Bull. Korean Math. Soc. **59** (2022), No. 3, pp. 547–566
https://doi.org/10.4134/BKMS.b201019
pISSN: 1015-8634 / eISSN: 2234-3016

THE CHARACTERISATION OF BMO VIA COMMUTATORS IN VARIABLE LEBESGUE SPACES ON STRATIFIED GROUPS

Dongli Liu, Jian Tan, and Jiman Zhao

Abstract. Let T be a bilinear Calderón-Zygmund operator,
$$b \in \cup_{q>1} L^q_{loc}(G).$$
We firstly obtain a constructive proof of the weak factorisation of Hardy spaces. Then we establish the characterization of BMO spaces by the boundedness of the commutator $[b,T]_j$ in variable Lebesgue spaces.

1. Introduction

The commutator of a Calderón-Zygmund operator T and a function b is defined by
$$[b,T](f)(x) = b(x)T(f)(x) - T(bf)(x).$$
In 1976, Coifman, Rochberg and Weiss [6] firstly established the characterisation of BMO via the boundedness of commutators. They proved if $b \in BMO(R^n)$, then $[b,T]$ is bounded on $L^p(R^n)$ $(1 < p < \infty)$. They also obtained
$$\sum_{j=1}^{n} \|[b,R_j] : L^p(R^n) \to L^p(R^n)\| \approx \|b\|_{BMO(R^n)},$$
where R_j denotes the j^{th} Riesz transform. In [18], Janson proved it is sufficient to show the boundedness of one of these commutators $[b,R_j]$. Uchiyama [34] characterized BMO in terms of the commutators of more general singular integral operators. Chanillo [4] obtained the characterization of BMO by the

Received December 7, 2020; Revised November 16, 2021; Accepted January 17, 2022.
2020 *Mathematics Subject Classification.* Primary 42B20, 42B35, 43A80.
Key words and phrases. Bilinear Calderón-Zygmund operators, variable Lebesgue spaces, stratified groups.
Jiman Zhao is supported by the National Key Research and Development Program of China (No.2020YFA0712900) and National Natural Science Foundation of China (Nos.11471040 and 11761131002). Jian Tan is supported by National Natural Science Foundation of China (No.11901309), Natural Science Foundation of Jiangsu Province of China (BK20180734), Natural Science Research of Jiangsu Higher Education Institutions of China (18KJB110022) and Nanjing University of Posts and Telecommunications Science Foundation (NY219114).

commutators of fractional integral operators. Since then, the research on the commutators has been paid much attention and has fruitful results on Euclidean spaces (see [3, 5, 15, 22, 25, 27, 31]) and on various settings (see [1, 2, 13, 17, 20, 23, 24, 28, 37]).

In 1931, Orlicz [30] introduced the variable Lebesgue spaces which are the generalization of the classical Lebesgue spaces. In the variable Lebesgue spaces, Cruz-Uribe et al. [8] studied the boundedness of some classical operators such as maximal operators, fractional integral operators, singular integral operators and commutators. Due to the applications to partial differential equations and the calculus of variations, many authors focus on the study of variable function spaces (see [7, 9, 19, 29, 33, 35, 36]).

Inspired by the above results, it is natural to ask whether the boundedness of the commutators in variable Lebesgue spaces can characterize BMO. Tan, Liu, and Zhao [32] gave an affirmative answer. The purpose of this paper is to extend the above result to stratified groups. On the Euclidean spaces, the authors [32] applied the techniques in Chaffee [3] and Janson [18] to prove the function belongs to BMO. In this paper, we will use the weak factorisation of Hardy spaces and the duality between BMO and the Hardy spaces to obtain the desired results.

This paper is organized as follows. In Section 2, we recall some basic properties of stratified groups and variable Lebesgue spaces. In Section 3, we firstly consider the properties of the characteristic functions in variable Lebesgue spaces (Lemma 3.3). Next, for any $H^1(G)$ atom a, we prove there exist functions f, g_1 and g_2 such that $\Pi_j(f, g_1, g_2) \to a$ in the sense of $H^1(G)$ norm (Lemma 3.6). Then we obtain the weak factorisation of Hardy spaces (Theorem 3.7). Finally, we obtain the characterization of BMO spaces by the boundedness of the commutator $[b, T]_j$ (Theorem 3.8).

Throughout this paper, the symbol $A \lesssim B$ denotes there exists a constant $C > 0$ such that $A \leq CB$, $A \approx B$ denotes $A \lesssim B$ and $B \lesssim A$. C always denotes a positive constant that is independent of the main parameters and may change from line to line. Constants with subscript will not change under different conditions. For any set $E \subset X$, χ_E denotes its characteristic function. Let $L_c^\infty(G)$ be the space of bounded functions with compact support. p' will always denote the conjugate of p.

