e_{11})v^*$. Arguing as in the proof of [21, Theorem 11], γ is an isomorphism from $F(\mathfrak{A}) \otimes e_{11}$ to $F(\mathfrak{B}) \otimes e_{11}$. Hence, $F(\mathfrak{A}) \cong F(\mathfrak{B})$. □

The following is a generalization of [52, Theorem 2.1].

Theorem 3.3. *Let X be a T_0-space and let $\mathcal{C}$ be a subcategory of $\mathfrak{C}^*$-$\mathfrak{alg}(X)$. Moreover, $\mathcal{C}$ is assumed to be closed under tensoring by $\mathsf{M}_2(\mathbb{C})$ and $\mathbb{K}$ and contains the canonical embeddings $\kappa_1 \colon \mathfrak{A} \to \mathsf{M}_2(\mathfrak{A})$ and $\kappa \colon \mathfrak{A} \to \mathfrak{A} \otimes \mathbb{K}$ as morphisms for every object $\mathfrak{A}$ in $\mathcal{C}$. Assume there is a functor $F \colon \mathcal{C} \to \mathcal{D}$ satisfying*

1. *For $\mathfrak{A}$ in $\mathcal{C}$, the embeddings $\kappa_1 \colon \mathfrak{A} \to \mathsf{M}_2(\mathfrak{A})$ $(x \mapsto x \otimes e_{11})$ and $\kappa \colon \mathfrak{A} \to \mathfrak{A} \otimes \mathbb{K}$ $(x \mapsto x \otimes e_{11})$, where $\{e_{ij}\}_{i,j=1}^{\infty}$ is a system of matrix units of $\mathbb{K}$ induce isomorphisms $F(\kappa_1)$ and $F(\kappa)$.*
2. *For all objects $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathcal{C}$ that are stable C^*-algebras, every isomorphism from $F(\mathfrak{A})$ to $F(\mathfrak{B})$ is induced by an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.*
3. *There exists a functor G from $\mathcal{D}$ to $\mathfrak{Ab}$ such that $G \circ F = K_0$.*

Assume that every X-equivariant isomorphism between objects in $\mathcal{C}$ is a morphism in $\mathcal{C}$ and that for objects $\mathfrak{A}$ in $\mathcal{C}$, $F(\mathrm{Ad}(u)|_{\mathfrak{A}}) = \mathrm{id}_{F(\mathfrak{A})}$ for every unitary $u \in \mathcal{M}(\mathfrak{A})$. If $\mathfrak{A}$ and $\mathfrak{B}$ are objects $\mathcal{C}$ that are unital C^*-algebras such that $\mathfrak{A}$ and $\mathfrak{B}$ both are either properly infinite or both have the stable weak cancellation property, and there is an isomorphism $\alpha \colon F(\mathfrak{A}) \to F(\mathfrak{B})$ such that $G(\alpha)([1_{\mathfrak{A}}]) = [1_{\mathfrak{B}}]$, then there exists an isomorphism $\phi \colon \mathfrak{A} \to \mathfrak{B}$ in $\mathcal{C}$ such that $F(\phi) = \alpha$.

Proof. The differences between the statement of [52, Theorem 2.1] and statement of the theorem are

(i) $\mathcal{C}$ is assumed to be a subcategory of $\mathfrak{C}^*$-$\mathfrak{alg}(X)$ instead of a subcategory of $\mathfrak{C}^*$-$\mathfrak{alg}$.

(ii) $\mathfrak{A}$ and $\mathfrak{B}$ might have the stable weak cancellation property instead of being properly infinite.

In the proof of [52, Theorem 2.1], properly infinite was used to ensure that $\psi(1_{\mathfrak{A}} \otimes e_{11})$ is Murray-von Neumann equivalent to $1_{\mathfrak{B}} \otimes e_{11}$, where $\psi \colon \mathfrak{A} \otimes \mathbb{K} \to \mathfrak{B} \otimes \mathbb{K}$ is the isomorphism from (2) that lifts the isomorphism from $F(\mathfrak{A} \otimes \mathbb{K})$ to $F(\mathfrak{B} \otimes \mathbb{K})$ that is induced by α. As in the proof of Theorem 3.2, we get that $\psi(1_{\mathfrak{A}} \otimes e_{11})$ is Murray-von Neumann equivalent to $1_{\mathfrak{B}} \otimes e_{11}$, using the fact that $\mathfrak{B}$ is either properly infinite or has the stable weak cancellation property. Arguing as in the proof of [52, Theorem 2.1], we get the desired result. □

4. Classification results

In this section, we show that $K_{X_2}^{+}(-)$ is a strong classification functor for a class of C^*-algebras with exactly one non-trivial ideal that contains the class of unital graph C^*-algebras with exactly one non-trivial ideal. We obtain our classification result by establishing an existence theorem and two uniqueness theorems. Recall that a *Kirchberg algebra* is a nuclear, purely infinite, separable, simple C^*-algebra.

Theorem 4.1 (Existence Theorem)**.** *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be separable, nuclear C^*-algebras over X_2 in $\mathcal{B}(X_2)$ and let $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ be an invertible element such that $\Gamma(x)_{Y,0}$ is an order isomorphism from $K_0(\mathfrak{A}_1[Y])$ to $K_0(\mathfrak{A}_2[Y])$ for all $Y \in \mathbb{LC}(X_2)$. Suppose $0 \to \mathfrak{A}_i[2] \to \mathfrak{A}_i \to \mathfrak{A}_i[1] \to 0$ is a full extension, $\mathfrak{A}_i[2]$ is a stable C^*-algebra, $\mathfrak{A}_i$ has real rank zero, and either*

(i) *$\mathfrak{A}_i[2]$ is a purely infinite simple C^*-algebra and $\mathfrak{A}_i[1]$ is an AF-algebra*; *or*

(ii) *$\mathfrak{A}_i[2]$ is an AF-algebra and $\mathfrak{A}_i[1]$ is a purely infinite simple C^*-algebra.*

Then there exists an X_2-equivariant homomorphism $\phi\colon \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KK(X_2; \phi) = KK(X_2; \mathrm{id}_{\mathfrak{A}_1} \otimes e_{11})^{-1} \times x \times KK(X_2; \mathrm{id}_{\mathfrak{A}_2} \otimes e_{11})$, and $\phi_{\{2\}}$ and $\phi_{\{1\}}$ are injective, where $\{e_{ij}\}_{i,j=1}^{\infty}$ is a system of matrix units of $\mathbb{K}$.

Proof. Set $y = KK(X_2; \mathrm{id}_{\mathfrak{A}_1} \otimes e_{11})^{-1} \times x \times KK(X_2; \mathrm{id}_{\mathfrak{A}_2} \otimes e_{11})$. Since $\mathfrak{A}_i[2] \otimes \mathbb{K}$ is either a nuclear, separable, stable, purely infinite C^*-algebra in the bootstrap class or a separable AF-algebra, it has finite nuclear dimension (cf. [60] and [39]), so it follows from [53, Corollary 3.5] that $\mathfrak{A}_i[2] \otimes \mathbb{K}$ satisfies the corona factorization property (see [40] for the definition of the corona factorization property). Since $\mathfrak{A}_i[k]$ is an AF-algebra or a Kirchberg algebra, $\mathfrak{A}_i[k]$ has the stable weak cancellation property. By [26, Lemma 3.15], $\mathfrak{A}_i$ has the stable weak cancellation property. Let $\mathfrak{e}_i$ be the extension

$$0 \to \mathfrak{A}_i[2] \otimes \mathbb{K} \to \mathfrak{A}_i \otimes \mathbb{K} \to \mathfrak{A}_i[1] \otimes \mathbb{K} \to 0.$$

By [22, Proposition 1.6], $\mathfrak{e}_i$ is a full extension since $0 \to \mathfrak{A}_i[2] \to \mathfrak{A}_i \to \mathfrak{A}_i[1] \to 0$ is a full extension and $\mathfrak{A}_i[2]$ is a stable C^*-algebra.

Case (i): *$\mathfrak{A}_i[2]$ is a purely infinite simple C^*-algebra and $\mathfrak{A}_i[1]$ is an AF-algebra.* Since y is invertible in $KK(X_2; \mathfrak{A}_1 \otimes \mathbb{K}, \mathfrak{A}_2 \otimes \mathbb{K})$, we have that $r_{X_2}^{\{1\}}(y)$ is invertible in $KK(\mathfrak{A}_1[1] \otimes \mathbb{K}, \mathfrak{A}_2[1] \otimes \mathbb{K})$ and $\Gamma(r_{X_2}^{\{1\}}(y)) = \Gamma(x)_{\{1\}}$ is an order isomorphism (up to canonical identification). Thus, by Elliott's classification [30], there exists an isomorphism $\psi_1\colon \mathfrak{A}_1[1] \otimes \mathbb{K} \to \mathfrak{A}_2[1] \otimes \mathbb{K}$ such that $KK(\psi_1) = r_{X_2}^{\{1\}}(y)$. Since y is invertible in $KK(X_2; \mathfrak{A}_1 \otimes \mathbb{K}, \mathfrak{A}_2 \otimes \mathbb{K})$, we have that $r_{X_2}^{\{2\}}(y)$ is invertible in $KK(\mathfrak{A}_1[2] \otimes \mathbb{K}, \mathfrak{A}_2[2] \otimes \mathbb{K})$. Thus, by Kirchberg-Phillips classification (see [38] and [49]), there exists an isomorphism $\psi_0\colon \mathfrak{A}_1[2] \otimes \mathbb{K} \to \mathfrak{A}_2[2] \otimes \mathbb{K}$ such that $KK(\psi_0) = r_{X_2}^{\{2\}}(y)$. By Proposition 2.13 and Proposition 2.14, there exists an X_2-equivariant isomorphism $\psi\colon \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KK(\psi_{\{k\}}) = r_{X_2}^{\{k\}}(y)$ for $k = 1, 2$.

Note that

$$0 \to i_{\{2\}}^{X_2}((\mathfrak{A}_i \otimes \mathbb{K})[2]) \xrightarrow{\lambda_i} \mathfrak{A}_i \otimes \mathbb{K} \xrightarrow{\beta_i} i_{\{1\}}^{X_2}((\mathfrak{A}_i \otimes \mathbb{K})[1]) \to 0$$

is a semi-split extension of C^*-algebras over X_2 (see [47, Definition 3.5]). Set

$$\mathfrak{I}_i = i_{\{2\}}^{X_2}((\mathfrak{A}_i \otimes \mathbb{K})[2]) \quad \text{and} \quad \mathfrak{B}_i = i_{\{1\}}^{X_2}((\mathfrak{A}_i \otimes \mathbb{K})[1]).$$

By [47, Theorem 3.6] (see also [8, Korollar 3.4.6]),

$$KK(X_2;\mathfrak{A}_1 \otimes \mathbb{K}, \mathfrak{I}_2) \xrightarrow{(\lambda_2)_*} KK(X_2;\mathfrak{A}_1 \otimes \mathbb{K}, \mathfrak{A}_2 \otimes \mathbb{K}) \xrightarrow{(\beta_2)_*} KK(X_2;\mathfrak{A}_1 \otimes \mathbb{K}, \mathfrak{B}_2)$$

is exact. By [47, Proposition 3.12], $KK(X_2;\mathfrak{A}_1 \otimes \mathbb{K}, \mathfrak{B}_2)$ and $KK(\mathfrak{A}_1[1] \otimes \mathbb{K}, \mathfrak{A}_2[1] \otimes \mathbb{K})$ are naturally isomorphic. Hence, there exists $z \in KK(X_2;\mathfrak{A}_1 \otimes \mathbb{K}, \mathfrak{I}_2)$ such that $y - KK(X_2;\psi) = z \times KK(X_2;\lambda_2)$ since $KK(\psi_{\{1\}}) = r_{X_2}^{\{1\}}(y)$.

By [47, Proposition 3.13], $KK(X_2;\mathfrak{A}_1 \otimes \mathbb{K}, \mathfrak{I}_2)$ and $KK(\mathfrak{A}_1 \otimes \mathbb{K}, (\mathfrak{A}_2 \otimes \mathbb{K})[2])$ are isomorphic. By Lemma 2.16, there exists a $*$-homomorphism $\eta\colon \mathfrak{A}_1 \otimes \mathbb{K} \to (\mathfrak{A}_2 \otimes \mathbb{K})[2]$ such that $KK(\eta) = \overline{z}$, where $\overline{z}$ is the image of z under the isomorphism $KK(X_2;\mathfrak{A}_1 \otimes \mathbb{K}, \mathfrak{I}_2) \cong KK(\mathfrak{A}_1 \otimes \mathbb{K}, (\mathfrak{A}_2 \otimes \mathbb{K})[2])$. Note that η induces an X_2-equivariant homomorphism $\eta\colon \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{I}_2$ such that $KK(X_2;\eta) = z$.

Set $\phi = \psi + (\lambda_2 \circ \eta)$, where the sum is the Cuntz sum in $\mathcal{M}(\mathfrak{A}_2 \otimes \mathbb{K})$. Then $\phi\colon \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ is an X_2-equivariant homomorphism such that $KK(X_2;\phi) = y$. Since $\psi_{\{2\}}$ and $\psi_{\{1\}}$ are injective $*$-homomorphisms, $\phi_{\{2\}}$ and $\phi_{\{1\}}$ are injective $*$-homomorphisms.

Case (ii): $\mathfrak{A}_i[2]$ *is an AF-algebra and* $\mathfrak{A}_i[1]$ *is a purely infinite simple C^*-algebra.* Since y is invertible in $KK(X_2;\mathfrak{A}_1 \otimes \mathbb{K}, \mathfrak{A}_2 \otimes \mathbb{K})$, we have that $r_{X_2}^{\{2\}}(y)$ is invertible in $KK(\mathfrak{A}_1[2] \otimes \mathbb{K}, \mathfrak{A}_2[2] \otimes \mathbb{K})$ and $\Gamma(r_{X_2}^{\{2\}}(y)) = \Gamma(x)_{\{2\}}$ is an order isomorphism (up to canonical identification). Thus, by Elliott's classification [30], there exists an isomorphism $\psi_0\colon \mathfrak{A}_1[2] \otimes \mathbb{K} \to \mathfrak{A}_2[2] \otimes \mathbb{K}$ such that $KK(\psi_0) = r_{X_2}^{\{2\}}(y)$. Since y is invertible in $KK(X_2;\mathfrak{A}_1 \otimes \mathbb{K}, \mathfrak{A}_2 \otimes \mathbb{K})$, we have that $r_{X_2}^{\{1\}}(y)$ is invertible in $KK(\mathfrak{A}_1[1] \otimes \mathbb{K}, \mathfrak{A}_2[1] \otimes \mathbb{K})$. Thus, by Kirchberg-Phillips classification (see [38] and [49]), there exists an isomorphism $\psi_1\colon \mathfrak{A}_1[1] \otimes \mathbb{K} \to \mathfrak{A}_2[1] \otimes \mathbb{K}$ such that $KK(\psi_1) = r_{X_2}^{\{1\}}(y)$. By Proposition 2.13 and Proposition 2.14, there exists an X_2-equivariant isomorphism $\psi\colon \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KK(\psi_{\{k\}}) = r_{X_2}^{\{k\}}(y)$ for $k = 1, 2$.

Note that

$$0 \to i_{\{2\}}^{X_2}((\mathfrak{A}_i \otimes \mathbb{K})[2]) \xrightarrow{\lambda_i} \mathfrak{A}_i \otimes \mathbb{K} \xrightarrow{\beta_i} i_{\{1\}}^{X_2}((\mathfrak{A}_i \otimes \mathbb{K})[1]) \to 0$$

is a semi-split extension of C^*-algebras over X_2 (see [47, Definition 3.5]). Set

$$\mathfrak{I}_i = i_{\{2\}}^{X_2}((\mathfrak{A}_i \otimes \mathbb{K})[2]) \quad \text{and} \quad \mathfrak{B}_i = i_{\{1\}}^{X_2}((\mathfrak{A}_i \otimes \mathbb{K})[1]).$$

By [47, Theorem 3.6] (see also [8, Korollar 3.4.6])

$$KK(X_2;\mathfrak{B}_1, \mathfrak{A}_2 \otimes \mathbb{K}) \xrightarrow{(\beta_1)^*} KK(X_2;\mathfrak{A}_1 \otimes \mathbb{K}, \mathfrak{A}_2 \otimes \mathbb{K}) \xrightarrow{(\lambda_1)^*} KK(X_2;\mathfrak{I}_1, \mathfrak{A}_2 \otimes \mathbb{K})$$

is exact. By [47, Proposition 3.12], $KK(X_2;\mathfrak{I}_1, \mathfrak{A}_2 \otimes \mathbb{K})$ and $KK(\mathfrak{A}_1[2] \otimes \mathbb{K}, \mathfrak{A}_2[2] \otimes \mathbb{K})$ are naturally isomorphic. Thus there exists $z \in KK(X_2;\mathfrak{B}_1, \mathfrak{A}_2 \otimes \mathbb{K})$ such that $y - KK(X_2;\psi) = KK(X_2;\beta_1) \times z$ since $KK(\psi_{\{2\}}) = r_{X_2}^{\{2\}}(y)$.

