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A SHARP INTEGRAL INEQUALITY FOR COMPACT LINEAR WEINGARTEN HYPERSURFACES

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Abstract. We establish a sharp integral inequality related to compact (without boundary) linear Weingarten hypersurfaces (immersed) in a locally symmetric Einstein manifold and we apply it to characterize totally umbilical hypersurfaces and isoparametric hypersurfaces with two distinct principal curvatures, one which is simple, in such an ambient space. Our approach is based on the ideas and techniques introduced by Alías and Meléndez in [4] for the case of hypersurfaces with constant scalar curvature in an Euclidean round sphere.

1. Introduction

In 1977, Cheng and Yau [8] studied the rigidity problem for hypersurfaces M^n with constant scalar curvature in a space form $\mathbb{Q}_c^{n+1}$ of constant sectional curvature c, introducing a self-adjoint second order differential operator, the so-called squared operator. By using Cheng-Yau's technique, Li [10] studied the pinching problem on the square norm of the second fundamental form for complete hypersurfaces with constant scalar curvature. Afterwards, Li [11] also studied the rigidity of oriented and without boundary compact hypersurfaces with nonnegative sectional curvature in a unit sphere $\mathbb{S}^{n+1}$ with scalar curvature proportional to the mean curvature.

Later on, Wei [17] investigated compact rotational hypersurfaces in $\mathbb{S}^{n+1}$, obtaining suitable integral formulas and applying them to characterize Clifford tori $\mathbb{S}^1(\sqrt{1-r^2})\times\mathbb{S}^{n-1}(r)$, $0<r<1$, under the assumption that some higher order mean curvature vanishes identically. In [12], Li, Suh and Wei obtained characterization results concerning (oriented and without boundary) compact

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linear Weingarten hypersurfaces in $\mathbb{S}^{n+1}$ (that is, whose mean and normalized scalar curvatures are linearly related). Next, the first author jointly with Aquino and Velásquez [5,6] established another characterization results related to complete linear Weingarten hypersurfaces in $\mathbb{Q}_c^{n+1}$, under appropriate restriction on the norm of the traceless part of the second fundamental form.

In [1], the first and second authors jointly with Alías and Meléndez extended these results for the context of complete linear Weingarten hypersurfaces in a locally symmetric Riemannian manifold obeying some standard curvature conditions (in particular, in a Riemannian space with constant sectional curvature). Under appropriate constrains on the scalar curvature function, they proved that such a hypersurface must be either totally umbilical or isometric to an isoparametric hypersurface with two distinct principal curvatures, one of them being simple.

More recently, Alías and Meléndez [4] studied the rigidity of compact hypersurfaces with constant scalar curvature in $\mathbb{S}^{n+1}$. In particular, they established a sharp Simons type integral inequality for the behavior of the norm of the traceless second fundamental form, with the equality characterizing the totally umbilical hypersurfaces and the Clifford tori $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r)$.

Proceeding with this picture, here we use the ideas and techniques of [4] to establish a sharp integral inequality related to compact linear Weingarten hypersurfaces in a locally symmetric Einstein manifold and we apply it to characterize totally umbilical hypersurfaces and isoparametric hypersurfaces with two distinct principal curvatures, one which is simple, in such an ambient space (see Theorem 4.1).

This manuscript is organized in the following way: In Section 2 we recall some basic facts concerning the geometry of hypersurfaces in a Riemannian manifold. In Section 3 we explain about the configuration of our ambient space and we quote some auxiliary results. Finally, in Section 4 we state and prove our sharp integral inequality.

2. Preliminaries

Along this work we will always deal with an n-dimensional, orientable and connected hypersurface M^n in a Riemannian manifold $\overline{M}^{n+1}$. We choose a local orthonormal frame $\{e_1, \ldots, e_{n+1}\}$ in $\overline{M}^{n+1}$ with dual coframe $\{\omega_1, \ldots, \omega_{n+1}\}$, such that, at each point, $e_1, \ldots, e_n$ are tangent to M^n and e_{n+1} is normal to M^n. We will use the following convention for the indices:

$$1 \leq A, B, C, \ldots \leq n+1 \quad \text{and} \quad 1 \leq i, j, k, \ldots \leq n.$$