2. Preliminary

Firstly, we recall some basic properties concerning stratified groups [12]. A Lie group G is called stratified if it is nilpotent, connected, and simple connected, and its Lie algebra $\mathfrak{g}$ is endowed with a vector space decomposition

$$\mathfrak{g} = \bigoplus_{i=1}^{m} V_i,$$

where

$$[V_1, V_i] = V_{i+1} \text{ for } 1 \le i < m, \quad \text{and} \quad [V_1, V_m] = 0.$$

The group G is identified with its Lie algebra $\mathfrak{g}$ via the exponential map which is a diffeomorphism from $\mathfrak{g}$ to G. The bi-invariant Haar measure μ on G is induced by the Lebesgue measure on its Lie algebra $\mathfrak{g}$.

A family of dilations on $\mathfrak{g}$ is a family $\{\delta_r : r > 0\}$ of algebra automorphisms of $\mathfrak{g}$ of the form $\delta_r = \exp(A \log r)$, where A is a diagonalizable linear operator with positive eigenvalues. Without loss of generality, we can assume the smallest eigenvalue of A is 1. The natural dilations on $\mathfrak{g}$ are defined by

$$\delta_r \left(\sum_{i=1}^{m} X_i \right) = \sum_{i=1}^{m} r^i X_i,$$

where $X_i \in V_i$ and $r > 0$. Moreover, let $d_1, d_2, \ldots, d_n$ be the eigenvalues of A, listed in increasing order and with each eigenvalue listed as many times as its multiplicity. Then the homogenous dimension of G is defined by $Q = \sum_{i=1}^{n} d_i$.

The homogenous norm on G is a continuous function $x \to \rho(x)$ from G to $[0, \infty)$. It is C^∞ on $G \backslash \{o\}$ and satisfies

(i) $\rho(x^{-1}) = \rho(x)$ for all $x \in G$;
(ii) $\rho(rx) = r\rho(x)$ for all $x \in G$ and $r > 0$;
(iii) $\rho(x) = 0$ if and only if $x = o$.

Where x^{-1} denotes the inverse of x and o denotes the identity element of G. Set

$$\rho(x, y) := \rho(x^{-1}y) = \rho(y^{-1}x), \qquad \forall\, x, y \in G.$$

Then there exists a constant $A_0 \ge 1$ such that

$$\rho(x, y) \le A_0(\rho(x, z) + \rho(z, y)), \qquad \forall\, x, y, z \in G.$$

With this norm, let $B(x, r) := \{y \in G : \rho(x, y) < r\}$ be the ball centred at x with radius r and $B_r := B(o, r)$. Clearly, we have $|B(x, r)| = r^Q$ for all $x \in G$.

Secondly, we recall the definition of variable Lebegue spaces on stratified groups. The function $p(\cdot) : G \to (0, \infty)$ is called the variable exponent. For a measurable subset $E \subset G$, set

$$p^+(E) := \sup_{x \in E} p(x), \qquad p^-(E) := \inf_{x \in E} p(x).$$

For conciseness, we abbreviate $p^+(G)$ and $p^-(G)$ to p^+ and p^-. Let $\mathscr{P}_1(G)$ be the set of measurable function $p(\cdot)$ such that

$$1 < p^- \le p^+ < \infty. \tag{2.1}$$

For a measurable function f,

$$\|f\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \int_G \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} d\mu(x) \le 1 \right\}.$$

The variable exponent Lebesgue space $L^{p(\cdot)}$ consists of those measurable functions f for which $\|f\|_{p(\cdot)} < \infty$.

Let $\Omega \subset G$. $p(\cdot)$ is locally log-Hölder continuous in Ω if for all $x, y \in \Omega$,

$$|p(x) - p(y)| \lesssim \frac{1}{\log(e + 1/\rho(x,y))}. \tag{2.2}$$

$p(\cdot)$ satisfies the log-Hölder decay condition with basepoint $o \in G$ if there exists $p_\infty \in \mathbb{R}$ such that for any $x \in \Omega$,

$$|p(x) - p_\infty| \lesssim \frac{1}{\log(e + \rho(o,x))}. \tag{2.3}$$

If (2.2) and (2.3) hold, then $p(\cdot)$ is log-Hölder continuous in Ω.

In fact, if $p(\cdot)$ is log-Hölder continuous in G, employing (2.2), then we have

$$|p(x) - p(y)| \lesssim \frac{1}{-\log(\rho(x,y))} \qquad \text{for } \rho(x,y) \leq \frac{1}{2}, \tag{2.4}$$

and (2.3) is equivalent to

$$|p(x) - p(y)| \lesssim \frac{1}{\log(e + \rho(o,x))} \qquad \text{for } \rho(o,y) \geq \rho(o,x). \tag{2.5}$$

Note that $p_\infty \equiv \lim\limits_{x\to\infty} p(x)$ exists in view of (2.5).