By [47, Proposition 3.13], $KK(X_2; \mathfrak{B}_1, \mathfrak{A}_2 \otimes \mathbb{K})$ and $KK((\mathfrak{A}_1 \otimes \mathbb{K})[1], \mathfrak{A}_2 \otimes \mathbb{K})$ are isomorphic. Therefore, by Lemma 2.16, there exists a $*$-homomorphism $\eta\colon (\mathfrak{A}_1 \otimes \mathbb{K})[1] \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KK(\eta) = \overline{z}$, where $\overline{z}$ is the image of z under the isomorphism $KK(X_2; \mathfrak{B}_1, \mathfrak{A}_2 \otimes \mathbb{K}) \cong KK((\mathfrak{A}_1 \otimes \mathbb{K})[1], \mathfrak{A}_2 \otimes \mathbb{K})$ (the existence of the $*$-homomorphism uses the fact that $\mathfrak{A}_2 \otimes \mathbb{K}$ is a properly infinite C^*-algebra which follows from [26, Proposition 3.21 and Corollary 3.22]). Note that η induces an X_2-equivariant homomorphism $\eta\colon \mathfrak{B}_1 \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KK(X_2; \eta) = z$.

Set $\phi = \psi + (\eta \circ \beta_1)$, where the sum is the Cuntz sum in $\mathcal{M}(\mathfrak{A}_2 \otimes \mathbb{K})$. Then ϕ is an X_2-equivariant homomorphism such that $KK(X_2; \phi) = y$. Since $\psi_{\{2\}}$ and $\psi_{\{1\}}$ are injective $*$-homomorphisms, $\phi_{\{2\}}$ and $\phi_{\{1\}}$ are injective $*$-homomorphisms. □

4.1. Strong classification of extensions of AF-algebras by purely infinite C^*-algebras

Definition 4.2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable C^*-algebras over X. Two X-equivariant homomorphisms $\phi, \psi\colon \mathfrak{A} \to \mathfrak{B}$ are said to be *approximately unitarily equivalent* if there exists a sequence of unitaries $\{u_n\}_{n=1}^{\infty}$ in $\mathcal{M}(\mathfrak{B})$ such that

$$\lim_{n\to\infty} \|u_n \phi(a) u_n^* - \psi(a)\| = 0$$

for all $a \in \mathfrak{A}$.

When we only consider separable C^*-algebras, this definition of approximate unitary equivalence is equivalent to the definition in [57, Definition 1.1.15], which says that the two homomorphisms are approximately unitarily equivalent if and only if for every $\varepsilon > 0$ and every finite subset $\mathcal{F}$ of $\mathfrak{A}$ there is a unitary $u \in \mathcal{M}(\mathfrak{B})$ such that $\|u\phi(a)u^* - \psi(a)\| < \varepsilon$ for all a in $\mathcal{F}$.

We now recall the definition of $KL(\mathfrak{A}, \mathfrak{B})$ from [55].

Definition 4.3. Let $\mathfrak{A}$ be a separable, nuclear C^*-algebra in $\mathcal{N}$ and let $\mathfrak{B}$ be a σ-unital C^*-algebra. Let

$$\mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B})) = \mathrm{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), K_1(\mathfrak{B})) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}), K_0(\mathfrak{B})).$$

Since $\mathfrak{A}$ is in $\mathcal{N}$, by [59], $\mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B}))$ can be identified as a sub-group of the group $KK(\mathfrak{A}, \mathfrak{B})$.

For abelian groups, G and H, let $\mathrm{Pext}^1_{\mathbb{Z}}(G, H)$ be the subgroup of $\mathrm{Ext}^1_{\mathbb{Z}}(G, H)$ of all pure extensions of G by H. Set

$$\mathrm{Pext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B})) = \mathrm{Pext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), K_1(\mathfrak{B})) \oplus \mathrm{Pext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}), K_0(\mathfrak{B})).$$

Define $KL(\mathfrak{A}, \mathfrak{B})$ as the quotient

$$KL(\mathfrak{A}, \mathfrak{B}) = KK(\mathfrak{A}, \mathfrak{B}) / \mathrm{Pext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B})).$$

Notation 4.4. Let $x \in KK(\mathfrak{A}, \mathfrak{B})$ be given. Then we denote the element $x + \mathrm{Pext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B}))$ in $KL(\mathfrak{A}, \mathfrak{B})$ by $KL(x)$.

Rørdam in [55] proved that if $\phi, \psi\colon \mathfrak{A} \to \mathfrak{B}$ are approximately unitarily equivalent, then $KL(\phi) = KL(\psi)$.

Theorem 4.5 (Uniqueness Theorem 1)**.** *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be separable, nuclear, stable C^*-algebras over X_2 of real rank zero such that for each i, $\mathfrak{A}_i[2]$ is a Kirchberg algebra in $\mathcal{N}$, $\mathfrak{A}_i[1]$ is an AF-algebra, and $\mathfrak{A}_i[2]$ is an essential ideal of $\mathfrak{A}_i$. Suppose $\phi, \psi\colon \mathfrak{A}_1 \to \mathfrak{A}_2$ are X_2-equivariant homomorphisms such that $KK(X_2; \phi) = KK(X_2; \psi)$, and $\phi_{\{2\}}$, $\phi_{\{1\}}$, $\psi_{\{2\}}$, and $\psi_{\{1\}}$ are injective $*$-homomorphisms. Then ϕ and ψ are approximately unitarily equivalent.*

Proof. Since $\mathfrak{A}_i[1]$ is an AF-algebra, we know that every finitely generated subgroup of $K_0(\mathfrak{A}_i[1])$ is torsion free (hence free), and $K_1(\mathfrak{A}_i[1]) = 0$. Thus, $\mathrm{Pext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}_i[1]), K_{*+1}(\mathcal{Q}(\mathfrak{A}_j[2]))) = \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}_i[1]), K_{*+1}(\mathcal{Q}(\mathfrak{A}_j[2])))$, which implies that

$$KL(\mathfrak{A}_i[1], \mathcal{Q}(\mathfrak{A}_j[2])) \cong \mathrm{Hom}(K_0(\mathfrak{A}_i[1]), K_0(\mathcal{Q}(\mathfrak{A}_j[2]))),$$

where the isomorphism sends x to $K_0(x)$.

Let $\mathfrak{e}_i$ denote the extension $0 \to \mathfrak{A}_i[2] \to \mathfrak{A}_i \to \mathfrak{A}_i[1] \to 0$. Since $\mathfrak{A}_i$ has real rank zero and $K_1(\mathfrak{A}_i[1]) = 0$, we have that $K_j(\tau_{\mathfrak{e}_i}) = 0$, where $\tau_{\mathfrak{e}_i}$ is the Busby invariant of $\mathfrak{e}_i$. Hence, $[\tau_{\mathfrak{e}_i}] = 0$ in $KL(\mathfrak{A}_i[1], \mathcal{Q}(\mathfrak{A}_i[2]))$. By [45, Corollary 6.7], $\mathfrak{e}_i$ is quasi-diagonal. In particular, there exists an approximate identity $\{e_k\}_{k=1}^{\infty}$ of $\mathfrak{A}_1[2]$ consisting of projections such that

$$\lim_{n\to\infty} \|e_k x - x e_k\| = 0$$

for all $x \in \mathfrak{A}_1$.

From [10, Theorem 2.6], it follows that $p\mathfrak{A}_1[2]p$ has an approximate identity consisting of projections for every projection $p \in \mathfrak{A}_1$. Since $\mathfrak{A}_1[1]$ is an AF-algebra and $\mathfrak{A}_1$ has real rank zero, it now follows from Lemma 2.17 that there exists a sequence of finite dimensional sub-C^*-algebras $\{\mathfrak{B}_k\}_{k=1}^{\infty}$ of $\mathfrak{A}_1$ such that $\mathfrak{B}_k \cap \mathfrak{A}_1[2] = \{0\}$, $\bigcup_{k=1}^{\infty}(\mathfrak{B}_k + \mathfrak{A}_1[2])$ is dense in $\mathfrak{A}_1$, and $\mathfrak{B}_k + \mathfrak{A}_1[2] \subseteq \mathfrak{B}_{k+1} + \mathfrak{A}_1[2]$.

We want to prove that ϕ is approximately unitarily equivalent to ψ. Since the C^*-algebras are all separable, we may show that for every $\epsilon > 0$ and every finite subset $\mathcal{F}$ of $\mathfrak{A}_1$, there exists a unitary $u \in \mathcal{M}(\mathfrak{A}_2)$ such that $\|u\phi(x)u^* - \psi(x)\| < \epsilon$ for all $x \in \mathcal{F}$ instead of using Definition 4.2. So let $\epsilon > 0$ and let $\mathcal{F}$ be a finite subset of $\mathfrak{A}_1$. Since $\mathfrak{B}_k + \mathfrak{A}_1[2] \subseteq \mathfrak{B}_{k+1} + \mathfrak{A}_1[2]$ and $\bigcup_{k=1}^{\infty}(\mathfrak{B}_k + \mathfrak{A}_1[2])$ is dense in $\mathfrak{A}_1$, we may assume that there exist $m \in \mathbb{N}$ and a finite subset of $\mathcal{G}_2$ of $\mathfrak{A}_1[2]$ and a finite subset $\mathcal{G}_1$ of $\mathfrak{B}_m$, such that every element y of $\mathcal{F}$ is written (uniquely) in the form $y_1 + y_2$, where y_1 is an element of $\mathcal{G}_1 \subseteq \mathfrak{B}_m$ and $y_2 \in \mathcal{G}_2 \subseteq \mathfrak{A}_1[2]$.

The algebra $\mathfrak{B}_m$ is a finite dimensional C^*-algebra, so let $(e^{\ell}_{i,j})$ be a system of matrix units generating this algebra. Thus every $y \in \mathfrak{B}_m$ has a (unique) representation

$$y = \sum_{i,j,\ell} \lambda^{\ell}_{i,j}(y) e^{\ell}_{i,j}.$$

Let

$$\alpha = \max_{i,j,\ell,y\in\mathcal{G}_1} |\lambda^\ell_{i,j}(y)|.$$

Since

$$\lim_{k\to\infty} \|e_k x - x e_k\| = 0$$

for all $x \in \mathfrak{A}_1$, every element x in $\mathfrak{A}_1$ is approximated by $(1-e_k)x(1-e_k)+e_kxe_k$, and moreover cutting down by $1-e_k$ gives a contractive, completely positive, linear map from $\mathfrak{B}_m$ to $(1-e_k)\mathfrak{B}_m(1-e_k)$ such that

$$(1-e_k)xy(1-e_k) - (1-e_k)x(1-e_k)y(1-e_k)$$

converges to zero.

Choose k_1 such that

$$\|x - (1-e_k)x(1-e_k) - e_kxe_k\| < \frac{\varepsilon}{9}$$

for all $k \geq k_1$ and all $x \in \mathcal{G}_1 \subseteq \mathfrak{B}_m \subseteq \mathfrak{A}_1$. Choose $k_2 \geq k_1$ such that

$$\|x - e_kxe_k\| < \frac{\varepsilon}{9}$$

for all $k \geq k_2$ and all $x \in \mathcal{G}_2 \subseteq \mathfrak{A}_1[2]$.

Let $e^{\ell,k}_{i,j}$ denote $(1-e_k)e^\ell_{i,j}(1-e_k)$. Then

$$e^{\ell,k}_{i,j}e^{\ell,k}_{j,\ell} - e^{\ell,k}_{i,\ell}$$

and

$$e^{\ell,k}_{i,j}e^{\ell',k}_{i',j'}$$

converge to zero, and $(e^{\ell,k}_{i,j})^* = e^{\ell,k}_{j,i}$. Every finite dimensional C^*-algebra is semiprojective. From [46, Theorem 14.1.4], it follows that the set of matrix unit relations (that generate the finite dimensional C^*-algebra $\mathfrak{B}_m$) is stable. Thus, it follows from [46, Definition 14.1.1] that there exist a $k_3 \geq k_2$ and a system of matrix units (f^ℓ_{ij}) in $(1-e_{k_3})\mathfrak{B}_m(1-e_{k_3})$ such that $\|f^\ell_{ij} - e^{\ell,k_3}_{ij}\| < \frac{\varepsilon}{9(\alpha+1)}$. Let $\mathfrak{D}$ denote the finite dimensional C^*-algebra generated by the system of matrix units $(f^\ell_{i,j})$. Then $\mathfrak{D}$ is a subset of $(1_{\mathcal{M}(\mathfrak{A}_1)} - e_{k_3})\mathfrak{A}_1(1_{\mathcal{M}(\mathfrak{A}_1)} - e_{k_3})$ and $\mathfrak{D} \cap \mathfrak{A}_1[2] = \{0\}$.

Moreover, for every $x \in \mathcal{F}$, we have that (recall that $x = x_1 + x_2$, where $x_1 \in \mathcal{G}_1$ and $x_2 \in \mathcal{G}_2$)

$$\begin{aligned}
&\left\| x_1 + x_2 - \sum_{i,j,\ell} \lambda^\ell_{i,j}(x_1) f^\ell_{i,j} - e_{k_3}x_1e_{k_3} - e_{k_3}x_2e_{k_3} \right\| \\
&= \|x_1 - (1-e_{k_3})x_1(1-e_{k_3}) - e_{k_3}x_1e_{k_3}\| \\
&\quad + \left\| (1-e_{k_3})x_1(1-e_{k_3}) - \sum_{i,j,\ell} \lambda^\ell_{i,j}(x_1) f^\ell_{i,j} \right\| + \|x_2 - e_{k_3}x_2e_{k_3}\|
\end{aligned}$$

$$
\begin{aligned}
&< \frac{\varepsilon}{9} + \sum_{i,j,\ell} \left|\lambda^{\ell}_{i,j}(x_1)\right| \cdot \left\| e^{\ell,k_3}_{i,j} - f^{\ell}_{i,j} \right\| + \frac{\varepsilon}{9} \\
&< \frac{\varepsilon}{9} + \sum_{i,j,\ell} \left|\lambda^{\ell}_{i,j}(x_1)\right| \cdot \frac{\varepsilon}{9(\alpha+1)} + \frac{\varepsilon}{9} \\
&< \frac{\varepsilon}{3}.
\end{aligned}
$$

Thus we have shown, that there are a k_3, a finite dimensional C^*-algebra $\mathfrak{D}$ with the above mentioned properties and a finite subset $\mathcal{H}$ of $e_{k_3}\mathfrak{A}_1[2]e_{k_3}$ such that for every $x \in \mathcal{F}$, there exist $y_1 \in \mathfrak{D}$ and $y_2 \in \mathcal{H}$ with

$$\|x - (y_1 + y_2)\| < \frac{\epsilon}{3}.$$

Set $\mathfrak{D} = \bigoplus_{l=1}^{s} \mathsf{M}_{n_l}$ and let as above $\{f^{\ell}_{ij}\}_{i,j=1}^{n_\ell}$ be a system of matrix units for M_{n_ℓ}. Let $\mathfrak{I}_\ell$ be the ideal in $\mathfrak{A}_1$ generated by f^{ℓ}_{11}. Since $\mathfrak{A}_i[2]$ is simple and $\mathfrak{A}_i[2]$ is an essential ideal of $\mathfrak{A}_i$, we have that $\mathfrak{A}_i[2] \subseteq \mathfrak{I}$ for every non-zero ideal $\mathfrak{I}$ of $\mathfrak{A}_i$. Thus, $\mathfrak{A}_1[2] \subseteq \mathfrak{I}_\ell$.

Let $\mathfrak{I}^{\phi}_{\ell}$ be the ideal in $\mathfrak{A}_2$ generated by $\phi(f^{\ell}_{11})$ and let $\mathfrak{I}^{\psi}_{\ell}$ be the ideal in $\mathfrak{A}_2$ generated by $\psi(f^{\ell}_{11})$. Since ϕ and ψ are X_2-equivariant homomorphisms and since $\phi_{\{1\}}$ and $\psi_{\{1\}}$ are injective $*$-homomorphisms, we have that $\phi(f^{\ell}_{11}) \notin \mathfrak{A}_2[2]$ and $\psi(f^{\ell}_{11}) \notin \mathfrak{A}_2[2]$. Therefore, $\mathfrak{A}_2[2] \subseteq \mathfrak{I}^{\phi}_{\ell}$ and $\mathfrak{A}_2[2] \subseteq \mathfrak{I}^{\psi}_{\ell}$. Since $K_0(\phi_{\{1\}}) = K_0(\psi_{\{1\}})$ and since $\mathfrak{A}_2[1]$ is an AF-algebra, we have that $\phi_{\{1\}}(\overline{f}^{\ell}_{11})$ is Murray-von Neumann equivalent to $\psi_{\{1\}}(\overline{f}^{\ell}_{11})$, where $\overline{f}^{\ell}_{11}$ is the image of f^{ℓ}_{11} in $\mathfrak{A}_1[1]$. Thus, they generate the same ideal in $\mathfrak{A}_2[1]$. Since $\mathfrak{A}_2[2] \subseteq \mathfrak{I}^{\phi}_{\ell}$ and $\mathfrak{A}_2[2] \subseteq \mathfrak{I}^{\psi}_{\ell}$ and since $\psi_{\{1\}}(\overline{f}^{\ell}_{11})$ and $\phi_{\{1\}}(\overline{f}^{\ell}_{11})$ generate the same ideal in $\mathfrak{A}_2[1]$, we have that $\mathfrak{I} := \mathfrak{I}^{\phi}_{\ell} = \mathfrak{I}^{\psi}_{\ell}$.