In this setting, denoting by $\{\omega_{AB}\}$ the connection forms of $\overline{M}^{n+1}$, we have that the structure equations of $\overline{M}^{n+1}$ are given by

$$d\omega_A = -\sum_i \omega_{Ai} \wedge \omega_i - \omega_{An+1} \wedge \omega_{n+1}, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2}\sum_{C,D} \overline{R}_{ABCD}\omega_C \wedge \omega_D,$$

where $\overline{R}_{ABCD}$ denotes the Riemannian curvature tensor. Moreover, if $\overline{R}_{AB}$ and $\overline{R}$ denote, respectively, the Ricci tensor and the scalar curvature of $\overline{M}^{n+1}$, we have

$$\overline{R}_{CD} = \sum_B \overline{R}_{BCDB} \quad \text{and} \quad \overline{R} = \sum_A \overline{R}_{AA}.$$

Now, restricting all the tensors to M^n, $\omega_{n+1} = 0$ on M^n. Hence, $0 = d\omega_{n+1} = -\sum_i \omega_{n+1i} \wedge \omega_i$ and we get

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j,\ h_{ij} = h_{ji}.$$

This gives the second fundamental form of M^n, $B = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j e_{n+1}$ and its square length $S = |B|^2 = \sum_{i,j} h_{ij}^2$. Furthermore, the mean curvature function H of M^n is defined by $H = \frac{1}{n}\sum_i h_{ii}$.

The structure equations of M^n are given by

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$
$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l.$$

As it is well-known, the *Gauss equation* of M^n is given by

$$R_{ijkl} = \overline{R}_{ijkl} + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{1}$$

where R_{ijkl} are the components of the curvature tensor of M^n.

From (1) it follows that the Ricci curvature and the normalized scalar curvature of M^n are given, respectively, by

$$R_{ij} = \sum_k \overline{R}_{ikjk} + nHh_{ij} - \sum_k h_{ik}h_{kj} \tag{2}$$

and

$$R = \frac{1}{n(n-1)}\sum_i R_{ii}. \tag{3}$$

From (2) and (3) we obtain

$$n(n-1)R = \sum_{i,j} \overline{R}_{ijij} + n^2H^2 - S. \tag{4}$$

3. Setup and some auxiliary results

Proceeding within the context of the previous section and inspired by Choi et al. [9, 15] and Nishikawa [14], we will assume that there exist constants c_1 and c_2 such that the sectional curvature $\overline{K}$ of the ambient space $\overline{M}^{n+1}$ satisfies the following two constraints

$$\overline{K}(\eta, v) = \frac{c_1}{n} \tag{5}$$

for non-zero vectors $\eta \in T^{\perp}M$ and $v \in TM$; and, for linearly independent vectors $u, v \in TM$,

$$\overline{K}(u, v) \geq c_2. \tag{6}$$

From now on, we consider $\overline{M}^{n+1}$ a locally symmetric Riemannian manifold. Recall that a Riemannian manifold is said locally symmetric when all the covariant derivative components $\overline{R}_{ABCD;E}$ of its curvature tensor vanish identically. Moreover, we say that a hypersurface M^n is linear Weingarten when their mean and normalized scalar curvatures are linearly related.

Remark 3.1. It is worth pointing out that the Riemannian space forms $\mathbb{Q}_c^{n+1}$ of constant sectional curvature $c \in \{0, 1, -1\}$ satisfy conditions (5) and (6) for $\frac{c_1}{n} = c_2 = c$. Just to mention other spaces having these properties, a standard computation proves that the Riemannian products $\mathbb{R}^{n-k} \times \mathbb{Q}_c^{k+1}$ are locally symmetric Riemannian manifolds which also satisfy the above conditions for a wide class of hypersurfaces (for more details, see [1, Remark 3.1]).

The scalar curvature $\overline{R}$ of a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying condition (5) is given by

$$\overline{R} = \sum_A \overline{R}_{AA} = \sum_{i,j} \overline{R}_{ijij} + 2\sum_i \overline{R}_{(n+1)i(n+1)i} = \sum_{i,j} \overline{R}_{ijij} + 2c_1. \tag{7}$$

Moreover, it is well known that the scalar curvature of a locally symmetric Riemannian manifold is constant. Thus, from (7) we have that $\sum_{i,j} \overline{R}_{ijij}$ is a constant naturally attached to a locally symmetric Riemannian manifold satisfying condition (5). In this setting, for the sake of simplicity, we will consider the constant

$$\overline{\mathcal{R}} := \frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijij}. \tag{8}$$

Now, let $\Psi = \sum_{i,j} \Psi_{ij} \omega_i \otimes \omega_j$ be the symmetric tensor field on M^n defined by

$$\Psi_{ij} = nH\delta_{ij} - h_{ij}. \tag{9}$$

Following Cheng-Yau [8] and using (9), we introduce the operator $\square$ associated to Ψ acting on any smooth function $u : M^n \to \mathbb{R}$ by

$$\square u = \sum_{i,j} \Psi_{ij} u_{ij} = \sum_{i,j} (nH\delta_{ij} - h_{ij}) u_{ij}, \tag{10}$$

where $u_{ij} = e_j(e_i(u))$.