Definition 2.1. T is said to be a bilinear Calderón-Zygmund operator on G if $T : L^{p_1}(G) \times L^{p_2}(G) \to L^p(G)$ for some $p_1, p_2 \in (1,\infty)$, $p \in [1,\infty)$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and for all $f_1, f_2 \in L^\infty(G)$ with bounded support, for all $x \notin \cap_{i=1}^2 \mathrm{supp}(f_i)$,

$$T(f_1,f_2)(x) = \int_{G\times G} K(x,y_1,y_2) f_1(y_1) f_2(y_2)\, d\mu(y_1)\, d\mu(y_2), \tag{2.6}$$

where K is a locally integral function defined on $G \times G \times G \backslash \{(x,y_1,y_2) : x = y_1 = y_2\}$ and satisfies

$$|K(y_0,y_1,y_2)| \lesssim \frac{1}{\left(\sum_{k,l=0}^2 \rho(y_k,y_l)\right)^{2Q}}, \tag{2.7}$$

$$|K(y_0,y_1,y_2) - K(y_0{}',y_1,y_2)| \lesssim \frac{\rho(y_0,y_0{}')^\beta}{\left(\sum_{k,l=0}^2 \rho(y_k,y_l)\right)^{2Q+\beta}}, \tag{2.8}$$

$$|K(y_0,y_1,y_2) - K(y_0,y_1{}',y_2)| \lesssim \frac{\rho(y_1,y_1{}')^\beta}{\left(\sum_{k,l=0}^2 \rho(y_k,y_l)\right)^{2Q+\beta}}, \tag{2.9}$$

and

$$|K(y_0,y_1,y_2) - K(y_0,y_1,y_2{}')| \lesssim \frac{\rho(y_2,y_2{}')^\beta}{\left(\sum_{k,l=0}^2 \rho(y_k,y_l)\right)^{2Q+\beta}}, \tag{2.10}$$

where $\beta > 0$ and $\rho(y_j,y_j{}') \leq \frac{1}{2} \max\limits_{0\leq k\leq 2} \rho(y_j,y_k)$ for all $j = 0,1,2$.

In analogy with the Euclidean spaces, we define the commutators of the bilinear Calderón-Zygmund operators as follows.

Definition 2.2. Suppose T is a bilinear Calderón-Zygmund operator on G. Then we define

$$[b,T]_1(f_1,f_2)(x) := b(x)T(f_1,f_2)(x) - T(bf_1,f_2)(x),$$
$$[b,T]_2(f_1,f_2)(x) := b(x)T(f_1,f_2)(x) - T(f_1,bf_2)(x).$$

Definition 2.3. Suppose T is a bilinear Calderón-Zygmund operator on G. Then T is said to be $2Q$-homogeneous if for all $x \in B(x_0,r)$,

$$|T(\chi_{B_1},\chi_{B_2})(x)| \gtrsim M^{-2Q},$$

where $B_1 = B(x_1,r)$ and $B_2 = B(x_2,r)$ denote two pairwise disjoint balls and satisfy $\rho(y_0,y_l) \approx Mr$ for all $y_0 \in B(x_0,r)$ and $y_l \in B_l (l=1,2)$, where $r>0$ and $M>0$.

Remark 2.4. On Euclidean spaces, according to [14] and [21], the bilinear Riesz transform $\overrightarrow{R_j}$ is a bilinear Calderón-Zygmund operator and $2n$-homogeneous. On stratified groups, for the linear case, Duong et al. [10] proved the Riesz transform R_j is Q-homogeneous.

Definition 2.5. A function a is called an L^∞ atom if it satisfies

(i) supp $a \subset B(x,r)$;
(ii) $\int_G a(x)d\mu(x) = 0$;
(iii) $\|a\|_{L^\infty(G)} \le |B|^{-1}$.

The Hardy space $H^1(G)$ is the set of functions of the form $f = \sum_{j=1}^\infty \lambda_j a_j$ with $\{\lambda_j\} \in l^1$ and a_j an L^∞ atom, the norm is defined by

$$\|f\|_{H^1(G)} := \inf \left\{ \sum_{j=1}^\infty |\lambda_j| : f = \sum_{j=1}^\infty \lambda_j a_j, \{\lambda_j\} \in l^1, a_j \text{ an } L^\infty \text{ atom} \right\},$$

where the infimum is taken over all decompositions of $f = \sum_{j=1}^\infty \lambda_j a_j$ above. Similarly, one has the definition via L^2 atom, meaning that the atom a is supported on a ball $B \subset G$, has mean value zero $\int_B a(x)d\mu(x) = 0$ and has a size condition $\|a\|_{L^2(G)} \le |B|^{-\frac{1}{2}}$.

In order to obtain the factorisation of $H^1(G)$ and inspired by [6,10,21], we define the operator Π_j.

Definition 2.6. Suppose T is a bilinear Calderón-Zygmund operator on G. Then we define

$$\Pi_1(f,g_1,g_2)(x) := f(x)T(g_1,g_2)(x) - g_1(x)T^{*1}(f,g_2)(x),$$
$$\Pi_2(f,g_1,g_2)(x) := f(x)T(g_1,g_2)(x) - g_2(x)T^{*2}(g_1,f)(x),$$

where T^{*j} is the j-transpose of T, that is, the kernel K^{*j} satisfying

$$K^{*1}(x,y_1,y_2) = K(y_1,x,y_2), \qquad K^{*2}(x,y_1,y_2) = K(y_2,y_1,x).$$

3. The main results and proofs

In this section, we mainly use the techniques in [10, 11, 21] to obtain the following lemmas which play important roles in obtaining the weak factorisation of Hardy space $H^1(G)$. Finally, we will give our main results, Theorem 3.7 and Theorem 3.8.