Note that the following diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & K_0(\mathfrak{A}_2[2]) & \longrightarrow & K_0(\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{I}/\mathfrak{A}_2[2]) \\
 & & \| & & \downarrow{\scriptstyle K_0(\iota)} & & \downarrow{\scriptstyle K_0(\overline{\iota})} \\
0 & \longrightarrow & K_0(\mathfrak{A}_2[2]) & \longrightarrow & K_0(\mathfrak{A}_2) & \longrightarrow & K_0(\mathfrak{A}_2[1])
\end{array}
$$

is commutative, the rows are exact, and ι and $\overline{\iota}$ are the canonical embeddings. Since $\mathfrak{A}_2[1]$ is an AF-algebra, $K_0(\overline{\iota})$ is injective. A diagram chase shows that $K_0(\iota)$ is injective. Since $KK(X_2;\phi) = KK(X_2;\psi)$, we have that $[\phi(f^{\ell}_{11})] = [\psi(f^{\ell}_{11})]$ in $K_0(\mathfrak{A}_2)$. Since $\phi(f^{\ell}_{11})$ and $\psi(f^{\ell}_{11})$ are elements of $\mathfrak{I}$ and $K_0(\iota)$ is injective, we have that $[\phi(f^{\ell}_{11})] = [\psi(f^{\ell}_{11})]$ in $K_0(\mathfrak{I})$. Since $\mathfrak{A}_i[1]$ is an AF-algebra and $\mathfrak{A}_i[2]$ is a Kirchberg algebra, they both have the stable weak cancellation property. By [26, Lemma 3.15], $\mathfrak{A}_i$ has the stable weak cancellation property. Thus, $\phi(f^{\ell}_{11})$ is Murray-von Neumann equivalent to $\psi(f^{\ell}_{11})$. Hence, there exists $v_\ell \in \mathfrak{A}_2$ such that $v_\ell^* v_\ell = \phi(f^{\ell}_{11})$ and $v_\ell v_\ell^* = \psi(f^{\ell}_{11})$.

Set

$$u_1 = \sum_{\ell=1}^{s} \sum_{i=1}^{n_\ell} \psi(f_{i1}^\ell) v_\ell \phi(f_{1i}^\ell).$$

Then, u_1 is a partial isometry in $\mathfrak{A}_2$ such that $u_1^* u_1 = \phi(1_{\mathfrak{D}})$, $u_1 u_1^* = \psi(1_{\mathfrak{D}})$, and $u_1 \phi(x) u_1^* = \psi(x)$ for all $x \in \mathfrak{D}$.

Let $\beta\colon e_{k_3}\mathfrak{A}_1[2]e_{k_3} \to \mathfrak{A}_1[2]$ denote the canonical embedding. Note that $KK(\phi_{\{2\}} \circ \beta) = KK(\psi_{\{2\}} \circ \beta)$, and the $*$-homomorphisms $\phi_{\{2\}} \circ \beta$ and $\psi_{\{2\}} \circ \beta$ are injective. Therefore, by [44, Theorem 6.7], there exists a partial isometry $u_2 \in \mathfrak{A}_2[2]$ such that $u_2^* u_2 = \phi(e_k)$, $u_2 u_2^* = \psi(e_k)$, and

$$\| u_2 \phi(x) u_2^* - \psi(x) \| < \frac{\epsilon}{3}$$

for all $x \in \mathcal{H}$.

By the proof of [26, Lemma 3.3], there exists a sequence of isometries $\{V_n\}_{n=1}^\infty$ in $\mathcal{M}(\mathfrak{A}_2)$ such that $\|V_n^* a V_n\| \to 0$ for all $a \in \mathfrak{A}_2$. Thus, for all projections $e \in \mathfrak{A}_2$, $\|V_n^*(V_n V_n^* - (1_{\mathcal{M}(\mathfrak{A}_2)} - e))V_n\| = \|1_{\mathcal{M}(\mathfrak{A}_2)} - V_n^*(1_{\mathcal{M}(\mathfrak{A}_2)} - e)V_n\| \to 0$. From [58, Propositions 2.2.4 and 2.2.7] it follows that for sufficiently large n, $1_{\mathcal{M}(\mathfrak{A}_2)} - e$ is Murray-von Neumann equivalent to $V_n V_n^*$, which is Murray-von Neumann equivalent to $V_n^* V_n = 1_{\mathcal{M}(\mathfrak{A}_2)}$. Consequently, $1_{\mathcal{M}(\mathfrak{A}_2)} - e$ is Murray-von Neumann equivalent to $1_{\mathcal{M}(\mathfrak{A}_2)}$ for all projections e in $\mathfrak{A}_2$. We now conclude that there exists $u_3 \in \mathcal{M}(\mathfrak{A}_2)$ such that $u_3^* u_3 = 1_{\mathcal{M}(\mathfrak{A}_2)} - (u_1 + u_2)^*(u_1 + u_2)$ and $u_3 u_3^* = 1_{\mathcal{M}(\mathfrak{A}_2)} - (u_1 + u_2)(u_1 + u_2)^*$. Set $u = u_1 + u_2 + u_3 \in \mathcal{M}(\mathfrak{A}_2)$. Then u is a unitary in $\mathcal{M}(\mathfrak{A}_2)$.

Let $x \in \mathcal{F}$. Choose $y_1 \in \mathfrak{D}$ and $y_2 \in \mathcal{H} \subseteq e_{k_3}\mathfrak{A}_1[2]e_{k_3}$ such that $\|x - (y_1 + y_2)\| < \frac{\epsilon}{3}$. Then

$$\begin{aligned} \|u\phi(x)u^* - \psi(x)\| &\leq \|u\phi(x)u^* - u\phi(y_1 + y_2)u^*\| \\ &\quad + \|u_1\phi(y_1)u_1^* + u_2\phi(y_2)u_2^* - \psi(y_1) - \psi(y_2)\| \\ &\quad + \|\psi(y_1 + y_2) - \psi(x)\| \\ &< \epsilon. \end{aligned}$$

Consequently, ϕ is approximately unitarily equivalent to ψ. □

Lemma 4.6. *Let $\mathfrak{A}$ be a separable C^*-algebra over a finite topological space X. Let u be a unitary in $\mathcal{M}(\mathfrak{A} \otimes \mathbb{K})$. Then $K_X(\mathrm{Ad}(u)|_{\mathfrak{A}\otimes\mathbb{K}}) = \mathrm{id}_{K_X(\mathfrak{A})}$.*

Proof. Since $\mathfrak{A}\otimes\mathbb{K}$ is stable, we have that there exists a strictly continuous path of unitaries $\{u_t\}$ in $\mathcal{M}(\mathfrak{A}\otimes\mathbb{K})$ such that $u_0 = u$ and $u_1 = 1_{\mathcal{M}(\mathfrak{A}\otimes\mathbb{K})}$ ([6, Proposition 12.2.2]). An easy verification shows that this is a homotopy invariance of X-equivariant homomorphisms. Thus it follows that $K_X(\mathrm{Ad}(u)|_{\mathfrak{A}\otimes\mathbb{K}}) = \mathrm{id}_{K_X(\mathfrak{A})}$. □

Theorem 4.7. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be separable C^*-algebras over X_2 in $\mathcal{B}(X_2)$ such that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are C^*-algebras with real rank zero and let $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ be an invertible element such that $\Gamma(x)_{Y,0}$ is an order isomorphism for all $Y \in$*

$\mathbb{LC}(X_2)$. *Suppose* $\mathfrak{A}_i[2]$ *is a Kirchberg algebra,* $\mathfrak{A}_i[1]$ *is an AF-algebra, and* $\mathfrak{A}_i[2]$ *is an essential ideal of* $\mathfrak{A}_i$. *Then there exists an* X_2*-equivariant isomorphism* $\phi\colon \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ *such that* $KL(\phi) = KL(g^1_{X_2}(y))$ *and* $K_{X_2}(\phi) = K_{X_2}(y)$, *where* $y = KK(X_2; \mathrm{id}_{\mathfrak{A}_1} \otimes e_{11})^{-1} \times x \times KK(X_2; \mathrm{id}_{\mathfrak{A}_2} \otimes e_{11})$.

Proof. Since $\mathfrak{A}_i[2]$ is a purely infinite simple C^*-algebra, $\mathfrak{A}_i[2]$ is either unital or stable. Since $\mathfrak{A}_i[2]$ is an essential ideal of $\mathfrak{A}_i$, $\mathfrak{A}_i[2]$ is non-unital (otherwise $\mathfrak{A}_i[2]$ would be isomorphic to a direct summand of $\mathfrak{A}_i$, which would contradict the assumption that $\mathfrak{A}_i[2]$ is an essential ideal of $\mathfrak{A}_i$). Therefore, $\mathfrak{A}_i[2]$ is stable. Moreover, $\mathcal{Q}(\mathfrak{A}_i[2])$ is simple (cf. [54, Theorem 3.2]), which implies that $0 \to \mathfrak{A}_i[2] \to \mathfrak{A}_i \to \mathfrak{A}_i[1] \to 0$ is a full extension. Since $\mathfrak{A}_i[2]$ and $\mathfrak{A}_i[1]$ are nuclear C^*-algebras, $\mathfrak{A}_i$ is a nuclear C^*-algebra.

Let $z \in KK(X_2; \mathfrak{A}_2 \otimes \mathbb{K}, \mathfrak{A}_1 \otimes \mathbb{K})$ such that $y \times z = [\mathrm{id}_{\mathfrak{A}_1 \otimes \mathbb{K}}]$ and $z \times y = [\mathrm{id}_{\mathfrak{A}_2 \otimes \mathbb{K}}]$. By Theorem 4.1, there exist X_2-equivariant homomorphisms $\psi_1 \colon \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ and $\psi_2 \colon \mathfrak{A}_2 \otimes \mathbb{K} \to \mathfrak{A}_1 \otimes \mathbb{K}$ such that $KK(X_2; \psi_1) = y$, $KK(X_2; \psi_2) = z$, and $(\psi_1)_i$ and $(\psi_2)_i$ are injective $*$-homomorphisms for all i. Using Theorem 4.5, $\psi_1 \circ \psi_2$ is approximately unitarily equivalent to $\mathrm{id}_{\mathfrak{A}_2 \otimes \mathbb{K}}$ and $\psi_2 \circ \psi_1$ is approximately unitarily equivalent to $\mathrm{id}_{\mathfrak{A}_1 \otimes \mathbb{K}}$. By [33, Proposition 6.3] and [32, Proposition 3.12], there exists an isomorphism $\phi\colon \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that ϕ and ψ_1 are approximately unitarily equivalent.

Let $\pi_2 \colon \mathfrak{A}_2 \otimes \mathbb{K} \to \mathfrak{A}_2[1] \otimes \mathbb{K}$ be the canonical quotient map. Then $\pi_2 \circ \phi|_{\mathfrak{A}_1[2] \otimes \mathbb{K}}$ is either zero or injective since $\mathfrak{A}_1[2] \otimes \mathbb{K}$ is simple. Since $\mathfrak{A}_1[2] \otimes \mathbb{K}$ is purely infinite and $\mathfrak{A}_2[1] \otimes \mathbb{K}$ is an AF-algebra, we must have that $\pi_2 \circ \phi|_{\mathfrak{A}_2[2] \otimes \mathbb{K}} = 0$. Thus, ϕ is an X_2-equivariant homomorphism. Similarly, ϕ^{-1} is an X_2-equivariant homomorphism. Hence, ϕ is an X_2-equivariant isomorphism. By construction, $KL(\phi) = KL(\psi_1) = KL(g^1_{X_2}(y))$. We claim that $K_{X_2}(\phi) = K_{X_2}(y)$. This is well-known for K_0 and K_1 of the whole algebra, and we might assume that everything is stable. Now it is easy to show that the induced homomorphisms on sub-quotients are also approximately unitarily equivalent, so the result follows. □

Corollary 4.8. *Let* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *be separable* C^**-algebras over* X_2 *in* $\mathcal{B}(X_2)$ *such that* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *are* C^**-algebras with real rank zero, and let* $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ *be an invertible element such that* $\Gamma(x)_{Y,0}$ *is an order isomorphism for all* $Y \in \mathbb{LC}(X_2)$. *Suppose* $\mathfrak{A}_i[2]$ *is a Kirchberg algebra,* $\mathfrak{A}_i[1]$ *is an AF-algebra,* $\mathfrak{A}_i[2]$ *is an essential ideal of* $\mathfrak{A}_i$, *and* $K_i(\mathfrak{A}[Y])$ *and* $K_i(\mathfrak{B}[Y])$ *are finitely generated for all* $Y \in \mathbb{LC}(X_2)$. *Then there exists an* X_2*-equivariant isomorphism* $\phi\colon \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ *such that* $KK(\phi) = KK(g^1_{X_2}(y))$ *and* $K_{X_2}(\phi) = K_{X_2}(y)$, *where* $y = KK(X_2; \mathrm{id}_{\mathfrak{A}_1} \otimes e_{11})^{-1} \times x \times KK(X_n; \mathrm{id}_{\mathfrak{A}_2} \otimes e_{11})$

Proof. This follows from Theorem 4.7 and the fact that if G is finitely generated, then $\mathrm{Pext}^1_{\mathbb{Z}}(G, H) = 0$. □

4.2. Strong classification of unital extensions of purely infinite by $\mathbb{K}$

We recall the following from [3, p. 341]. Let $\psi\colon \mathfrak{A} \to B(\mathcal{H})$ be a representation of $\mathfrak{A}$. Let $\mathcal{H}_e$ denote the subspace of $\mathcal{H}$ spanned by the ranges of all compact operators in $\psi(\mathfrak{A})$. Since $\psi(\mathfrak{A}) \cap \mathbb{K}$ is an ideal of $\psi(\mathfrak{A})$, we have that $\mathcal{H}_e$ reduces $\psi(\mathfrak{A})$, and so the decomposition $\mathcal{H} = \mathcal{H}_e \oplus \mathcal{H}_e^{\perp}$ induces a decomposition of ψ into sub-representations $\psi = \psi_e \oplus \psi'$. The summand ψ_e, considered as a representation of $\mathfrak{A}$ on $\mathcal{H}_e$, will be called the *essential part of* ψ and $\mathcal{H}_e$ is called the *essential subspace* for ψ.

Let $\mathfrak{B}$ be a tight C^*-algebra over X_2. Consider the essential extension

$$\mathfrak{e}_{\mathfrak{B}} : 0 \to \mathfrak{B}[2] \to \mathfrak{B} \to \mathfrak{B}[1] \to 0.$$

If $\tau_{\mathfrak{e}_{\mathfrak{B}}} : \mathfrak{B}[1] \to \mathcal{Q}(\mathfrak{B}[2])$ is the Busby invariant of $\mathfrak{e}_{\mathfrak{B}}$, then there exists an injective $*$-homomorphism $\sigma_{\mathfrak{e}_{\mathfrak{B}}} : \mathfrak{B} \to \mathcal{M}(\mathfrak{B}[2])$ such that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{B}[2] & \longrightarrow & \mathfrak{B} & \xrightarrow{\pi_{\mathfrak{B}}} & \mathfrak{B}[1] & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle \sigma_{\mathfrak{e}_{\mathfrak{B}}}} & & \downarrow{\scriptstyle \tau_{\mathfrak{e}_{\mathfrak{B}}}} & & \\
0 & \longrightarrow & \mathfrak{B}[2] & \longrightarrow & \mathcal{M}(\mathfrak{B}[2]) & \xrightarrow[\overline{\pi}_{\mathfrak{B}}]{} & \mathcal{Q}(\mathfrak{B}[2]) & \longrightarrow & 0
\end{array}$$

commutes. If $\mathfrak{B}[2] \cong \mathbb{K}$, let $\eta_{\mathfrak{B}} : \mathcal{M}(\mathfrak{B}[2]) \to B(\ell^2)$ be the isomorphism extending the isomorphism $\mathfrak{B}[2] \cong \mathbb{K}$ and let $\overline{\eta}_{\mathfrak{B}} : \mathcal{Q}(\mathfrak{B}[2]) \to B(\ell^2)/\mathbb{K}$ be the induced isomorphism.