Now assume M^n is a linear Weingarten hypersurface in $\overline{M}^{n+1}$ with $R = aH + b$, $a, b \in \mathbb{R}$. Taking into account (10), we will consider the following Cheng-Yau's modified operator

$$Lu = \square u - \frac{n-1}{2} a \Delta u, \tag{11}$$

where the Laplacian of u is defined by $\Delta u = \sum_k u_{kk}$.

In other words,

$$L(u) = \operatorname{tr}(P \circ \nabla^2 u), \tag{12}$$

with

$$P = \left(nH - \frac{n-1}{2} a \right) I - B, \tag{13}$$

where I is the identity in the algebra of smooth vector fields on M^n and $\nabla^2 u$ stands for the self-adjoint linear operator metrically equivalent to the Hessian of u.

Remark 3.2. From (4) and (8), since $R = aH + b$, we have that

$$n^2 H^2 = S + n(n-1)(aH + b - \overline{\mathcal{R}}). \tag{14}$$

Since the curvature condition (5) is satisfied, we have $\overline{\mathcal{R}}$ is constant. In the case that $b > \overline{\mathcal{R}}$, it follows from (14) that $H(p) \neq 0$ for every $p \in M^n$. In this case, we choose on M^n the orientation such that $H > 0$.

Our first auxiliary lemma corresponds to a sufficient criteria of ellipticity for the operator L, whose proof can be found in [1, Lemma 3.4].

Lemma 3.3. *Let M^n, $n \geq 2$, be a linear Weingarten hypersurface in a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying* (5)*, such that $R = aH + b$. Let μ_- and μ_+ be, respectively, the minimum and the maximum of the eigenvalues of the operator P defined in* (13) *at every point $p \in M^n$. If $b > \overline{\mathcal{R}}$, where $\overline{\mathcal{R}}$ is defined in* (8)*, then the operator L defined in* (12) *is elliptic, with*

$$\mu_- > 0 \quad \text{and} \quad \mu_+ < 2nH - (n-1)a.$$

Given $\Phi_{ij} = h_{ij} - H\delta_{ij}$, we will also consider the following symmetric tensor field on M^n

$$\Phi = \sum_{i,j} \Phi_{ij} \omega_i \otimes \omega_j.$$

Let $|\Phi|^2=\sum_{i,j}\Phi_{ij}^2$ be the square of the length of Φ. From (4) and (8) we get

$$|\Phi|^2 = S - nH^2 = n(n-1)H^2 + n(n-1)\left(\overline{\mathcal{R}} - R\right). \tag{15}$$

Moreover, taking a (local) frame $\{e_1,\ldots,e_n\}$ at $p\in M^n$ such that

$$h_{ij}=\lambda_i\delta_{ij} \quad \text{and} \quad \Phi_{ij}=\kappa_i\delta_{ij},$$

it is not difficult to verify the following algebraic relations

$$\sum_i \kappa_i = 0, \quad \sum_i \kappa_i^2 = |\Phi|^2 \quad \text{and} \quad \sum_i \kappa_i^3 = \sum_i \lambda_i^3 - 3H|\Phi|^2 - nH^3. \tag{16}$$

Hence, from equations (15) and (16), we have

$$\begin{aligned} nH\sum_i \lambda_i^3 - S^2 &= -(|\Phi|^2+nH^2)^2 + nH\sum_i \kappa_i^3 + 3nH^2|\Phi|^2 + n^2H^4 \\ &= -|\Phi|^4 + nH^2|\Phi|^2 + nH\sum_i \kappa_i^3. \end{aligned} \tag{17}$$

In particular, when $n=2$, since Φ is traceless, we have that $\sum_i \kappa_i^3=0$. Thus, from (17) we obtain

$$2H\sum_i \lambda_i^3 - S^2 = -|\Phi|^4 + 2H^2|\Phi|^2. \tag{18}$$

From equation (18), we see that [1, Proposition 3.7] also works for $n=2$. So, we have the following lower estimate for the operator L acting on $|\Phi|^2$.