Lemma 3.1. *Suppose* $p(\cdot)$ *satisfies* (2.1), (2.4) *and* (2.5). *If* $f \in L^{p(\cdot)}$ *and* $\|f\|_{p(\cdot)} > 0$, *then*

$$\int_G \left(\frac{|f(x)|}{\|f\|_{p(\cdot)}} \right)^{p(x)} d\mu(x) = 1. \tag{3.1}$$

Proof. It follows from the Fatou's lemma and the definition of $L^{p(\cdot)}$. □

Lemma 3.2. *Let* $p(\cdot)$ *satisfy* (2.1). *Then the followings are equivalent*:

(i) $p(\cdot)$ *satisfies* (2.4);

(ii) *for a given ball* B *and* $x \in B$, *we have*

$$|B|^{p(x)-p^+(B)} \lesssim 1, \qquad |B|^{p^-(B)-p(x)} \lesssim 1.$$

Proof. Let $\overline{B}$ be the closure of B and $r(B)$ be the radius of B. If $r(B) \geq \frac{1}{4A_0}$, employing $0 \leq p^+(B) - p(x) \leq p^+ - p^-$, then

$$|B|^{p(x)-p^+(B)} \leq (4A_0)^{Q(p^+(B)-p(x))} \lesssim (4A_0)^{Q(p^+-p^-)} \lesssim 1. \tag{3.2}$$

If $r(B) < \frac{1}{4A_0}$, then for any $y \in \overline{B}$, $\rho(x, y) \leq 2A_0 r(B) < \frac{1}{2}$. Since $p(\cdot)$ is continuous, then there exists $y \in \overline{B}$ such that $p(y) = p^+(B)$. Employing the fact that $0 \leq p^+(B) - p(x) = p(y) - p(x) \leq p^+ - p^-$ and (2.4), then we have

$$\begin{aligned} |B|^{p(x)-p^+(B)} &\leq (2A_0)^{Q|p(x)-p(y)|} \left(\rho(x, y)\right)^{-Q|p(x)-p(y)|} \\ &\lesssim [\rho(x, y)]^{\frac{-QC_2}{-\log(\rho(x,y))}} \\ &\lesssim \exp\left\{ \frac{-QC_2}{-\log(\rho(x, y))} \log(\rho(x, y)) \right\} \lesssim 1, \end{aligned} \tag{3.3}$$

where C_2 is the constant such that (2.4) holds. Combining (3.2) and (3.3), then we obtain the first inequality in (ii). The proof of the second is similar.

Fix $x, y \in G$ such that $\rho(x, y) \leq \frac{1}{2}$, then there exists a ball B such that $x, y \in B$ and $r(B) < \rho(x, y) + \varepsilon$. Employing (ii), then we have

$$\begin{aligned} 1 &\gtrsim |B|^{-|p(x)-p(y)|} \gtrsim (\rho(x, y) + \varepsilon)^{-Q|p(x)-p(y)|} \\ &\gtrsim \exp\{-Q|p(x) - p(y)| \log(\rho(x, y) + \varepsilon)\}. \end{aligned}$$

Let $\varepsilon \to 0$, then we obtain (2.4). □

Lemma 3.3. *Suppose* $p(\cdot)$ *satisfies* (2.1), (2.4) *and* (2.5). *Then*

(I) *for the ball* $B = B(x, r)$ *and* $|B| \leq 1$, *we have*

$$|B|^{\frac{1}{p^-(B)}} \sim |B|^{\frac{1}{p^+(B)}} \sim |B|^{\frac{1}{p(x)}} \sim \|\chi_B\|_{p(\cdot)}. \tag{3.4}$$

(II) *for the ball* $B = B(x, r)$ *and* $|B| \geq 1$, *we have*

$$\|\chi_B\|_{p(\cdot)} \sim |B|^{\frac{1}{p_\infty}}. \tag{3.5}$$

Proof. (I) By Lemma 3.2, for the ball B and any $y \in B$, we have

$$|B|^{p(y)} \lesssim |B|^{p^+(B)}, \qquad |B|^{p^-(B)} \lesssim |B|^{p(y)}.$$

If $|B| \leq 1$, then

$$|B|^{p(y)} \sim |B|^{p^+(B)} \sim |B|^{p^-(B)} \sim |B|^{p(x)}.$$

So we have

$$\int_G \left(\frac{|\chi_B(x)|}{|B|^{\frac{1}{p(x)}}} \right)^{p(y)} d\mu(y) = \int_B |B|^{\frac{-p(y)}{p(x)}} d\mu(y) \sim \int_B |B|^{-1} d\mu(y) = 1.$$

Then by Lemma 3.1, we have

$$\|\chi_B\|_{p(\cdot)} \sim |B|^{\frac{1}{p(x)}}.$$

Then we obtain (3.4).