Lemma 4.9. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable, tight C^*-algebras over X_2 such that $\mathfrak{A}[2] \cong \mathfrak{B}[2] \cong \mathbb{K}$. Let $\psi_1, \psi_2 \colon \mathfrak{A} \to \mathfrak{B}$ be full X_2-equivariant homomorphisms such that $K_0((\psi_1)_{\{2\}}) = K_0((\psi_2)_{\{2\}})$ and $\eta_{\mathfrak{B}} \circ \sigma_{\mathfrak{e}_{\mathfrak{B}}} \circ \psi_i$ is a non-degenerate representation of $\mathfrak{A}$. Then there exists a sequence of unitaries $\{U_n\}_{n=1}^{\infty}$ in $\mathcal{M}(\mathfrak{B}[2])$ such that*

$$U_n(\sigma_{\mathfrak{e}_{\mathfrak{B}}} \circ \psi_1)(a)U_n^* - (\sigma_{\mathfrak{e}_{\mathfrak{B}}} \circ \psi_2)(a) \in \mathfrak{B}[2]$$

for all $a \in \mathfrak{A}$ and for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \|U_n(\sigma_{\mathfrak{e}_{\mathfrak{B}}} \circ \psi_1)(a)U_n^* - (\sigma_{\mathfrak{e}_{\mathfrak{B}}} \circ \psi_2)(a)\| = 0$$

for all $a \in \mathfrak{A}$.

Proof. We argue as in the proof of [42, Lemma 2.8]. Set $\sigma_i = \eta_{\mathfrak{B}} \circ \sigma_{\mathfrak{e}_{\mathfrak{B}}} \circ \psi_i$. By assumption, $\sigma_i \colon \mathfrak{A} \to B(\ell^2)$ is a non-degenerate representation of $\mathfrak{A}$. We claim that there exists a sequence of unitaries $\{V_n\}_{n=1}^{\infty}$ in $B(\ell^2)$ such that $V_n\sigma_1(a)V_n^* - \sigma_2(a) \in \mathbb{K}$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \|V_n\sigma_1(a)V_n^* - \sigma_2(a)\| = 0$$

for all $a \in \mathfrak{A}$. This will be a consequence of [3, Theorem 5(iii)].

Let $\rho\colon \mathfrak{A} \to B(\ell^2)$ be the unique irreducible faithful representation defined by the isomorphism $\mathfrak{A}[2] \cong \mathbb{K}$ (ρ is unique since $\mathfrak{A}[2]$ is an ideal so ρ is the unique irreducible representation that extends the irreducible representation given by

$\mathfrak{A}[2] \cong \mathbb{K}$; see e.g. [2, Theorem 1.3.4]). Since ψ_i is a full X_2-equivariant homomorphism, ψ_i is injective. Since $\psi_i, \sigma_{\mathfrak{e}_{\mathfrak{B}}}, \eta_{\mathfrak{B}}$ are injective $*$-homomorphisms, σ_i is injective. Therefore, $\ker(\sigma_1) = \ker(\sigma_2) = \{0\}$. Let $\pi\colon B(\ell^2) \to B(\ell^2)/\mathbb{K}$ be the canonical surjective $*$-homomorphism. Note that

$$\begin{aligned}\pi \circ \sigma_i &= \pi \circ \eta_{\mathfrak{B}} \circ \sigma_{\mathfrak{e}_{\mathfrak{B}}} \circ \psi_i = \overline{\eta}_{\mathfrak{B}} \circ \overline{\pi}_{\mathfrak{B}} \circ \sigma_{\mathfrak{e}_{\mathfrak{B}}} \circ \psi_i \\ &= \overline{\eta}_{\mathfrak{B}} \circ \tau_{\mathfrak{e}_{\mathfrak{B}}} \circ \pi_{\mathfrak{B}} \circ \psi_i = \overline{\eta}_{\mathfrak{B}} \circ \tau_{\mathfrak{e}_{\mathfrak{B}}} \circ (\psi_i)_{\{1\}} \circ \pi_{\mathfrak{A}}.\end{aligned}$$

It now follows that $\ker(\pi \circ \sigma_1) = \ker(\pi \circ \sigma_2) = \mathfrak{A}[2]$ since $\overline{\eta}_{\mathfrak{B}}$, $\tau_{\mathfrak{e}_{\mathfrak{B}}}$, and $(\psi_i)_{\{1\}}$ are injective $*$-homomorphisms.

Let H_1 be the essential subspace of σ_1. Since $\sigma_1(\mathfrak{A}[2]) \subseteq \mathbb{K}$ and since $\sigma_1(x) \notin \mathbb{K}$ for each $x \notin \mathfrak{A}[2]$, we have that $H_1 = \overline{\sigma_1(\mathfrak{A}[2])\ell^2}$. Similarly, we have that $H_2 = \overline{\sigma_2(\mathfrak{A}[2])\ell^2}$, where H_2 is the essential subspace of σ_2.

We will show that $\sigma_1(-)|_{H_1}$ is unitarily equivalent to $\sigma_2(-)|_{H_2}$. Let e be a minimal projection of $\mathfrak{A}[2] \cong \mathbb{K}$. Since $\sigma_1(e)$ is a projection in $\mathbb{K}$, it has finite rank. Suppose $\sigma_1(e)$ has rank k. Standard representation theory now implies that $\left(\sigma_1|_{\mathfrak{A}[2]}(-)\right)|_{H_1}$ is unitarily equivalent to the direct sum of k copies of $\rho|_{\mathfrak{A}[2]}$. Since $\mathfrak{A}[2]$ is an ideal, representations of $\mathfrak{A}[2]$ extend uniquely to representations of $\mathfrak{A}$. Therefore, $\sigma_1(-)|_{H_1}$ is unitarily equivalent to the direct sum of k copies of ρ. Since $K_0((\psi_1)_{\{2\}}) = K_0((\psi_2)_{\{2\}})$, we have that $\sigma_1(e)$ is Murray-von Neumann equivalent to $\sigma_2(e)$. Hence, $\sigma_2(e)$ has rank k. Using a similar argument as in the previous case $\sigma_2(-)|_{H_2}$ is unitarily equivalent to the direct sum of k copies of ρ.

The above paragraph implies that $\sigma_2(-)|_{H_2}$ and $\sigma_1(-)|_{H_1}$ are unitarily equivalent. Since $\ker(\sigma_1) = \ker(\sigma_2)$ and $\ker(\pi \circ \sigma_1) = \ker(\pi \circ \sigma_2)$ by [3, Theorem 5(iii)], there exists a sequence of unitaries $\{V_n\}_{n=1}^{\infty}$ in $B(\ell^2)$ such that $V_n\sigma_1(a)V_n^* - \sigma_2(a) \in \mathbb{K}$ for all $n \in \mathbb{N}$ and for all $a \in \mathfrak{A}$, and

$$\lim_{n\to\infty} \|V_n\sigma_1(a)V_n^* - \sigma_2(a)\| = 0$$

for all $a \in \mathfrak{A}$.

Set $U_n = \eta_{\mathfrak{B}}^{-1}(V_n)$. Then $\{U_n\}_{n=1}^{\infty}$ is a sequence of unitaries in $\mathcal{M}(\mathfrak{B}[2])$ such that $U_n(\sigma_{\mathfrak{e}_{\mathfrak{B}}} \circ \psi_1)(a)U_n^* - (\sigma_{\mathfrak{e}_{\mathfrak{B}}} \circ \psi_2)(a) \in \mathfrak{B}[2]$ for all $n \in \mathbb{N}$ and for all $a \in \mathfrak{A}$, and

$$\lim_{n\to\infty} \|U_n(\sigma_{\mathfrak{e}_{\mathfrak{B}}} \circ \psi_1)(a)U_n^* - (\sigma_{\mathfrak{e}_{\mathfrak{B}}} \circ \psi_2)(a)\| = 0$$

for all $a \in \mathfrak{A}$. □

Definition 4.10. A C^*-algebra $\mathfrak{A}$ is called *weakly semiprojective* if for every sequence $(\mathfrak{B}_n)_{n=1}^{\infty}$ of C^*-algebras and for every $*$-homomorphism ϕ from $\mathfrak{A}$ to $\prod_{n=1}^{\infty} \mathfrak{B}_n / \bigoplus_{n=1}^{\infty} \mathfrak{B}_n$ there exist an $N \in \mathbb{N}$ and a $*$-homomorphism $\widetilde{\phi}$ from $\mathfrak{A}$

to $\prod_{n=N}^{\infty}\mathfrak{B}_n$ such that the following diagram commutes

$$\begin{array}{ccc} & & \prod_{n=N}^{\infty}\mathfrak{B}_n \\ & \overset{\widetilde{\phi}}{\nearrow} & \big\downarrow \rho_N \\ \mathfrak{A} & \xrightarrow[\phi]{} & \prod_{n=1}^{\infty}\mathfrak{B}_n/\bigoplus_{n=1}^{\infty}\mathfrak{B}_n \end{array} \qquad \begin{array}{c} (b_N, b_{N+1}, \dots) \\ \big\downarrow \\ [(0,\dots,0,b_N,b_{N+1},\dots)]. \end{array}$$

A C^*-algebra $\mathfrak{A}$ is called *semiprojective* if for every C^*-algebra $\mathfrak{B}$ and every increasing sequence $\mathfrak{I}_1 \trianglelefteq \mathfrak{I}_2 \trianglelefteq \cdots \trianglelefteq \mathfrak{B}$ of ideals of $\mathfrak{B}$ and for every $*$-homomorphism ϕ from $\mathfrak{A}$ to $\mathfrak{B}/\overline{\cup_{n=1}^{\infty}\mathfrak{I}_n}$ there exist an $N \in \mathbb{N}$ and a $*$-homomorphism $\widetilde{\phi}$ from $\mathfrak{A}$ to $\mathfrak{B}/\mathfrak{I}_N$ such that the following diagram commutes

$$\begin{array}{ccc} & & \mathfrak{B}/\mathfrak{I}_N \\ & \overset{\widetilde{\phi}}{\nearrow} & \big\downarrow \rho_N \\ \mathfrak{A} & \xrightarrow[\phi]{} & \mathfrak{B}/\overline{\bigcup_{n=1}^{\infty}\mathfrak{I}_n} \end{array} \qquad (\mathfrak{I}_1 \trianglelefteq \mathfrak{I}_2 \trianglelefteq \cdots \trianglelefteq \mathfrak{B}).$$

Lemma 4.11. *Let $\mathfrak{A}_0$ be a unital, separable, nuclear, tight C^*-algebra over X_2 such that $\mathfrak{A}_0[2] \cong \mathbb{K}$ and $\mathfrak{A}_0$ has the stable weak cancellation property. Set $\mathfrak{A} = \mathfrak{A}_0 \otimes \mathbb{K}$. Suppose $\beta\colon \mathfrak{A} \to \mathfrak{A}$ is a full X_2-equivariant homomorphism such that $K_{X_2}(\beta) = K_{X_2}(\mathrm{id}_{\mathfrak{A}})$ and $\beta_{\{1\}} = \mathrm{id}_{\mathfrak{A}[1]}$. Then there exists a sequence of contractive, completely positive, linear maps $\{\alpha_n\colon \mathfrak{A} \to \mathfrak{A}\}_{n=1}^{\infty}$ such that*

(1) *$\alpha_n|_{e_n\mathfrak{A}e_n}$ is a $*$-homomorphism for all $n \in \mathbb{N}$ and*
(2) *for all $a \in \mathfrak{A}$,*
$$\lim_{n\to\infty}\|\alpha_n \circ \beta(a) - a\| = 0$$

where $e_n = \sum_{k=1}^{n} 1_{\mathfrak{A}_0} \otimes e_{kk}$ and $\{e_{ij}\}_{i,j=1}^{\infty}$ is a system of matrix units for $\mathbb{K}$. If, in addition, $\mathfrak{A}$ is assumed to be weakly semiprojective, then α_n can be chosen to be a $*$-homomorphism for all $n \in \mathbb{N}$.

Proof. For each $n \in \mathbb{N}$, let $\mathfrak{e}_n$ be the extension $0 \to e_n\mathfrak{A}[2]e_n \to e_n\mathfrak{A}e_n \to \overline{e}_n\mathfrak{A}[1]\overline{e}_n \to 0$, where $\overline{e}_n$ is the image of e_n under the canonical surjective $*$-homomorphism from $\mathfrak{A}$ to $\mathfrak{A}[1]$. Let $\mathfrak{e}$ be the extension $0 \to \mathfrak{A}[2] \to \mathfrak{A} \to \mathfrak{A}[1] \to 0$.

Since β is a full X_2-equivariant homomorphism and the ideal in $\mathfrak{A}$ generated by e_n is $\mathfrak{A}$, we have that the ideal in $\mathfrak{A}$ generated by $\beta(e_n)$ is $\mathfrak{A}$. Since $K_{X_2}(\beta) = K_{X_2}(\mathrm{id}_{\mathfrak{A}})$, we have that $[\beta(e_n)] = [e_n]$ in $K_0(\mathfrak{A})$. It now follows that $\beta(e_n)$ and e_n are Murray-von Neumann equivalent since $\mathfrak{A}_0$ has the stable weak cancellation property. Since $\mathfrak{A}$ is stable, there exists a unitary v_n in the unitization of $\mathfrak{A}$ such that $v_n\beta(e_n)v_n^* = e_n$ (use e.g. [58, Proposition 2.2.8(i)]).

Since $e_n\mathfrak{A}[2]e_n$ is a non-unital hereditary sub-C^*-algebra of $\mathfrak{A}[2]$, we have that $e_n\mathfrak{A}[2]e_n$ is isomorphic to a non-unital hereditary sub-C^*-algebra of $\mathbb{K}$. Therefore, $e_n\mathfrak{A}[2]e_n \cong \mathfrak{A}[2] \cong \mathbb{K}$. Set $\mathfrak{A}_n = e_n\mathfrak{A}e_n$ and define $\beta_n\colon \mathfrak{A}_n \to \mathfrak{A}_n$ by $\beta_n(x) = \mathrm{Ad}(v_n)\circ\beta(x)$. Then β_n is a unital, full X_2-equivariant homomorphism.

Since $\eta_{\mathfrak{A}_n} \circ \sigma_{\mathfrak{e}_n} \circ \beta_n$ is a unital representation of $\mathfrak{A}_n$, the closed subspace of ℓ^2 generated by $\left\{(\eta_{\mathfrak{A}_n} \circ \sigma_{\mathfrak{e}_n} \circ \beta_n)(x)\xi \,:\, x \in \mathfrak{A}_n, \xi \in \ell^2\right\}$ is ℓ^2. Therefore, $\eta_{\mathfrak{A}_n} \circ \sigma_{\mathfrak{e}_n} \circ \beta_n$ is a non-degenerate representation of $\mathfrak{A}_n$. Similarly, $\eta_{\mathfrak{A}_n} \circ \sigma_{\mathfrak{e}_n}$ is a non-degenerate representation of $\mathfrak{A}_n$.

Since $K_{X_2}(\beta) = K_{X_2}(\mathrm{id}_{\mathfrak{A}})$ and the X_2-equivariant embedding of $\mathfrak{A}_n$ as a sub-algebra of $\mathfrak{A}$ induces an isomorphism in ideal related K-theory, we have that $K_{X_2}(\beta_n) = K_{X_2}(\mathrm{id}_{\mathfrak{A}_n})$ (cf. Lemma 2.15 and Lemma 4.6). By Lemma 4.9, there exists a sequence of unitaries $W_{k,n} \in \mathcal{M}(\mathfrak{A}_n[2])$ such that

$$(\mathrm{Ad}(W_{k,n}) \circ \sigma_{\mathfrak{e}_n} \circ \beta_n)(x) - \sigma_{\mathfrak{e}_n}(x) \in \mathfrak{A}_n[2]$$

for all $x \in \mathfrak{A}_n$ and for all $k \in \mathbb{N}$, and

$$\lim_{k\to\infty} \|(\mathrm{Ad}(W_{k,n}) \circ \sigma_{\mathfrak{e}_n} \circ \beta_n)(x) - \sigma_{\mathfrak{e}_n}(x)\| = 0$$

for all $x \in \mathfrak{A}_n$.