Lemma 3.4. *Let M^n, $n\geq 2$, be a linear Weingarten hypersurface in a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying* (5) *and* (6)*, such that $R = aH+b$ with $b>\overline{\mathcal{R}}$, where $\overline{\mathcal{R}}$ is defined in* (8)*. Then*

$$L(|\Phi|^2) \geq 2(n-1)|\Phi|^2\varphi_{a,b,c}(|\Phi|)\sqrt{\frac{|\Phi|^2}{n(n-1)} + b - \overline{\mathcal{R}} + \frac{a^2}{4}},$$

where

$$\begin{aligned} \varphi_{a,b,c}(x) = & -\frac{n-2}{n-1}x^2 + \left(na - \frac{n(n-2)}{\sqrt{n(n-1)}}x\right)\sqrt{\frac{x^2}{n(n-1)} + b - \overline{\mathcal{R}} + \frac{a^2}{4}} \\ & - \frac{n(n-2)}{\sqrt{n(n-1)}}\frac{a}{2}x + n\left(\frac{a^2}{2} + b - \overline{\mathcal{R}} + c\right) \end{aligned} \tag{19}$$

and $c = 2c_2 - \frac{c_1}{n}$.

4. A sharp integral inequality

We observe that the operator L defined in (11) is a divergence-type operator when the ambient space is an Einstein manifold. Indeed, from (10) we have that

$$\square u = \operatorname{tr}(P_1 \circ \nabla^2 u), \tag{20}$$

where $P_1 = nHI - B$. So, choosing a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n and using the standard notation $\langle\, , \rangle$ for the (induced) metric of M^n, from (20) we get

$$\Box u = \sum_{i=1}^{n} \langle P_1(\nabla_{e_i} \nabla u), e_i \rangle. \tag{21}$$

Thus, from (21) with a straightforward computation we have

$$\mathrm{div}(P_1 \nabla u) = \langle \mathrm{div} P_1, \nabla u \rangle + \Box u, \tag{22}$$

where

$$\mathrm{div} P_1 = \mathrm{tr}(\nabla P_1) = \sum_{i=1}^{n} (\nabla_{e_i} P_1) e_i.$$

Hence, from [3, Lemma 25] (see also [2, Lemma 3.1]) we have

$$\langle \mathrm{div} P_1, \nabla u \rangle = \overline{\mathrm{Ric}}(N, \nabla u), \tag{23}$$

where $\overline{\mathrm{Ric}}$ stands for the Ricci tensor of $\overline{M}^{n+1}$ and N denotes the unit normal vector field of M^n. So, assuming that the ambient space $\overline{M}^{n+1}$ is an Einstein manifold, from (23) we get

$$\langle \mathrm{div} P_1, \nabla u \rangle = 0.$$

Thus, in this case, from (22) we conclude that

$$\Box u = \mathrm{div}(P_1(\nabla u)).$$

Moreover, returning to the operator L, we get

$$L(u) = \mathrm{div}(P(\nabla u)), \tag{24}$$

where P is defined in (13).

At this point, it is natural to ask oneself about the existence of Einstein manifolds which are locally symmetric. In this direction, Tod [16] showed that four-dimensional Einstein manifolds which are also D'Atri spaces are necessarily locally symmetric. Later on, Brendle [7] proved that a compact Einstein manifold of dimension $n \geq 4$ having nonnegative isotropic curvature must be locally symmetric, extending a previous result of Micallef and Wang for $n = 4$ (see [13, Theorem 4.4]). See also [18] for another sufficient conditions for an Einstein manifold to be locally symmetric.

Now, we are in position to establish our sharp integral inequality concerning compact linear Weingarten hypersurface in a locally symmetric Einstein manifold, which follows closely the ideas and techniques introduced by Alías and Meléndez in reference [4] for the case of hypersurfaces with constant scalar curvature in an Euclidean round sphere.

Theorem 4.1. *Let M^n, $n \geq 2$, be a compact linear Weingarten hypersurface in a locally symmetric Einstein manifold $\overline{M}^{n+1}$ satisfying* (5) *and* (6)*, such that $R = aH + b$ with $b > \overline{\mathcal{R}}$, where $\overline{\mathcal{R}}$ is defined in* (8)*. Then, for every $q \geq 2$,*

$$\int_M |\Phi|^{q+2} \varphi_{a,b,c}(|\Phi|) dM \leq 0, \tag{25}$$

where the real function $\varphi_{a,b,c}$ is defined by (19)*. Moreover, if the equality holds in* (25)*, assuming in addition when $c < 0$ that $b > \overline{\mathcal{R}} - c$ and $b - \overline{\mathcal{R}} + c + \sqrt{-c}a > 0$, then*

(i) *either M^n is a totally umbilical hypersurface,*
(ii) *or*

$$|\Phi|^2 = \alpha(n, a, b, c, \overline{\mathcal{R}}),$$

where $\alpha(n, a, b, c, \overline{\mathcal{R}})$ is a positive constant depending on n, a, b, c and $\overline{\mathcal{R}}$, and M^n is isometric to an isoparametric hypersurface with two distinct principal curvatures, one of which is simple.