(II) Cover G with a collection of balls of the same radius $s < 1$. Then there exists a countable subcover which we denote by E. Now we pick one ball from E and denote it by B_0. Then we can pick another ball B_1 from E which does not intersect with B_0. Again, we can pick a third ball B_2 which does not intersect with B_0 and B_1. In the same way, we can pick a sequences of balls $B_3, B_4, B_5, \ldots$. By Zorn's lemma, we know there is a maximal disjoint subcollection of E, which can be written as

$$F = \{B_j\}_{j=0}^{\infty} \subset E.$$

By the construction of F, if we pick an arbitrary ball $B' \in E$, then we can find a ball $B_i \in F$ such that $B_i \cap B' \neq \varnothing$. Moreover, we have $B' \subset 3A_0^2 B_i$. For any $x \in G$, there exist $B \in E$ and $B_k \in F$ such that $x \in B \subset 3A_0^2 B_k$, thus

$$G = \bigcup_{B_j \in F} 3A_0^2 B_j = \bigcup_{j=0}^{\infty} 3A_0^2 B_j.$$

Set $F^* = \{3A_0^2 B_j\}_{j=0}^{\infty}$. For any point $x \in G$, denote the set of balls that belong to F^* and contain x by $G(x) = \{3A_0^2 B_{jk}\}_{k=1}^{N}$. It is easy to check $B_{jk} \subset B_{4A_0^3 s}(x)$ and $N \leq (4A_0^3)^Q$. So the set $\{3A_0^2 B_j\}_{j=0}^{\infty}$ covers any point of G at most $(4A_0^3)^Q$ times. Let us denote the set of all balls that belong to F^* and intersect with $B_{rs}(z)$ by $\{3A_0^2 B_{ji}\}_{i=1}^{M}$, where r is an arbitrary positive constant. Then we have $B_{ji} \subset B_{(4+r)A_0^4 s}(z)$ and $M \leq [(4+r)A_0^4]^Q$.

Now we rearrange the sequence of balls $\{3A_0^2 B_j\}_{j=0}^{\infty}$ such that

$$dist(o, 3A_0^2 B_i) \geq dist(o, 3A_0^2 B_j) \quad \text{if } \ i > j.$$

For a sequence $\{q_j\}_{j=0}^{\infty}$ in $(1,\infty)$, define

$$\left\|\{x_j\}_{j=0}^{\infty}\right\|_{l^{(q_j)}} := \inf\left\{\lambda>0:\ \sum_{j=0}^{\infty}\left(\frac{|x_j|}{\lambda}\right)^{q_j}\leq 1\right\}.$$

For a sequence of measurable sets $\{E_j\}_{j=0}^{\infty}$ in G, write

$$\|f\|_{p(\cdot),(E_j)} = \left\|\{\|f\|_{L^{p(\cdot)}(E_j)}\}_{j=0}^{\infty}\right\|_{l^{p_\infty}}.$$

Set $q_j = p^+(3A_0^2B_j)$ or $p^-(3A_0^2B_j)$ and $q_\infty = p_\infty$. We have discussed that there are at most $[(4+j)A_0^4]^Q$ balls satisfying $dist(o,3A_0^2B_{jl}) < js$. Then we can find a number k to be large enough such that $e^{k-1}s\geq A_0^4$ and $k\geq 1$. Thus for a positive constant m, we have

$$\begin{aligned}|q_{[(4+j)A_0^4]^Q+m} - q_\infty| &\lesssim \frac{1}{\log(e+dist(o,3A_0^2B_{[(4+j)A_0^4]^Q+m}))}\\ &\lesssim \frac{1}{\log(e+js)} \lesssim \frac{1}{\log(e+A_0^4j)^{2Q+1}}\\ &\lesssim \frac{1}{\log((e+A_0^4j)^{2Q}+(e+A_0^4j)^{2Q})}\\ &\lesssim \frac{1}{\log(e+(e+A_0^8j^2+2eA_0^4j)^Q)}\\ &\lesssim \frac{1}{\log(e+(A_0^4(j+5))^Q)},\end{aligned}$$

where $j\geq 1$. Hence for all $l\in(((4+j)A_0^4)^Q,((5+j)A_0^4)^Q]\cap N$, we have

$$|q_l - q_\infty| \leq \frac{C_3}{\log(e+l)}.$$

Let $C_4=\max\{|q_i-q_\infty|\log(e+i) : i=0,1,2,\dots,(5A_0^4)^Q\}$ and $\widetilde{C}=\max\{C_3,C_4\}$. Then we have

$$|q_j - q_\infty| \leq \frac{\widetilde{C}}{\log(e+j)} \quad \text{for all } \ j\in N\cup\{0\}.$$