There exists an isomorphism $\psi_n \colon \mathcal{M}(\mathfrak{A}_n[2]) \to \sigma_{\mathfrak{e}}(e_n)\mathcal{M}(\mathfrak{A}[2])\sigma_{\mathfrak{e}}(e_n)$ such that $\psi_n|_{\mathfrak{A}_n[2]} = \mathrm{id}_{e_n\mathfrak{A}[2]e_n}$ and $\psi_n \circ \sigma_{\mathfrak{e}_n}(x) = \sigma_{\mathfrak{e}}(x)$ for all $x \in \mathfrak{A}_n$ (use [37, Corollary 1.1.15] and the universality of $\sigma_{\mathfrak{e}_n}$). Thus, for each k, we get a partial isometry $\widetilde{W}_{k,n} = \psi_n(W_{k,n})$ in $\mathcal{M}(\mathfrak{A}[2])$ such that $\widetilde{W}_{k,n}^*\widetilde{W}_{k,n} = \widetilde{W}_{k,n}\widetilde{W}_{k,n}^* = \sigma_{\mathfrak{e}}(e_n)$ and

$$(\mathrm{Ad}(\widetilde{W}_{k,n}) \circ \sigma_{\mathfrak{e}} \circ \mathrm{Ad}(v_n) \circ \beta)(x) - \sigma_{\mathfrak{e}}(x) \in \mathfrak{A}[2]$$

for all $x \in \mathfrak{A}_n$ and for all $k \in \mathbb{N}$, and

$$\lim_{k\to\infty} \|(\mathrm{Ad}(\widetilde{W}_{k,n}) \circ \sigma_{\mathfrak{e}} \circ \mathrm{Ad}(v_n) \circ \beta)(x) - \sigma_{\mathfrak{e}}(x)\| = 0$$

for all $x \in \mathfrak{A}_n$ (note that the maps $\mathrm{Ad}(\widetilde{W}_{k,n})$ are not necessarily $*$-homomorphisms on their whole domain).

Set $V_{k,n} = (\widetilde{W}_{k,n} + 1_{\mathcal{M}(\mathfrak{A}[2])} - \sigma_{\mathfrak{e}}(e_n))\sigma_{\mathfrak{e}}^{\dagger}(v_n)$, where $\sigma_{\mathfrak{e}}^{\dagger}$ is the unital extension of $\sigma_{\mathfrak{e}}$. Thus, $V_{k,n}$ is a unitary in $\mathcal{M}(\mathfrak{A}[2])$. Since $\sigma_{\mathfrak{e}} \circ \mathrm{Ad}(v_n) \circ \beta = \mathrm{Ad}(\sigma_{\mathfrak{e}}^{\dagger}(v_n)) \circ \sigma_{\mathfrak{e}} \circ \beta$, we have that

$$(\mathrm{Ad}(V_{k,n}) \circ \sigma_{\mathfrak{e}} \circ \beta)(x) - \sigma_{\mathfrak{e}}(x) \in \mathfrak{A}[2]$$

for all $x \in \mathfrak{A}_n$ and for all $k \in \mathbb{N}$, and

$$\lim_{k\to\infty} \|(\mathrm{Ad}(V_{k,n}) \circ \sigma_{\mathfrak{e}} \circ \beta)(x) - \sigma_{\mathfrak{e}}(x)\| = 0$$

for all $x \in \mathfrak{A}_n$. Consequently,

$$(\mathrm{Ad}(V_{k,n}) \circ \sigma_{\mathfrak{e}} \circ \beta)(x) \in \sigma_{\mathfrak{e}}(\mathfrak{A}_n) + \mathfrak{A}[2] \subseteq \sigma_{\mathfrak{e}}(\mathfrak{A})$$

for all $x \in \mathfrak{A}_n$. Since $\beta_{\{1\}} = \mathrm{id}_{\mathfrak{A}[1]}$, we have that $x - \beta(x) \in \mathfrak{A}[2]$ for all $x \in \mathfrak{A}_n$. Therefore,

$$\begin{aligned}\mathrm{Ad}(V_{k,n})(\sigma_{\mathfrak{e}}(x)) &= \mathrm{Ad}(V_{k,n}) \circ \sigma_{\mathfrak{e}}(x - \beta(x)) + \mathrm{Ad}(V_{k,n}) \circ \sigma_{\mathfrak{e}} \circ \beta(x) \\ &\in \sigma_{\mathfrak{e}}(\mathfrak{A}_n) + \mathfrak{A}[2]\end{aligned}$$

for all $x \in \mathfrak{A}_n$. So, $\mathrm{Ad}(V_{k,n})(\sigma_{\mathfrak{e}}(x)) \in \sigma_{\mathfrak{e}}(\mathfrak{A})$ for all $x \in \mathfrak{A}_n$. Thus, $\alpha_{k,n} = \sigma_{\mathfrak{e}}^{-1} \circ (\mathrm{Ad}(V_{k,n}) \circ \sigma_{\mathfrak{e}})|_{\mathfrak{B}_n}$ is a $*$-homomorphism from $\mathfrak{B}_n$ to $\mathfrak{A}$, where $\mathfrak{B}_n$ is the sub-C^*-algebra generated by $\mathfrak{A}_n$ and $\beta(\mathfrak{A}_n)$.

Since

$$\lim_{k\to\infty} \|(\mathrm{Ad}(V_{k,n}) \circ \sigma_{\mathfrak{e}} \circ \beta)(x) - \sigma_{\mathfrak{e}}(x)\| = 0$$

for all $x \in \mathfrak{A}_n$ and $\mathfrak{A}_n \subseteq \mathfrak{A}_{n+1}$, there exists a strictly increasing sequence $\{k(n)\}_{n=1}^{\infty}$ of positive integers such that

$$\lim_{n\to\infty} \|\alpha_{k(n),n} \circ \beta(x) - x\| = 0$$

for all $x \in \bigcup_{n=1}^{\infty} \mathfrak{A}_n$. Let $\{a_1, a_2, \ldots\}$ be a dense sequence in $\mathfrak{A}$. Set

$$\mathcal{F}_n = \{e_i a_k e_i, \beta(e_i a_k e_i) \ : \ i, k = 1, 2, \ldots, n\} \subseteq \mathfrak{B}_n.$$

Using the fact that $\mathfrak{A}$ is nuclear and Arveson's Extension Theorem, there exists a contractive, completely positive, linear map $\alpha_n \colon \mathfrak{A} \to \mathfrak{A}$ such that

$$\|\alpha_{k(n),n}(x) - \alpha_n(x)\| < \frac{1}{n}$$

for all $x \in \mathcal{F}_n$. Let $\epsilon > 0$ and let $x \in \mathfrak{A}$. Since $\{e_n\}_{n=1}^{\infty}$ is an approximate identity for $\mathfrak{A}$ and since $\{a_1, a_2, \ldots\}$ is a dense sequence in $\mathfrak{A}$, there are $m, N \in \mathbb{N}$ with $N \geq m$ such that $\|x - e_N a_m e_N\| < \frac{\epsilon}{4}$, $\frac{1}{N} < \frac{\epsilon}{4}$. Since $e_N a_m e_N \in \mathcal{F}_N$, we may choose $M \geq N$ such that $\|\alpha_{k(n),n} \circ \beta(e_N a_m a_N) - e_N a_m a_N\| < \frac{\epsilon}{4}$ for all $n \geq M$. Hence, for all $n \geq M$

$$\begin{aligned} \|\alpha_n \circ \beta(x) - x\| &\leq \|\alpha_n \circ \beta(x) - \alpha_n \circ \beta(e_N a_m e_N)\| \\ &\quad + \|\alpha_n \circ \beta(e_N a_m e_N) - \alpha_{k(n),n} \circ \beta(e_N a_m e_N)\| \\ &\quad + \|\alpha_{k(n),n} \circ \beta(e_N a_m e_N) - e_N a_m e_N\| \\ &\quad + \|e_N a_m e_N - x\| \\ &< \epsilon. \end{aligned}$$

Thus, we have that

$$\lim_{n\to\infty} \|\alpha_n \circ \beta(x) - x\| = 0$$

for all $x \in \mathfrak{A}$. We have just proved the first part of the lemma.

We now show that α_n can be chosen to be a $*$-homomorphism provided that $\mathfrak{A}$ is weakly semiprojective. Suppose $\mathfrak{A}$ is weakly semiprojective. Since α_n is a $*$-homomorphism on $e_n A e_n$ and since $\bigcup_n e_n A e_n$ is dense in A, the contractive completely positive linear map $(\alpha_n) \colon \mathfrak{A} \to \prod_{n=1}^{\infty} \mathfrak{B} / \bigoplus_{n=1}^{\infty} \mathfrak{B}$ is a $*$-homomorphism. Since $\mathfrak{A}$ is a weakly semiprojective, there exists $*$-homomorphism $\phi = (\phi_n) \colon A \to \prod_{n=1}^{\infty} \mathfrak{B}$ that lifts (α_n). Consequently,

$$\lim_{n\to\infty} \|\phi_n(a) - \alpha_n(a)\| = 0$$

for all $a \in A$. By replacing α_n with ϕ_n with get the desired result. $\square$

To prove a uniqueness theorem involving tight C^*-algebras $\mathfrak{A}$ over X_2, we require that $\mathfrak{A}[1]$ belongs to a class of C^*-algebras whose injective $*$-homomorphisms between two objects in this class are classified by KK. To be more precise, we make the following definition.

Definition 4.12. We will be interested in classes $\mathcal{C}$ of separable, nuclear, simple C^*-algebras satisfying the following property that if $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$ and $\phi, \psi\colon \mathfrak{A} \otimes \mathbb{K} \to \mathfrak{B} \otimes \mathbb{K}$ are two injective $*$-homomorphisms such that $KK(\phi) = KK(\psi)$, then ϕ and ψ are approximately unitarily equivalent.

All Kirchberg algebras and all simple C^*-algebras with tracial rank zero satisfy the property of Definition 4.12.

Remark 4.13.

(1) By [49, Theorem 4.1.3] if $\mathcal{C}$ is the class of Kirchberg algebras, then $\mathcal{C}$ satisfies the property in Definition 4.12.
(2) Let $\mathcal{C}$ be the class of unital, separable, nuclear, simple tracially AF C^*-algebras in $\mathcal{N}$. Then using [43, Theorem 2.3] we get that $\mathcal{C}$ satisfies the property in Definition 4.12.

Theorem 4.14 (Uniqueness Theorem 2)**.** *Let $\mathcal{C}$ be a class of C^*-algebras satisfying the property in Definition 4.12 and let $\mathfrak{A}$ be a unital, separable, nuclear, tight C^*-algebra over X_2 such that $\mathfrak{A}[2] \cong \mathbb{K}$ and $\mathfrak{A}[1] \in \mathcal{C}$. Suppose $\mathfrak{A} \otimes \mathbb{K}$ is semiprojective and $\mathfrak{A}$ has the stable weak cancellation property. Let $\phi\colon \mathfrak{A} \otimes \mathbb{K} \to \mathfrak{A} \otimes \mathbb{K}$ be a full X_2-equivariant homomorphism such that $KK(X_2; \phi) = KK(X_2; \mathrm{id}_{\mathfrak{A}\otimes\mathbb{K}})$. Then there exists a sequence of full X_2-equivariant endomorphisms $\{\alpha_n\colon \mathfrak{A}\otimes\mathbb{K} \to \mathfrak{A}\otimes\mathbb{K}\}_{n=1}^{\infty}$ such that $KK(X_2; \alpha_n) = KK(X_2; \mathrm{id}_{\mathfrak{A}\otimes\mathbb{K}})$ and*

$$\lim_{n\to\infty} \|(\alpha_n \circ \phi)(x) - x\| = 0$$

for all $x \in \mathfrak{A} \otimes \mathbb{K}$.

Proof. Set $\mathfrak{B} = \mathfrak{A} \otimes \mathbb{K}$. Note that $\mathfrak{B}$ is a tight C^*-algebra over X_2 with $\mathfrak{B}[2] \cong \mathbb{K}$. Throughout the proof, $\pi\colon \mathfrak{B} \to \mathfrak{B}[1]$ will denote the canonical surjective $*$-homomorphism. Note that $KK(\phi_{\{1\}}) = KK(\mathrm{id}_{\mathfrak{B}[1]})$ since $KK(X_2; \phi) = KK(X_2; \mathrm{id}_{\mathfrak{B}})$. Since $\mathfrak{A}[1] \in \mathcal{C}$, there exists a sequence of unitaries $\{z_k\}_{k=1}^{\infty}$ in $\mathcal{M}(\mathfrak{B}[1])$ such that

$$\lim_{k\to\infty} \|z_k \phi_{\{1\}}(\pi(b)) z_k^* - \pi(b)\| = 0$$

for all $b \in \mathfrak{B}$. Using the fact that ϕ is an X_2-equivariant homomorphism, we have that $\pi \circ \phi = \phi_{\{1\}} \circ \pi$, and hence

$$\lim_{k\to\infty} \|z_k (\pi \circ \phi(b)) z_k^* - \pi(b)\| = 0$$

for all $b \in \mathfrak{B}$.

Let $\overline{\pi}\colon \mathcal{M}(\mathfrak{B}) \to \mathcal{M}(\mathfrak{B}[1])$ be the surjective $*$-homomorphism induced by π which exists by [1, Theorem 4.2]. Since $\mathfrak{B}$ is stable, we have that $\mathfrak{B}[1]$

is stable. Thus, the unitary group of $\mathcal{M}(\mathfrak{B}[1])$ is path-connected with the norm topology (cf. [11]), which implies that every unitary in $\mathcal{M}(\mathfrak{B}[1])$ lifts to a unitary in $\mathcal{M}(\mathfrak{B})$ (use e.g. [58, Lemma 2.1.7(i)]). Hence, there exists a sequence of unitaries $\{w_k\}_{k=1}^{\infty}$ in $\mathcal{M}(\mathfrak{B})$ such that $\overline{\pi}(w_k) = z_k$. So,

$$\lim_{k\to\infty} \|(\pi((\mathrm{Ad}(w_k) \circ \phi)(b)) - \pi(b)\| = 0$$

for all $b \in \mathfrak{B}$.

Since $\mathfrak{B}$ is semiprojective, by [13, Proposition 2.2] (see [46]), there exist a sequence of $*$-homomorphisms $\{\beta_\ell\colon \mathfrak{B} \to \mathfrak{B}\}_{\ell=1}^{\infty}$ and a strictly increasing sequence $\{k(\ell)\}_{\ell=1}^{\infty}$ of positive integers such that $\pi \circ \beta_\ell = \pi$ and

$$\lim_{\ell\to\infty} \|\mathrm{Ad}(w_{k(\ell)}) \circ \phi(b) - \beta_\ell(b)\| = 0$$

for all $b \in \mathfrak{B}$.

By Remark 2.5, there exists $N_1 \in \mathbb{N}$ such that β_ℓ is a full X_2-equivariant homomorphism for all $\ell \geq N_1$. By [13, Proposition 2.3], we may choose $N_2 \geq N_1$ such that for all $\ell \geq N_2$, we have that β_ℓ and $\mathrm{Ad}(w_{k(\ell)}) \circ \phi$ are homotopic. It follows from Lemma 2.18, [14, Theorem 5.2] and Lemma 4.6 that $KK(X_2; \beta_\ell) = KK(X_2; \mathrm{Ad}(w_{k(\ell)}) \circ \phi) = KK(X_2; \phi) = KK(X_2; \mathrm{id}_{\mathfrak{B}})$ (β_ℓ and $\mathrm{Ad}(w_{k(\ell)}) \circ \phi$ are homotopic so by Lemma 2.18 they are homotopic as X_2-equivariant maps, and thus they induce the same map in $KK(X_2)$).

Let $\ell \geq N_2$. Note that $(\beta_\ell)_{\{1\}} = \mathrm{id}_{\mathfrak{B}[1]}$ since $\pi \circ \beta_\ell = \pi$. Since $\mathfrak{A}$ is semiprojective, by [12, Corollary 3.6] (also see [46, Chapter 19]), $\mathfrak{A}$ is weakly semiprojective. Hence, by Lemma 4.11, there exists a sequence of $*$-homomorphisms $\{\alpha_{m,\ell}\colon \mathfrak{B} \to \mathfrak{B}\}_{m=1}^{\infty}$ such that

$$\lim_{m\to\infty} \|\alpha_{m,\ell} \circ \beta_\ell(x) - x\| = 0$$

for all $x \in \mathfrak{B}$. Since β_ℓ and $\mathrm{id}_{\mathfrak{B}}$ are full X_2-equivariant homomorphisms, by Remark 2.5, there exists N_3 such that for each $m \geq N_3$, we have that $\alpha_{m,\ell}$ is a full X_2-equivariant homomorphism. Moreover, by [13, Proposition 2.3], we can choose $N_3 \geq N_2$ such that $\alpha_{m,\ell} \circ \beta_\ell$ and $\mathrm{id}_{\mathfrak{B}}$ are homotopic for all $m \geq N_3$. It follows from Lemma 2.18 and [14, Theorem 5.2] that $KK(X_2; \alpha_{m,\ell} \circ \beta_\ell) = KK(X_2; \mathrm{id}_{\mathfrak{B}})$ for all $m \geq N_3$. Consequently, $KK(X_2; \alpha_{m,\ell}) = KK(X_2; \mathrm{id}_{\mathfrak{B}})$ for all $m \geq N_3$ since $KK(X_2; \beta_\ell) = KK(X_2; \mathrm{id}_{\mathfrak{B}})$.