Proof. Considering $u = |\Phi|^2$, from Lemma 3.4 we have

$$L(u) \geq 2(n-1)u\varphi_{a,b,c}(\sqrt{u})\sqrt{\frac{u}{n(n-1)} + b - \overline{\mathcal{R}} + \frac{a^2}{4}}.$$

Now, since $u \geq 0$ and $b > \overline{\mathcal{R}}$, we obtain

$$u^{\frac{q+2}{2}}\varphi_{a,b,c}(\sqrt{u}) \leq \sqrt{\frac{n}{n-1}}\frac{u^{\frac{q}{2}}}{\sqrt{4u + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2)}}L(u)$$

for every real number q. Besides that, the compactness of M^n guarantees that we can integrate both sides of the previous equation and gets

$$\begin{aligned} &\int_M u^{\frac{q+2}{2}}\varphi_{a,b,c}(\sqrt{u}) dM \\ &\leq \sqrt{\frac{n}{n-1}}\int_M \frac{u^{\frac{q}{2}}}{\sqrt{4u + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2)}}L(u) dM. \end{aligned} \tag{26}$$

But, from (24) we have

$$f(u)L(u) = \operatorname{div}(f(u)P(\nabla u)) - f'(u)\langle P(\nabla u), \nabla u\rangle \tag{27}$$

for every smooth function f. We can integrate both sides and use the Stoke's Theorem, yielding

$$\int_M f(u)L(u) dM = -\int_M f'(u)\langle P(\nabla u), \nabla u\rangle dM$$

for every smooth function f. In our case, we choose

$$f(t) = \frac{t^{\frac{q}{2}}}{\sqrt{4t + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2)}} \quad \text{for} \quad t \geq 0. \tag{28}$$

For this reason, using the fact that $b > \overline{\mathcal{R}}$, we achieve in

$$f'(t) = \frac{(q-1)4t^{\frac{q}{2}} + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2)qt^{\frac{q-2}{2}}}{2(4t + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2))^{\frac{3}{2}}} \geq 0 \tag{29}$$

for every real number $q \geq 2$ and $t \geq 0$. Putting (28) and (29) into (26), we can estimate

$$\int_M u^{\frac{q+2}{2}} \varphi_{a,b,c}(\sqrt{u}) dM \leq -\sqrt{\frac{n}{n-1}} \int_M f'(u)\langle P(\nabla u), \nabla u\rangle dM \leq 0, \tag{30}$$

since P is positive definite by Lemma 3.3. Therefore,

$$\int_M u^{\frac{q+2}{2}} \varphi_{a,b,c}(\sqrt{u}) dM \leq 0.$$

This proves inequality (25). For the second part of our theorem, if the equality holds in (25), from (30) we obtain

$$\int_M f'(u)\langle P(\nabla u), \nabla u\rangle dM = 0. \tag{31}$$

Consequently, we get from (29) that

$$f'(u) = \frac{(q-1)4u^{\frac{q}{2}} + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2)qu^{\frac{q-2}{2}}}{2(4u + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2))^{\frac{3}{2}}} \geq 0,$$

with equality if and only if $u = 0$ and $q > 2$. Besides that, since $b > \overline{\mathcal{R}}$, we know from Lemma 3.3 that

$$\langle P(\nabla u), \nabla u\rangle \geq 0,$$

with equality if and only if $\nabla u = 0$. Well, from (31), we gain

$$f'(u)\langle P(\nabla u), \nabla u\rangle = 0.$$

Thus, the function $u = |\Phi|^2$ must be constant. In the case that $|\Phi| = 0$, M^n must be totally umbilical. Otherwise, $|\Phi|$ is a positive constant and the equality in (25) implies $\varphi_{a,b,c}(|\Phi|) = 0$. Therefore, we can finish our proof applying [1, Theorem 4.3] and, taking into account our additional assumptions when $c < 0$, [1, Theorem 4.5]. □

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