According to [16], then we have $l^{(q_j)}\cong l^{q_\infty}$. Next we will show that

$$\|f\|_{p(\cdot),(3A_0^2B_j)} \sim \|f\|_{p(\cdot)}. \tag{3.6}$$

Let $q_j = p^+(3A_0^2B_j)$. Since $\|\cdot\|_{p(\cdot),(3A_0^2B_j)}$ is homogeneous and it is easy to check that $\|f\|_{p(\cdot)} = 0 \Rightarrow \|f\|_{p(\cdot),(3A_0^2B_j)} = 0$. Thus we will only need to consider the case $\|f\|_{p(\cdot)}=1$. By Lemma 3.1 and the definition of q_j, we have

$$\int_{3A_0^2B_j}\frac{|f(x)|^{p(x)}}{\|f\|_{L^{p(\cdot)}(3A_0^2B_j)}^{p(x)}}d\mu(x) = 1 \leq \int_{3A_0^2B_j}\frac{|f(x)|^{p(x)}}{\|f\|_{L^{p(\cdot)}(3A_0^2B_j)}^{q_j}}d\mu(x).$$

Then

$$\sum_{j=0}^{\infty} \|f\|^{q_j}_{L^{p(\cdot)}(3A_0^2B_j)} \leq (4A_0^3)^Q \int_G |f(x)|^{p(x)} d\mu(x) = (4A_0^3)^Q.$$

Thus

$$\|f\|_{p(\cdot),\,(3A_0^2B_j)} \approx \left\| \{\|f\|_{L^{p(\cdot)}(3A_0^2B_j)}\}_{j=0}^{\infty} \right\|_{l^{(q_j)}} \leq (4A_0^3)^{\frac{Q}{p^-}} \lesssim \|f\|_{p(\cdot)}.$$

Let $q_j = p^-(3A_0^2B_j)$. Similarly, we have $\|f\|_{p(\cdot),\,(3A_0^2B_j)} \gtrsim \|f\|_{p(\cdot)}$. So we obtain (3.6). Let $f = \chi_B$. Then have

$$\|\chi_B\|_{p(\cdot)} \sim \|\chi_B\|_{p(\cdot),\,(3A_0^2B_j)}. \tag{3.7}$$

Finally, we will prove

$$\|\chi_B\|_{p(\cdot),\,(3A_0^2B_j)} \sim |B|^{\frac{1}{p_\infty}}. \tag{3.8}$$

Note that G can be regarded as a doubling metric space, by the dyadic cubes and the invariant properties of G under dilation and translations. If we fixed a ball B with $|B| \geq 1$, then there exists a dyadic cube Q^0 such that

$$(18A_0^2C')^{-1}B \subset Q^0 \subset (3A_0^2)^{-1}B,$$

where $C' \geq 1$. There also exists a sequence of dyadic cubes $\{Q_j^k\}_{j=0}^{\infty}$ and balls B_j such that for all $k \in \mathbb{Z}$,

$$G = \bigcup_{j=0}^{\infty} Q_j^k \quad \text{and} \quad B_j \subset Q_j^k \subset 6C'B_j,$$

where $\{Q_j^k\}_{j=0}^{\infty}$ are pairwise disjoint cubes and either $Q_j^k \subseteq Q^0$ or $Q_j^k \cap Q^0 = \varnothing$. It is obvious that $|Q_j^k| \sim |B_j| \sim |B| \sim s^Q$. Let $F = \{B_j\}_{j=0}^{\infty}$. We choose the radius of B_j to be small enough such that $|3A_0^2B_j| \ll 1$. By (3.4), we have

$$|3A_0^2B_j|^{\frac{1}{p(x_j)}} \sim \|\chi_{3A_0^2B_j}\|_{p(\cdot)} = \|1\|_{L^{p(\cdot)}(3A_0^2B_j)} \sim (3A_0^2s)^{\frac{Q}{p(x_j)}}.$$

Denote the sequence of cubes that include in Q^0 and belong to $V = \{Q_j^k\}_{j=0}^{\infty}$ by $W = \{Q_{ju}^k\}_{u=1}^{N}$. Then

$$|\cup_{u=1}^{N} Q_{ju}^k| = \sum_{u=1}^{N} |Q_{ju}^k| = |Q^0| \sim N|B_{ju}| \sim Ns^Q.$$

Since $B_{ju} \subset Q_{ju}^k \subseteq Q^0 \subset (3A_0^2)^{-1}B$, we have

$$\begin{aligned}\sum_{j=0}^{\infty} \|\chi_B\|^{p_\infty}_{L^{p(\cdot)}(3A_0^2B_j)} &\geq \sum_{u=1}^{N} \|1\|^{p_\infty}_{L^{p(\cdot)}(3A_0^2B_{ju})} \sim \sum_{u=1}^{N} |3A_0^2B_{ju}|^{\frac{p_\infty}{p(x_j)}} \\ &\geq \sum_{u=1}^{N} |Q_{ju}^k|^{\frac{p_\infty}{p^-}} \sim Ns^{\frac{Qp_\infty}{p^-}} \sim |B|s^{Q\left(\frac{p_\infty}{p^-}-1\right)}.\end{aligned} \tag{3.9}$$