Let $\mathcal{F}$ be a finite subset of $\mathfrak{B}$ and $\epsilon > 0$. Then there exists $\ell \geq N_2$ such that

$$\|\mathrm{Ad}(w_{k(\ell)}) \circ \phi(b) - \beta_\ell(b)\| < \frac{\epsilon}{2}$$

for all $b \in \mathcal{F}$. Moreover, there exists $m \geq N_3$ such that

$$\|\alpha_{m,\ell} \circ \beta_\ell(b) - b\| < \frac{\epsilon}{2}$$

for all $b \in \mathcal{F}$. Set $\alpha_1 = \mathrm{Ad}(w_{k(\ell)})|_{\mathfrak{B}}$ and $\alpha = \alpha_{m,\ell} \circ \alpha_1$. Since $w_{k(\ell)}$ is a unitary in $\mathcal{M}(\mathfrak{B})$, we have that α_1 is an automorphism of $\mathfrak{B}$ and $KK(X_2; \alpha_1) = KK(X_2; \mathrm{id}_{\mathfrak{B}})$. Therefore, α is a full X_2-equivariant homomorphism. Since

$\ell \geq N_2$ and $m \geq N_3$, we have that $KK(X_2; \alpha_{m,\ell}) = KK(X_2; \mathrm{id}_{\mathfrak{B}})$. Therefore, $KK(X_2; \alpha) = KK(X_2; \mathrm{id}_{\mathfrak{B}})$. Let $b \in \mathcal{F}$. Then

$$\begin{aligned}\|\alpha \circ \phi(b) - b\| &= \|\alpha_{m,\ell} \circ \mathrm{Ad}(w_{k(\ell)}) \circ \phi(b) - b\| \\ &\leq \|\alpha_{m,\ell} \circ \mathrm{Ad}(w_{k(\ell)}) \circ \phi(b) - \alpha_{m,\ell} \circ \beta_\ell(b)\| + \|\alpha_{m,\ell} \circ \beta_\ell(b) - b\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

We have just shown that for every $\epsilon > 0$ and for every finite subset $\mathcal{F}$ of $\mathfrak{B}$, there exists a full X_2-equivariant homomorphism $\alpha\colon \mathfrak{B} \to \mathfrak{B}$ such that $KK(X_2; \alpha) = KK(X_2; \mathrm{id}_{\mathfrak{B}})$ and

$$\|\alpha \circ \phi(b) - b\| < \epsilon$$

for all $b \in \mathcal{F}$. Since $\mathfrak{B}$ is a separable C^*-algebra, there exists a sequence of full X_2-equivariant homomorphisms $\{\alpha_n\colon \mathfrak{B} \to \mathfrak{B}\}_{n=1}^{\infty}$ such that $KK(X_2; \alpha_n) = KK(X_2; \mathrm{id}_{\mathfrak{B}})$ and

$$\lim_{n\to\infty} \|\alpha_n \circ \phi(b) - b\| = 0$$

for all $b \in \mathfrak{B}$. □

Theorem 4.15. *Let $\mathcal{C}$ be a class of C^*-algebras satisfying the property in Definition 4.12, and let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be unital, separable, nuclear, tight C^*-algebras over X_2 such that $\mathfrak{A}_i[2] \cong \mathbb{K}$ and $\mathfrak{A}_i[1] \in \mathcal{C}$ for all i. Suppose $\mathfrak{A}_i \otimes \mathbb{K}$ is semiprojective and $\mathfrak{A}_i$ has the stable weak cancellation property for all i. If there exist full X_2-equivariant homomorphisms $\phi\colon \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ and $\psi\colon \mathfrak{A}_2 \otimes \mathbb{K} \to \mathfrak{A}_1 \otimes \mathbb{K}$ such that $KK(X_2; \phi \circ \psi) = KK(X_2; \mathrm{id}_{\mathfrak{A}_2 \otimes \mathbb{K}})$ and $KK(X_2; \psi \circ \phi) = KK(X_2; \mathrm{id}_{\mathfrak{A}_1 \otimes \mathbb{K}})$, then for any finite subset $\mathcal{F}$ and $\epsilon > 0$, there exists an isomorphism $\gamma\colon \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KK(X_2; \gamma) = KK(X_2; \phi)$ and*

$$\|\gamma(x) - \phi(x)\| < \epsilon$$

for all $x \in \mathcal{F}$.

Proof. Let $\{\overline{\mathcal{F}}_n\}_{n=1}^{\infty}$ be a sequence of finite subsets of $\mathfrak{A}_1 \otimes \mathbb{K}$ such that $\overline{\mathcal{F}}_n \subseteq \overline{\mathcal{F}}_{n+1}$ and $\bigcup_{n=1}^{\infty} \overline{\mathcal{F}}_n$ is dense in $\mathfrak{A}_1 \otimes \mathbb{K}$ and let $\{\overline{\mathcal{G}}_n\}_{n=1}^{\infty}$ be a sequence of finite subsets of $\mathfrak{A}_2 \otimes \mathbb{K}$ such that $\overline{\mathcal{G}}_n \subseteq \overline{\mathcal{G}}_{n+1}$ and $\bigcup_{n=1}^{\infty} \overline{\mathcal{G}}_n$ is dense in $\mathfrak{A}_2 \otimes \mathbb{K}$.

Let $\epsilon > 0$ and $\mathcal{F}$ be a finite subset of $\mathfrak{A}_1$. Set $\mathcal{F}_1 = \mathcal{F} \cup \overline{\mathcal{F}}_1$ and choose $m_1 \in \mathbb{N}$ such that $\sum_{k=m_1}^{\infty} \frac{1}{2^k} < \epsilon$. By Theorem 4.14, there exists a full X_2-equivariant homomorphism $\alpha_1\colon \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_1 \otimes \mathbb{K}$ such that $KK(X_2; \alpha_1) = KK(X_2; \mathrm{id}_{\mathfrak{A}_1 \otimes \mathbb{K}})$ and

$$\|\alpha_1 \circ \psi \circ \phi(a) - a\| < \frac{1}{2^{m_1+1}}$$

for all $a \in \mathcal{F}_1$. Set $\phi_1 = \phi$ and $\psi_1 = \alpha_1 \circ \psi$. Then $KK(X_2; \psi_1) = KK(X_2; \psi)$ and $\|\psi_1 \circ \phi_1(a) - a\| < \frac{1}{2^{m_1+1}}$ for all $a \in \mathcal{F}_1$.

Set $\mathcal{G}_1 = \overline{\mathcal{G}}_1 \cup \phi(\mathcal{F}_1)$. Note that $KK(X_2; \phi \circ \psi_1) = KK(X_2; \phi \circ \psi) = KK(X_2; \mathrm{id}_{\mathfrak{A}_2 \otimes \mathbb{K}})$. Hence, by Theorem 4.14, there exists a full X_2-equivariant homomorphism $\beta_1\colon \mathfrak{A}_2\otimes\mathbb{K} \to \mathfrak{A}_2\otimes\mathbb{K}$ such that $KK(X_2;\beta_1) = KK(X_2;\mathrm{id}_{\mathfrak{A}_2\otimes\mathbb{K}})$ and

$$\|\beta_1 \circ \phi \circ \psi_1(x) - x\| < \frac{1}{2^{m_1+1}}$$

for all $x \in \mathcal{G}_1$. Set $\phi_2 = \beta_1 \circ \phi$. Then $KK(X_2;\phi_2) = KK(X_2;\phi)$ and

$$\|\phi_2 \circ \psi_1(x) - x\| < \frac{1}{2^{m_1+1}}$$

for all $x \in \mathcal{G}_1$. Note that for each $x \in \mathcal{F}_1$, we have that

$$\begin{aligned}\|\phi(x) - \phi_2(x)\| &\leq \|\phi_1(x) - \phi_2 \circ \psi_1(\phi_1(x))\| + \|\phi_2 \circ \psi_1(\phi_1(x)) - \phi_2(x)\| \\ &< \frac{1}{2^{m_1+1}} + \|\psi_1 \circ \phi_1(x) - x\| < \frac{1}{2^{m_1}}.\end{aligned}$$

Set $\mathcal{F}_2 = \overline{\mathcal{F}}_2 \cup \psi_1(\mathcal{G}_1) \cup \mathcal{F}_1$. Note that $KK(X_2;\psi\circ\phi_2) = KK(X_2;\psi\circ\phi) = KK(X_2;\mathrm{id}_{\mathfrak{A}_1\otimes\mathbb{K}})$. Hence, by Theorem 4.14, there exists a full X_2-equivariant homomorphism $\alpha_2\colon \mathfrak{A}_1\otimes\mathbb{K} \to \mathfrak{A}_1\otimes\mathbb{K}$ such that $KK(X_2;\alpha_2) = KK(X_2;\mathrm{id}_{\mathfrak{A}_1\otimes\mathbb{K}})$ and

$$\|\alpha_2 \circ \psi \circ \phi_2(a) - a\| < \frac{1}{2^{m_1+2}}$$

for all $a \in \mathcal{F}_2$. Set $\psi_2 = \alpha_2 \circ \psi$. Then $KK(X_2;\psi_2) = KK(X_2;\psi)$ and

$$\|\psi_2 \circ \phi_2(a) - a\| < \frac{1}{2^{m_1+2}}$$

for all $x \in \mathcal{F}_2$.

Set $\mathcal{G}_2 = \overline{\mathcal{G}}_2 \cup \phi_2(\mathcal{F}_2) \cup \mathcal{G}_1$. Note that $KK(X_2;\phi\circ\psi_2) = KK(X_2;\phi\circ\psi) = KK(X_2;\mathrm{id}_{\mathfrak{A}_2\otimes\mathbb{K}})$. Hence, by Theorem 4.14, there exists a full X_2-equivariant homomorphism $\beta_2\colon \mathfrak{A}_2\otimes\mathbb{K} \to \mathfrak{A}_2\otimes\mathbb{K}$ such that $KK(X_2;\beta_2) = KK(X_2;\mathrm{id}_{\mathfrak{A}_2\otimes\mathbb{K}})$ and

$$\|\beta_2 \circ \phi \circ \psi_2(x) - x\| < \frac{1}{2^{m_1+2}}$$

for all $x \in \mathcal{G}_2$. Set $\phi_3 = \beta_2 \circ \phi$. Then $KK(X_2;\phi_3) = KK(X_2;\phi)$ and

$$\|\phi_3 \circ \psi_2(x) - x\| < \frac{1}{2^{m_1+2}}$$

for all $x \in \mathcal{G}_2$. Note that for all $x \in \mathcal{F}_2$, we have that

$$\begin{aligned}\|\phi_2(x) - \phi_3(x)\| &\leq \|\phi_2(x) - \phi_3 \circ \psi_2(\phi_2(x))\| + \|\phi_3 \circ \psi_2(\phi_2(x)) - \phi_3(x)\| \\ &< \frac{1}{2^{m_1+2}} + \|\psi_2(\phi_2(x)) - x\| < \frac{1}{2^{m_1+1}}.\end{aligned}$$

Continuing this process, we have constructed a sequence $\{\mathcal{F}_n\}_{n=1}^\infty$ of finite subsets of $\mathfrak{A}_1 \otimes \mathbb{K}$, a sequence $\{\mathcal{G}_n\}_{n=1}^\infty$ of finite subsets of $\mathfrak{A}_2 \otimes \mathbb{K}$, a sequence of full X_2-equivariant homomorphisms $\{\phi_n\colon \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}\}_{n=1}^\infty$, and a sequence of full X_2-equivariant homomorphisms $\{\psi_n\colon \mathfrak{A}_2 \otimes \mathbb{K} \to \mathfrak{A}_1 \otimes \mathbb{K}\}_{n=1}^\infty$ such that

(1) $KK(X_2;\phi_n) = KK(X_2;\phi)$ for all $n \in \mathbb{N}$ and $\phi_1 = \phi$;

(2) $KK(X_2;\psi_n) = KK(X_2;\psi)$ for all $n \in \mathbb{N}$;
(3) $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ and $\overline{\mathcal{F}}_n \subseteq \mathcal{F}_n$;
(4) $\mathcal{G}_n \subseteq \mathcal{G}_{n+1}$ and $\overline{\mathcal{G}}_n \subseteq \mathcal{G}_n$;
(5) for each $x \in \mathcal{F}_n$ and for each $y \in \mathcal{G}_n$

$$\|\psi_n \circ \phi_n(x) - x\| < \frac{1}{2^{m_1+n}} \quad \text{and} \quad \|\phi_{n+1} \circ \psi_n(y) - y\| < \frac{1}{2^{m_1+n}};$$

(6) for each $x \in \mathcal{F}_n$,

$$\|\phi_n(x) - \phi_{n+1}(x)\| < \frac{1}{2^{m_1+n-1}}.$$

Since $\bigcup_{n=1}^{\infty} \overline{\mathcal{F}}_n$ is dense in $\mathfrak{A}_1 \otimes \mathbb{K}$ and $\overline{\mathcal{F}}_n \subseteq \mathcal{F}_n$, we have that $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is dense in $\mathfrak{A}_1 \otimes \mathbb{K}$. Similarly, $\bigcup_{n=1}^{\infty} \mathcal{G}_n$ is dense in $\mathfrak{A}_2 \otimes \mathbb{K}$. Therefore, using a standard approximate intertwining argument (cf. e.g. [57, §2.3]), there exists an isomorphism $\gamma\colon \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that

$$\|\gamma(a) - \phi_n(a)\| < \sum_{k=m_1+n-1}^{\infty} \frac{1}{2^k}$$

for all $a \in \mathcal{F}_n$. Since $\mathcal{F} \subseteq \mathcal{F}_1$, we have that

$$\|\phi(x) - \gamma(x)\| = \|\phi_1(x) - \gamma(x)\| < \sum_{k=m_1}^{\infty} \frac{1}{2^k} < \epsilon$$

for all $x \in \mathcal{F}$. Since

$$\lim_{n\to\infty} \sum_{k=m_1+n-1}^{\infty} \frac{1}{2^k} = 0,$$

we have that

$$\lim_{n\to\infty} \|\gamma(a) - \phi_n(a)\| = 0$$

for all $a \in \mathfrak{A}_1 \otimes \mathbb{K}$. Since $\mathfrak{A}_1 \otimes \mathbb{K}$ is semiprojective, by [13, Proposition 2.3], there exists $N \in \mathbb{N}$ such that γ and ϕ_N are homotopic. Hence, by Lemma 2.18 and [14, Theorem 5.2], $KK(X_2;\gamma) = KK(X_2;\phi_N) = KK(X_2;\phi)$. □

4.3. Unital classification

We now combine the above results with the Meta-theorem of Section 3 (see Theorem 3.3) to get a strong classification for a class of unital C^*-algebras which includes all unital graph C^*-algebras with exactly one non-trivial ideal.

Corollary 4.16. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be unital, tight C^*-algebras over X_n of real rank zero such that $\mathfrak{A}_i[n]$ is a Kirchberg algebra in $\mathcal{N}$ and $\mathfrak{A}_i[1, n-1]$ is an AF-algebra for all i. Let $x \in KK(X_2;\mathfrak{A}_1,\mathfrak{A}_2)$ be invertible such that $K_{X_n}(x)_Y$ is an order isomorphism for each $Y \in \mathbb{LC}(X_n)$ and $K_{X_n}(x)_{X_n}([1_{\mathfrak{A}_1}]) = [1_{\mathfrak{A}_2}]$ in $K_0(\mathfrak{A}_2)$. Then there exists an isomorphism $\phi\colon \mathfrak{A} \to \mathfrak{B}$ such that $K_{X_n}(\phi) = K_{X_n}(x)$.*

Proof. Since $\mathfrak{A}_i[1]$ and $\mathfrak{A}_i[2]$ are separable and nuclear, we have that $\mathfrak{A}_i$ is separable and nuclear. Since $\mathfrak{A}_i[1, n-1]$ is an AF-algebra and $\mathfrak{A}_i[n]$ is a Kirchberg algebra, they both have the stable weak cancellation property. By [26, Lemma 3.15], $\mathfrak{A}_i$ has the stable weak cancellation property. By Lemma 4.6, for each tight C^*-algebra $\mathfrak{A}$ over X_n, we have that $K_{X_n}(\mathrm{Ad}(u)|_{\mathfrak{A}}) = K_{X_n}(\mathrm{id}_{\mathfrak{A}})$ for every unitary $u \in \mathcal{M}(\mathfrak{A})$. Theorem 4.7 shows that $K_{X_n}(-)$ satisfies (2) of Theorem 3.3, while a computation shows that $K_{X_n}(-)$ satisfies the remaining conditions. The corollary now follows from Theorem 3.3. ☐

Corollary 4.17. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be tight unital C^*-algebras over X_2 such that $\mathfrak{A}_i[2] \cong \mathbb{K}$ and $\mathfrak{A}_i[1]$ is a Kirchberg algebra in $\mathcal{N}$ for all i. Assume that each $\mathfrak{A}_i \otimes \mathbb{K}$ is semiprojective.*

(1) *Let $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ be invertible such that $K_{X_2}(x)_Y$ is an order isomorphism for each $Y \in \mathbb{LC}(X_2)$ and $K_{X_2}(x)_{X_2}([1_{\mathfrak{A}_1}]) = [1_{\mathfrak{A}_2}]$ in $K_0(\mathfrak{A}_2)$. Then there exists an isomorphism $\gamma\colon \mathfrak{A}_1 \to \mathfrak{A}_2$ such that $KK(X_2; \gamma) = x$.*

(2) *Let $\mathfrak{B}_i = \mathfrak{A}_i \otimes \mathbb{K}$, and let $x \in KK(X_2; \mathfrak{B}_1, \mathfrak{B}_2)$ be invertible such that $K_{X_2}(x)_Y$ is an order isomorphism for each $Y \in \mathbb{LC}(X_2)$. Then there exists an isomorphism $\gamma\colon \mathfrak{B}_1 \to \mathfrak{B}_2$ such that $KK(X_2; \gamma) = x$.*

Proof. Let $i \in \{1, 2\}$. Since $\mathfrak{A}_i[1]$ and $\mathfrak{A}_i[2]$ are separable and nuclear, we have that $\mathfrak{A}_i$ is separable and nuclear. Since $\mathfrak{A}_i[2]$ and $\mathfrak{A}_i[1]$ have real rank zero and $K_1(\mathfrak{A}_i[2]) = 0$, by [10, Proposition 3.15], we have that $\mathfrak{A}_i$ has real rank zero. Since $\mathfrak{A}_i[2]$ is an AF-algebra and $\mathfrak{A}_i[1]$ is a Kirchberg algebra, they both have the stable weak cancellation property. Therefore, by [26, Lemma 3.15], $\mathfrak{A}_i$ has the stable weak cancellation property.

By [22, Lemma 1.5], the extension $0 \to \mathfrak{A}_i[2] \to \mathfrak{A}_i \to \mathfrak{A}_i[1] \to 0$ is full, and hence by [22, Proposition 1.6], $0 \to \mathfrak{A}_i[2] \otimes \mathbb{K} \to \mathfrak{A}_i \otimes \mathbb{K} \to \mathfrak{A}_i[1] \otimes \mathbb{K} \to 0$ is full. The stable case of the corollary now follows from Theorem 4.1(ii) and Theorem 4.15, and then the unital case follows from Theorem 3.3. ☐

It is an open question to determine if every unital, separable, nuclear, tight C^*-algebra $\mathfrak{A}$ over X_2 with $\mathfrak{A}[2] \cong \mathbb{K}$ and $\mathfrak{A}[1]$ a Kirchberg algebra in $\mathcal{N}$ with finitely generated K-theory is semiprojective. The following results show that under some K-theoretical conditions, $\mathfrak{A}$ is semiprojective.

Lemma 4.18. *Let E be a graph with finitely many vertices such that $C^*(E)$ is a tight C^*-algebra over X_2 with $C^*(E)[1]$ being purely infinite. Then $C^*(E)$ and $C^*(E) \otimes \mathbb{K}$ are semiprojective.*

Proof. The fact that $C^*(E)$ is semiprojective follows from the results of [18, Corollary 7.12]. By [29, Proposition 6.4], $C^*(E)[2]$ is stable. Since $C^*(E)$ is a unital C^*-algebra, by [22, Lemma 1.5], the extension $\mathfrak{e} : 0 \to C^*(E)[2] \to C^*(E) \to C^*(E)[1] \to 0$ is a full extension. By [26, Proposition 3.21 and Corollary 3.22], $C^*(E)$ is properly infinite. Therefore, by [7, Theorem 4.1], $C^*(E) \otimes \mathbb{K}$ is semiprojective. ☐

Proposition 4.19. *Let $\mathfrak{A}$ be a unital, separable, nuclear, tight C^*-algebra over X_2. If we have that $\mathfrak{A}[2] \cong \mathbb{K}$ and $\mathfrak{A}[1]$ is a Kirchberg algebra in $\mathcal{N}$ such that* $\operatorname{rank}(K_1(\mathfrak{A}[1])) \leq \operatorname{rank}(K_0(\mathfrak{A}[1]))$*, $K_1(\mathfrak{A}[1])$ is free, and the K-groups of $\mathfrak{A}[i]$ are finitely generated, then $\mathfrak{A}$ and $\mathfrak{A} \otimes \mathbb{K}$ are semiprojective.*

Proof. By [22, Lemma 1.5], $\mathfrak{e} : 0 \to \mathfrak{A}[2] \to \mathfrak{A} \to \mathfrak{A}[1] \to 0$ is a full extension. By [26, Corollary 3.22], $K_0(\mathfrak{A})_+ = K_0(\mathfrak{A})$. By [19, Theorem 6.4], there exists a graph E with finitely many vertices such that $K^+_{X_2}(\mathfrak{A}) \cong K^+_{X_2}(C^*(E))$ such that $C^*(E)$ is a tight C^*-algebra over X_2. Since E has finitely many vertices, $C^*(E)$ is unital. Since $K^+_{X_2}(\mathfrak{A}) \cong K^+_{X_2}(C^*(E))$, we have that $C^*(E)[1]$ is a Kirchberg algebra. By [22, Theorem 3.9], we have that $\mathfrak{A} \otimes \mathbb{K} \cong C^*(E) \otimes \mathbb{K}$. Thus, $C^*(E)$ and $C^*(E) \otimes \mathbb{K}$ are semiprojective by Lemma 4.18. Hence, by [7, Proposition 2.7], $\mathfrak{A}$ and $\mathfrak{A} \otimes \mathbb{K}$ are semiprojective. ☐

Corollary 4.20. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be unital, tight C^*-algebras over X_2 such that $\mathfrak{A}_i[2] \cong \mathbb{K}$ and $\mathfrak{A}_i[1]$ is a Kirchberg algebra in $\mathcal{N}$ such that* $\operatorname{rank}(K_1(\mathfrak{A}[1])) \leq \operatorname{rank}(K_0(\mathfrak{A}[1]))$*, $K_1(\mathfrak{A}[1])$ is free, and the K-groups of $\mathfrak{A}_i$ are finitely generated. Let $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ be invertible such that $K_{X_2}(x)_Y$ is an order isomorphism for each $Y \in \mathbb{LC}(X_2)$ and $K_{X_2}(x)_{X_2}([1_{\mathfrak{A}_1}]) = [1_{\mathfrak{A}_2}]$ in $K_0(\mathfrak{A}_2)$. Then there exists an isomorphism $\gamma\colon \mathfrak{A}_1 \to \mathfrak{A}_2$ such that $KK(X_2; \gamma) = x$.*

Proof. This follows from Proposition 4.19 and Corollary 4.17. ☐

5. Applications

Let E be a graph satisfying Condition (K) (in particular, if $C^*(E)$ has finitely many ideals, then E satisfies Condition (K)). Let $\mathfrak{I}_1, \mathfrak{I}_2$ be ideals of $C^*(E)$ such that $\mathfrak{I}_1 \subseteq \mathfrak{I}_2$ and $\mathfrak{I}_2/\mathfrak{I}_1$ is simple. Then by [61, Theorem 5.1] and [4, Corollary 3.5], $\mathfrak{I}_2/\mathfrak{I}_1$ is a simple graph C^*-algebra. Hence, $\mathfrak{I}_2/\mathfrak{I}_1$ is either a Kirchberg algebra or an AF-algebra.

5.1. Classification of graph C^*-algebras with exactly one ideal

Lemma 5.1. *Let E be a graph with finitely many vertices such that $C^*(E)$ is a simple AF-algebra. Then $C^*(E) \otimes \mathbb{K} \cong \mathbb{K}$. Consequently, if F is a graph with finitely many vertices such that $C^*(F)$ is a tight C^*-algebra over X_2 and $C^*(F)[2]$ is an AF-algebra, then $C^*(F)[2] \cong \mathbb{K}$.*

Proof. We claim that E is a finite graph. By [15, Corollary 2.13 and Corollary 2.15], E has no cycles, and for every vertex v_0 that emits infinitely many edges and for each vertex v, there exists a path from v to v_0. Since E has no cycles, we have that every vertex of E emits only finitely many edges. Hence, E is a finite graph. By [50, Proposition 1.18], $C^*(E) \cong \mathsf{M}_n$.

We now prove the second statement. First note that $C^*(F)[2]$ is a simple AF-algebra. Since $C^*(F)[2]$ is stably isomorphic to $C^*(E)$ for a subgraph of E, $C^*(F)[2] \otimes \mathbb{K} \cong C^*(E) \otimes \mathbb{K}$ for some graph E with finitely many vertices. Since $C^*(E)$ is a simple AF-algebra, we have that $C^*(E) \otimes \mathbb{K} \cong \mathbb{K}$. Hence,

$C^*(F)[2] \otimes \mathbb{K} \cong \mathbb{K}$ which implies that $C^*(F)[2] \cong \mathsf{M}_n$ or $C^*(F)[2] \cong \mathbb{K}$. Since $C^*(F)[2]$ is a non-unital C^*-algebra ($C^*(E)$ is a tight C^*-algebra over X_2), we have that $C^*(F)[2] \cong \mathbb{K}$. □

Definition 5.2. For a C^*-algebra $\mathfrak{A}$, set

$$\Sigma\mathfrak{A} = \{x \in K_0(\mathfrak{A}) \,:\, x = [p] \text{ for some projection } p \text{ in } \mathfrak{A}\}.$$

Let $\mathfrak{B}$ be a C^*-algebra. An order isomorphism $\alpha\colon K_0(\mathfrak{A}) \to K_0(\mathfrak{B})$ is *scale preserving* if one of the following holds:

(1) $\mathfrak{A}$ is unital if and only if $\mathfrak{B}$ unital and $\alpha([1_{\mathfrak{A}}]) = [1_{\mathfrak{B}}]$.
(2) $\mathfrak{A}$ is non-unital if and only if $\mathfrak{B}$ is non-unital and $\alpha(\Sigma\mathfrak{A}) = \Sigma\mathfrak{B}$.

Theorem 5.3. *Let E_1 and E_2 be graphs with finitely many vertices and $C^*(E_i)$ is a tight C^*-algebra over X_2. If $\alpha\colon K^+_{X_2}(C^*(E_1)) \to K^+_{X_2}(C^*(E_2))$ is an isomorphism such that α_Y is scale preserving for all $Y \in \mathbb{LC}(X_2)$, then there exists an isomorphism $\phi\colon C^*(E_1) \to C^*(E_2)$ such that $K_{X_2}(\phi) = \alpha$.*

Proof. Since E_i has finitely many vertices, $C^*(E_1)$ and $C^*(E_2)$ are unital C^*-algebras. Since every simple sub-quotient of a graph C^*-algebra is either an AF-algebra or a Kirchberg algebra and since α is an order isomorphism on the K_0-groups of the simple sub-quotients, we have that $C^*(E_1)[i]$ is an AF-algebra if and only if $C^*(E_2)[i]$ is an AF-algebra. Moreover, $C^*(E_1)$ is an AF-algebra if and only if $C^*(E_2)$ is an AF-algebra.

Case 1: Suppose $C^*(E_1)$ and $C^*(E_2)$ are AF-algebras. Hence, the result follows from Elliott's classification of AF-algebras [30].

Case 2: Suppose $C^*(E_1)$ and $C^*(E_2)$ are not AF-algebras.

Subcase 2.1: Suppose $C^*(E_1)[1]$ and $C^*(E_2)[1]$ are AF-algebras. By Corollary 4.16 and Corollary 2.11, there exists an isomorphism $\phi\colon C^*(E_1) \to C^*(E_2)$ such that $K_{X_2}(\phi) = \alpha$.

Subcase 2.2: Suppose $C^*(E_1)[1]$ and $C^*(E_2)[1]$ are Kirchberg algebras. Since $C^*(E_i)$ is not an AF-algebra, either $C^*(E_i)[2]$ is a Kirchberg algebra or an AF-algebra for all i.

Suppose $C^*(E_i)[2]$ is a Kirchberg algebra for all i. By [52, Theorem 2.4], there exists an isomorphism $\phi\colon C^*(E_1) \to C^*(E_2)$ such that $K_{X_2}(\phi) = \alpha$. Suppose $C^*(E_i)[2]$ is an AF-algebra for all i. Then, by Lemma 5.1, $C^*(E_i)[2] \cong \mathbb{K}$. By Corollary 4.20 and Corollary 2.11, there exists an isomorphism $\phi\colon C^*(E_1) \to C^*(E_2)$ such that $K_{X_2}(\phi) = \alpha$. □

Remark 5.4. In earlier versions of the paper, there was a corollary claiming to resolve the full classification problem for graph algebras with exactly one ideal, since the case solved in the theorem above was the only one thought to be open at the time. However, as explained in detail in [27, 31], the invariant used in that result fails to be complete even in the graph C^*-algebra case. The issue has now been fully resolved in [36] with the introduction of a refined invariant.

5.2. Classification of graph C^*-algebras with more than one ideal

We generalized Theorem 5.3 above to arbitrary primitive ideal spaces in [28], essentially completing the classification program for all unital graph C^*-algebras. The proof in [28] relies heavily on methods and ideas from symbolic dynamics, and draws, among many other things, on ideas from Section 3 of the paper at hand. The general, non-unital, case remains open and we end by presenting results solving special cases of it that are corollaries to the work above.

For a tight C^*-algebra $\mathfrak{A}$ over X_n, the finite and infinite simple sub-quotients of $\mathfrak{A}$ are *separated* if there exists $U \in \mathbb{O}(X_n)$ such that either

(1) $\mathfrak{A}(U)$ is an AF-algebra and $\mathfrak{A}(X_n \setminus U) \otimes \mathcal{O}_\infty \cong \mathfrak{A}(X_n \setminus U)$ or
(2) $\mathfrak{A}(X_n \setminus U)$ is an AF-algebra and $\mathfrak{A}(U) \otimes \mathcal{O}_\infty \cong \mathfrak{A}(U)$.

In [24], the authors proved that if $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are graph C^*-algebras that are tight C^*-algebras over X_n such that the finite and infinite simple sub-quotients are separated, then $\mathfrak{A}_1 \otimes \mathbb{K} \cong \mathfrak{A}_2 \otimes \mathbb{K}$ if and only if $K^+_{X_n}(\mathfrak{A}_1) \cong K^+_{X_n}(\mathfrak{A}_2)$. We will show in this section that under mild K-theoretical conditions, we may remove the separated condition for the case $n = 3$.

Lemma 5.5. *Let $n \geq 3$ be given and let E be a graph such that $C^*(E)$ is a tight C^*-algebra over X_n.*

(i) *If $C^*(E)[n]$ and $C^*(E)[1]$ are purely infinite and $C^*(E)[2, n-1]$ is an AF-algebra, then*

$$\mathfrak{e}_1 : 0 \to C^*(E)[2, n] \otimes \mathbb{K} \to C^*(E) \otimes \mathbb{K} \to C^*(E)[1] \otimes \mathbb{K} \to 0$$

is a full extension.

(ii) *Let, moreover, $3 \leq k \leq n$. If $C^*(E)[k, n]$ and $C^*(E)[1, k-2]$ are AF-algebras and $C^*(E)[k-1]$ is purely infinite, then*

$$\mathfrak{e}_2 : 0 \to C^*(E)[k, n] \otimes \mathbb{K} \to C^*(E) \otimes \mathbb{K} \to C^*(E)[1, k-1] \otimes \mathbb{K} \to 0$$

is a full extension.

Proof. Suppose $C^*(E)[n]$ and $C^*(E)[1]$ are purely infinite and $C^*(E)[2, n-1]$ is an AF-algebra. Note that $C^*(E)[1, n-1]/C^*(E)[2, n-1] \cong C^*(E)[1]$ and $C^*(E)[2, n-1]$ is the largest ideal of $C^*(E)[1, n-1]$ which is an AF-algebra. Since $C^*(E)[1, n-1]$ is isomorphic to a graph C^*-algebra of a graph with no breaking vertices, by using [29, Proposition 3.10] in the same way as in the proof of [24, Proposition 5.5], we get that

$$0 \to C^*(E)[2, n-1] \otimes \mathbb{K} \to C^*(E)[1, n-1] \otimes \mathbb{K} \to C^*(E)[1] \otimes \mathbb{K} \to 0$$

is a full extension. Since $C^*(E)[n] \otimes \mathbb{K}$ is a purely infinite simple C^*-algebra, we have that

$$0 \to C^*(E)[n] \otimes \mathbb{K} \to C^*(E)[2, n] \otimes \mathbb{K} \to C^*(E)[2, n-1] \otimes \mathbb{K} \to 0$$

is a full extension (since the corona algebra is simple, cf. [54, Theorem 3.2]). Hence, by [23, Proposition 3.2], $\mathfrak{e}_1$ is a full extension ($C^*(E)[2,n] \otimes \mathbb{K}$ satisfies the corona factorization property by [24, Proposition 6.1]).