On the other hand, suppose $B = B_r(z)$, from our previous discussion. There are at most $\left\lfloor \left(\frac{(r+4s)A_0^4}{s}\right)^Q \right\rfloor$ balls in F^* that intersect with B, where $\lfloor \cdot \rfloor$ denotes the floor function. Denote them by $\{3A_0^2 B_{jl}\}_{l=1}^M$. Then we have

$$\begin{aligned} \sum_{j=0}^{\infty} \|\chi_B\|_{L^{p(\cdot)}(3A_0^2B_j)}^{p_\infty} &\leq \sum_{l=1}^{M} \|1\|_{L^{p(\cdot)}(3A_0^2B_{jl})}^{p_\infty} \\ &\lesssim \left(\frac{(r+4s)A_0^4}{s}\right)^Q (3A_0^2 s)^{\frac{Qp_\infty}{p^+}}. \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10), we obtain (3.8). □

Lemma 3.4 ([10])**.** *Let $r > 0$, $0 < \eta \ll 1$. Suppose f is a function satisfying*

(i) $\int_G f(x)d\mu(x) = 0$;

(ii) $|f(x)| \leq \chi_{B(x_1,\eta r)}(x) + \chi_{B(x_2,\eta r)}(x)$, *where* $\rho(x_1, x_2) = r$.

Then we have

$$\|f\|_{H^1(G)} \lesssim \eta^Q r^Q \log \frac{1}{\eta}.$$

Lemma 3.5. *Let $1 \leq j \leq 2$. Suppose T is a bilinear Calderón-Zygmund operator. And for $b \in BMO(G)$, $[b, T]_j$ is bounded from $L^{p_1(\cdot)}(G) \times L^{p_2(\cdot)}(G) \to L^{p(\cdot)}(G)$ for some $p_1(\cdot)$, $p_2(\cdot) \in \mathscr{P}_1(G)$ and $p(\cdot)$ satisfying*

$$\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)} < 1, \qquad \forall\, x \in G.$$

Moreover,

$$\left\| [b, T]_j : L^{p_1(\cdot)} \times L^{p_2(\cdot)} \to L^{p(\cdot)} \right\| \lesssim \|b\|_{BMO}.$$

Then for any fixed $f \in L_c^\infty \cap L^{p'(\cdot)}$, $g_1 \in L_c^\infty \cap L^{p_1(\cdot)}$, $g_2 \in L_c^\infty \cap L^{p_2(\cdot)}$, we have

$$\Pi_j(f, g_1, g_2) \in H^1(G). \tag{3.11}$$

Moreover, there is a constant C, independent of the functions f, g_1, g_2, such that

$$\|\Pi_j(f, g_1, g_2)\|_{H^1} \leq C\|f\|_{L^{p'(\cdot)}} \|g_1\|_{L^{p_1(\cdot)}} \|g_2\|_{L^{p_2(\cdot)}}. \tag{3.12}$$

Proof. Since $f, g_1, g_2 \in L_c^\infty(G)$, for any $q_1, q_2 \in (1, \infty)$ and $q \in [1, \infty)$ with $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, we have that $f \in L^{q'}(G)$, $g_1 \in L^{q_1}(G)$ and $g_2 \in L^{q_2}(G)$. Applying the definition of Π_1, the Hölder's inequality and the property of the Calderón-Zygmund operators, we have

$$\begin{aligned} \|\Pi_1(f, g_1, g_2)\|_{L^1} &\lesssim \|f\|_{L^{q'}} \|T(g_1, g_2)\|_{L^q} + \|g_1\|_{L^{q_1}} \|T^{*1}(f, g_2)\|_{L^{q_1{}'}} \\ &\lesssim \|f\|_{L^{q'}} \|g_1\|_{L^{q_1}} \|g_2\|_{L^{q_2}}. \end{aligned}$$

Similarly we obtain $\|\Pi_2(f,g_1,g_2)\|_{L^1} \lesssim \|f\|_{L^{q'}}\|g_1\|_{L^{q_1}}\|g_2\|_{L^{q_2}}$. We also have

$$\int_G \Pi_j(f,g_1,g_2)(x)d\mu(x) = 0. \tag{3.13}$$

In practice, $\Pi_j(f,g_1,g_2) \in L^2(G)$ and has compact support. Hence, $\Pi_j(f,g_1,g_2)$ is a multiple of 2-atom in $H^1(G)$. Then we obtain (3.11).

For $b \in BMO(G)$, we consider the inner product

$$\langle b, \Pi_j(f,g_1,g_2)\rangle := \int_G b(x)\Pi_j(f,g_1,g_2)(x)d\mu(x).$$

Without loss of generality, we assume $\mathrm{supp}(\Pi_j(f,g_1,g_2)) \subset B_j$. Thus

$$\left|\int_G b(x)\Pi_j(f,g_1,g_2)(x)d\mu(x)\right| \lesssim |B_j|^{\frac{1}{2}}\|b\|_{BMO}\|\Pi_j(f,g_1,g_2)\|_{L^2(B_j)} < \infty.$$

Hence, $\langle b, \Pi_j(f,g_1,g_2)\rangle$ is well defined for $j=1,2$. In fact, we have

$$\int_G b(x)g_1(x)T^{*1}(f,g_2)(x)d\mu(x) = \langle f, T(bg_1,g_2)\rangle.$$

Thus

$$\langle b, \Pi_1(f,g_1,g_2)\rangle = \langle f, bT(g_1,g_2)\rangle - \langle f, T(bg_1,g_2)\rangle = \langle f, [b,T]_1(g_1,g_2)\rangle.$$