Suppose $C^*(E)[k,n]$ and $C^*(E)[1,k-2]$ are AF-algebras and $C^*(E)[k-1]$ is purely infinite. Note that $C^*(E)[k,n]$ is the largest ideal of $C^*(E)[k-1,n]$ such that $C^*(E)[k,n]$ is an AF-algebra and $C^*(E)[k-1,n]/C^*(E)[k,n] \cong C^*(E)[k-1]$ is purely infinite. Since $C^*(E)[k-1,n]\otimes\mathbb{K}$ is isomorphic to a graph C^*-algebra of a graph with no breaking vertices, by using [29, Proposition 3.10] in the same way as in the proof of [24, Proposition 5.5], we get that

$$0 \to C^*(E)[k,n] \otimes \mathbb{K} \to C^*(E)[k-1,n] \otimes \mathbb{K} \to C^*(E)[k-1] \otimes \mathbb{K} \to 0$$

is a full extension. By [24, Proposition 5.4], $\mathfrak{e}_2$ is a full extension. □

Theorem 5.6. *Let E_1 and E_2 be graphs such that $C^*(E_i)$ is a tight C^*-algebra over X_n with $n \geq 3$. Suppose*

(i) *$C^*(E_i)[n]$ and $C^*(E_i)[1]$ are purely infinite*;
(ii) *$C^*(E_i)[2,n-1]$ is an AF-algebra*; *and*
(iii) *$KK^1(C^*(E_1)[1], C^*(E_2)[2,n]) = KL^1(C^*(E_1)[1], C^*(E_2)[2,n])$.*

Then we have that $C^*(E_1)\otimes\mathbb{K} \cong C^*(E_2)\otimes\mathbb{K}$ if and only if $K^+_{X_n}(C^*(E_1)\otimes\mathbb{K}) \cong K^+_{X_n}(C^*(E_2)\otimes\mathbb{K})$.

Proof. Let $\mathfrak{e}_i$ be the extension

$$0 \to C^*(E_i)[2,n] \otimes \mathbb{K} \to C^*(E_i) \otimes \mathbb{K} \to C^*(E_i)[1] \otimes \mathbb{K} \to 0.$$

By Lemma 5.5(i), $\mathfrak{e}_i$ is a full extension. Suppose $\alpha\colon K^+_{X_n}(C^*(E_1) \otimes \mathbb{K}) \to K^+_{X_n}(C^*(E_2) \otimes \mathbb{K})$ is an isomorphism. Lift α to an invertible element $x \in KK(X_n; C^*(E_1) \otimes \mathbb{K}, C^*(E_2) \otimes \mathbb{K})$ (cf. Corollary 2.11). Note that $r^{[2,n]}_{X_n}(x)$ is invertible in $KK([2,n]; C^*(E_1)[2,n] \otimes \mathbb{K}, C^*(E_2)[2,n] \otimes \mathbb{K})$ and $r^{\{1\}}_{X_n}(x)$ is invertible in $KK(C^*(E_1)[1] \otimes \mathbb{K}, C^*(E_2)[1] \otimes \mathbb{K})$. By Theorem 4.7, there exists an isomorphism $\phi_0\colon C^*(E_1)[2,n]\otimes\mathbb{K} \to C^*(E_2)[2,n]\otimes\mathbb{K}$ such that $KL(\phi_0) = z$, where z is the invertible element of $KL(C^*(E_1)[2,n] \otimes \mathbb{K}, C^*(E_2)[2,n] \otimes \mathbb{K})$ induced by $r^{[2,n]}_{X_n}(x)$. By the Kirchberg-Phillips classification ([38] and [49]), there exists an isomorphism $\phi_2\colon C^*(E_1)[1] \otimes \mathbb{K} \to C^*(E_2)[1] \otimes \mathbb{K}$ such that $KK(\phi_2) = r^{\{1\}}_{X_n}(x)$.

For each $i = 1,2$, we will consider $C^*(E_i)$ as a C^*-algebra over X_2 by setting $C^*(E_i)[2] = C^*(E_i)[2,n]$ and $C^*(E_i)[1,2] = C^*(E_i)$. Let y be the invertible element in $KK(X_2, C^*(E_1), C^*(E_2))$ induced by x. Note that $r^{\{1\}}_{X_2}(y) = r^{\{1\}}_{X_n}(x) = KK(\phi_2)$ and $KL(r^{\{2\}}_{X_2}(y)) = z = KL(\phi_0)$ in $KL(C^*(E_1)[2,n], C^*(E_2)[2,n])$. By [24, Theorem 3.3],

$$r^{\{1\}}_{X_2}(y) \times [\tau_{\mathfrak{e}_2}] = [\tau_{\mathfrak{e}_1}] \times r^{\{2\}}_{X_2}(y)$$

in $KK^1(C^*(E_1)[1] \otimes \mathbb{K}, C^*(E_2)[2,n] \otimes \mathbb{K})$, where $\mathfrak{e}_i$ is the extension

$$0 \to C^*(E_i)[2,n] \otimes \mathbb{K} \to C^*(E_i) \otimes \mathbb{K} \to C^*(E_i)[1] \otimes \mathbb{K} \to 0.$$

Thus,

$$KL(\phi_2) \times [\tau_{\mathfrak{e}_2}] = [\tau_{\mathfrak{e}_1}] \times KL(\phi_0)$$

in $KL^1(C^*(E_1)[1] \otimes \mathbb{K}, C^*(E_2)[2,n] \otimes \mathbb{K})$. Since we have that $KL^1(C^*(E_1)[1] \otimes \mathbb{K}, C^*(E_2)[2,n] \otimes \mathbb{K}) = KK^1(C^*(E_1)[1] \otimes \mathbb{K}, C^*(E_2)[2,n] \otimes \mathbb{K})$,

$$KK(\phi_2) \times [\tau_{\mathfrak{e}_2}] = [\tau_{\mathfrak{e}_1}] \times KK(\phi_0)$$

in $KK^1(C^*(E_1)[1] \otimes \mathbb{K}, C^*(E_2)[2,n] \otimes \mathbb{K})$. By Proposition 2.13 and Proposition 2.14, $C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K}$. □

Theorem 5.7. *Let E_1 and E_2 be graphs such that $C^*(E_i)$ is a tight C^*-algebra over X_n with $n \geq 3$. Suppose*

(i) *$C^*(E_i)[k,n]$ and $C^*(E_i)[1,k-2]$ are AF-algebras*;
(ii) *$C^*(E_i)[k-1]$ is purely infinite*; *and*
(iii) *$KK^1(C^*(E_1)[1,k-1], C^*(E_2)[k,n]) = KL^1(C^*(E_1)[1,k-1], C^*(E_2)[k,n])$,*

where $3 \leq k \leq n$. Then $C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K}$ if and only if $K^+_{X_n}(C^*(E_1) \otimes \mathbb{K}) \cong K^+_{X_n}(C^*(E_2) \otimes \mathbb{K})$.

Proof. Let $\mathfrak{e}_i$ denote the extension $0 \to C^*(E_i)[k,n] \otimes \mathbb{K} \to C^*(E_i) \otimes \mathbb{K} \to C^*(E_i)[1,k-1] \otimes \mathbb{K} \to 0$. By Lemma 5.5(ii), $\mathfrak{e}_i$ is a full extension. Suppose $\alpha\colon K^+_{X_n}(C^*(E_1) \otimes \mathbb{K}) \to K^+_{X_n}(C^*(E_2) \otimes \mathbb{K})$ is an isomorphism. Lift α to an invertible element $x \in KK(X_n; C^*(E_1) \otimes \mathbb{K}, C^*(E_2) \otimes \mathbb{K})$ (cf. Corollary 2.11). Note that $r^{[k,n]}_{X_n}(x)$ is invertible in $KK([k,n]; C^*(E_1)[k,n] \otimes \mathbb{K}, C^*(E_2)[k,n] \otimes \mathbb{K})$ and $r^{[1,k-1]}_{X_n}(x)$ is invertible in $KK([1,k-1]; C^*(E_1)[1,k-1], C^*(E_2)[1,k-1])$. By Theorem 4.7, there exists an isomorphism $\phi_2\colon C^*(E_1)[1,k-1] \otimes \mathbb{K} \to C^*(E_2)[1,k-1] \otimes \mathbb{K}$ such that $KL(\phi_2) = z_2$, where z_2 is the invertible element in $KL(C^*(E_1)[1,k-1], C^*(E_2)[1,k-1])$ induced by $r^{[1,k-1]}_{X_n}(x)$. By Elliott's classification [30], there exists an isomorphism $\phi_0\colon C^*(E_1)[k,n] \otimes \mathbb{K} \to C^*(E_2)[k,n] \otimes \mathbb{K}$ such that $KK(\phi_0) = z_0$, where z_0 is the invertible element in $KK(C^*(E_1)[k,n] \otimes \mathbb{K}, C^*(E_2)[k,n] \otimes \mathbb{K})$ induced by $r^{[k,n]}_{X_n}(x)$.

We will consider $C^*(E_i)$ as a C^*-algebra over X_2 by setting $C^*(E_i)[2] = C^*(E_i)[k,n]$ and $C^*(E_i)[1,2] = C^*(E_i)$. Let y be the invertible element in $KK(X_2, C^*(E_1), C^*(E_2))$ induced by x. Note that $KL(r^{\{1\}}_{X_2}(y)) = z_2 = KL(\phi_2)$ and $r^{\{2\}}_{X_2}(y) = z_0 = KK(\phi_0)$. By [24, Theorem 3.3],

$$r^{\{1\}}_{X_2}(y) \times [\tau_{\mathfrak{e}_2}] = [\tau_{\mathfrak{e}_1}] \times r^{\{2\}}_{X_2}(y)$$

in $KK^1(C^*(E_1)[1,k-1] \otimes \mathbb{K}, C^*(E_2)[k,n] \otimes \mathbb{K})$, where $\mathfrak{e}_i$ is the extension

$$0 \to C^*(E_i)[k,n] \otimes \mathbb{K} \to C^*(E_i) \otimes \mathbb{K} \to C^*(E_i)[1,k-1] \otimes \mathbb{K} \to 0.$$

Thus,

$$KL(\phi_2) \times [\tau_{\mathfrak{e}_2}] = [\tau_{\mathfrak{e}_1}] \times KL(\phi_0)$$

in $KL^1(C^*(E_1)[1,k-1]\otimes\mathbb{K}, C^*(E_2)[k,n]\otimes\mathbb{K})$. Since $KL^1(C^*(E_1)[1,k-1]\otimes\mathbb{K}, C^*(E_2)[k,n]\otimes\mathbb{K}) = KK^1(C^*(E_1)[1,k-1]\otimes\mathbb{K}, C^*(E_2)[k,n]\otimes\mathbb{K})$,

$$KK(\phi_2)\times[\tau_{\mathfrak{e}_2}] = [\tau_{\mathfrak{e}_1}]\times KK(\phi_0)$$

in $KK^1(C^*(E_1)[1,k-1]\otimes\mathbb{K}, C^*(E_2)[k,n]\otimes\mathbb{K})$. By Proposition 2.13 and Proposition 2.14, $C^*(E_1)\otimes\mathbb{K}\cong C^*(E_2)\otimes\mathbb{K}$, $C^*(E_1)\otimes\mathbb{K}\cong C^*(E_2)\otimes\mathbb{K}$. □

Theorem 5.8. *Let E_1 and E_2 be graphs such that $C^*(E_i)$ is a tight C^*-algebra over X_3. Suppose $K_0(C^*(E_1)[1])$ is the direct sum of cyclic groups if $C^*(E_1)[1]$ is purely infinite and $K_0(C^*(E_1)[1,2])$ is the direct sum of cyclic groups if $C^*(E_1)[1]$ is an AF-algebra. Then $C^*(E_1)\otimes\mathbb{K}\cong C^*(E_2)\otimes\mathbb{K}$ if and only if $K^+_{X_3}(C^*(E_1))\cong K^+_{X_3}(C^*(E_2))$.*

Proof. The "only if" direction is clear. Suppose $K^+_{X_3}(C^*(E_1))\cong K^+_{X_3}(C^*(E_2))$. Suppose $C^*(E_1)[1]$ is purely infinite. Then $K_0(C^*(E_1)[1])$ is the direct sum of cyclic groups. Thus, $\operatorname{Pext}^1_{\mathbb{Z}}(K_0(C^*(E_1)[1]), K_0(C^*(E_2)[2])) = 0$. Since $K_1(C^*(E_1)[1])$ is a free group, $\operatorname{Pext}^1_{\mathbb{Z}}(K_1(C^*(E_1)[1]), K_1(C^*(E_2)[2])) = 0$. Thus

$$KK^1(C^*(E_1)[1], C^*(E_2)[2,3]) = KL^1(C^*(E_1)[1], C^*(E_2)[2,3]).$$

Suppose $C^*(E_1)[1]$ is an AF-algebra. Then $K_0(C^*(E_1)[1,2])$ is the direct sum of cyclic groups. Thus, $\operatorname{Pext}^1_{\mathbb{Z}}(K_0(C^*(E_1)[1,2]), K_0(C^*(E_2)[3])) = 0$. Since $K_1(C^*(E_1)[1,2])$ is free, $\operatorname{Pext}^1_{\mathbb{Z}}(K_1(C^*(E_1)[1,2]), K_1(C^*(E_2)[3])) = 0$. Therefore,

$$KK^1(C^*(E_1)[1,2], C^*(E_2)[3]) = KL^1(C^*(E_1)[1,2], C^*(E_2)[3]).$$

Case 1: Suppose the finite and infinite simple sub-quotients of $C^*(E_1)$ are separated. Then the finite and infinite simple sub-quotients of $C^*(E_2)$ are separated. Hence, by [24, Theorem 6.8], $C^*(E_1)\otimes\mathbb{K}\cong C^*(E_2)\otimes\mathbb{K}$.

Case 2: Suppose the finite and infinite simple sub-quotients of $C^*(E_1)$ are not separated. Then the finite and infinite simple sub-quotients of $C^*(E_2)$ are not separated.

Subcase 2.1: Suppose that $C^*(E_1)[3]$ and $C^*(E_1)[1]$ are purely infinite and $C^*(E_1)[2]$ is an AF-algebra. Then $C^*(E_2)[3]$ and $C^*(E_2)[1]$ are purely infinite and $C^*(E_2)[2]$ is an AF-algebra. Then by the above paragraph we have that $KK^1(C^*(E_1)[1], C^*(E_2)[2,3]) = KL^1(C^*(E_1)[1], C^*(E_2)[2,3])$. Hence, by Theorem 5.6, $C^*(E_1)\otimes\mathbb{K}\cong C^*(E_2)\otimes\mathbb{K}$.

Subcase 2.2: Suppose that $C^*(E_1)[3]$ and $C^*(E_1)[1]$ are AF-algebras and $C^*(E_1)[2]$ is purely infinite. Then $C^*(E_2)[3]$ and $C^*(E_2)[1]$ are AF-algebras and $C^*(E_2)[2]$ is purely infinite. Then by the above paragraph we have that

$$KK^1(C^*(E_1)[1,2], C^*(E_2)[3]) = KL^1(C^*(E_1)[1,2], C^*(E_2)[3]).$$

Hence, by Theorem 5.7, $C^*(E_1)\otimes\mathbb{K}\cong C^*(E_2)\otimes\mathbb{K}$. □

Corollary 5.9. *Let E_1 and E_2 be graphs such that $C^*(E_i)$ is a tight C^*-algebra over X_3. Suppose that $K_0(C^*(E_i))$ is finitely generated. Then $C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K}$ if and only if $K^+_{X_3}(C^*(E_1)) \cong K^+_{X_3}(C^*(E_2))$.*

Proof. Since $C^*(E_i)$ is a C^*-algebra of real rank zero, the canonical surjective $*$-homomorphisms $\pi_1 \colon C^*(E_i) \to C^*(E_i)[1]$ and $\pi_2 \colon C^*(E_i) \to C^*(E)[1, 2]$ induce surjective homomorphisms $K_0(\pi_1) \colon K_0(C^*(E_i)) \to K_0(C^*(E_i)[1])$ and $K_0(\pi_2) \colon K_0(C^*(E_i)) \to K_0(C^*(E_i)[1, 2])$. Hence, $K_0(C^*(E_i)[1])$ is finitely generated since $K_0(C^*(E_i))$ is finitely generated. The corollary now follows from Theorem 5.8. □

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