Similarly, we also obtain $\langle b, \Pi_2(f,g_1,g_2)\rangle = \langle f, [b,T]_2(g_1,g_2)\rangle$. So we have

$$|\langle b, \Pi_j(f,g_1,g_2)\rangle| \lesssim \|b\|_{BMO}\|f\|_{L^{p'(\cdot)}}\|g_1\|_{L^{p_1(\cdot)}}\|g_2\|_{L^{p_2(\cdot)}}.$$

By the duality between $H^1(G)$ and $BMO(G)$, we have

$$\begin{aligned}\|\Pi_j(f,g_1,g_2)\|_{H^1(G)} &\approx \sup_{b\in BMO(G):\|b\|_{BMO}\le 1} |\langle b, \Pi_j(f,g_1,g_2)\rangle| \\ &\lesssim \|f\|_{L^{p'(\cdot)}}\|g_1\|_{L^{p_1(\cdot)}}\|g_2\|_{L^{p_2(\cdot)}}.\end{aligned}$$

□

Lemma 3.6. *Suppose T is a bilinear Calderón-Zygmund operator which is $2Q$-homogeneous. Let $p_1(\cdot)$, $p_2(\cdot) \in \mathscr{P}_1(G)$ and be log-Hölder continuous. Let $p(\cdot)$ satisfy*

$$\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)} < 1, \qquad \forall\, x \in G.$$

For every $H^1(G)$ atom a and for all $\varepsilon > 0$, for all $j = 1,2$, there exist functions $f \in L_c^\infty \cap L^{p'(\cdot)}$, $g_1 \in L_c^\infty \cap L^{p_1(\cdot)}$, $g_2 \in L_c^\infty \cap L^{p_2(\cdot)}$ and a number $M = M(\varepsilon)$ such that

$$\|a - \Pi_j(f,g_1,g_2)\|_{H^1(G)} < \varepsilon, \tag{3.14}$$

and

$$\|f\|_{L^{p'(\cdot)}}\|g_1\|_{L^{p_1(\cdot)}}\|g_2\|_{L^{p_2(\cdot)}} \lesssim M^{2Q}. \tag{3.15}$$

Proof. Let a be an $H^1(G)$ atom, supported in $B_0 = B(x_0, r) \subset G$, satisfying

$$\int_G a(x)d\mu(x) = 0 \quad \text{and} \quad \|a\|_{L^\infty(G)} \leq |B_0|^{-1}.$$

For x_0, select $x_1 \in G$ such that $\rho(x_0, x_1) = Mr$, and then select $x_2 \in G$ such that $\rho(x_2, x_1) = 2A_0 r$. Let $B_1 = B(x_1, r)$, $B_2 = B(x_2, r)$. Then $B_1 \cap B_2 = \varnothing$. For any $y_0 \in B_0$, $y_1 \in B_1$ and $y_2 \in B_2$, if $M \geq 5A_0^3$, then for $l = 1, 2$, $\rho(y_0, y_l) \approx Mr$. Since T is $2Q$-homogeneous, for any $x \in B_0$,

$$|T(\chi_{B_1}, \chi_{B_2})(x)| \gtrsim M^{-2Q}. \tag{3.16}$$

Set $g_1(x) = \chi_{B_1}(x)$, $g_2(x) = \chi_{B_2}(x)$ and $f(x) = \frac{a(x)}{T(g_1,g_2)(x_0)}$, then we have

$$\begin{aligned} & a(x) - \Pi_1(f, g_1, g_2)(x) \\ = & \left(a(x) - \frac{a(x)T(\chi_{B_1}, \chi_{B_2})(x)}{T(\chi_{B_1}, \chi_{B_2})(x_0)}\right) \\ & + \left(\chi_{B_1}(x)T^{*1}\left(\frac{a}{T(\chi_{B_1}, \chi_{B_2})(x_0)}, \chi_{B_2}\right)(x)\right) \\ =: & W_1(x) + W_2(x). \end{aligned} \tag{3.17}$$

It is easy to verify that if $M \geq 10A_0^3$, then for all $y_0 \in B_0$, we have $\rho(y_0, y_2) \geq 2r$. Since the atom a is supported in the ball B_0, for $x \in B_0$, we have

$$\rho(x_0, x) < r \leq \frac{1}{2}\rho(x_0, y_2).$$

Employing (2.8), we have

$$|K(x_0, y_1, y_2) - K(x, y_1, y_2)| \lesssim \frac{\rho(x_0, x)^\beta}{\rho(x_0, y_1)^{2Q+\beta}} \lesssim \frac{r^\beta}{(Mr)^{2Q+\beta}}.$$

By (3.16), the properties of the atom a and the above inequality, we obtain

$$\begin{aligned} & |W_1(x)| \\ = & |a(x)| \left| \frac{T(\chi_{B_1}, \chi_{B_2})(x_0) - T(\chi_{B_1}, \chi_{B_2})(x)}{T(\chi_{B_1}, \chi_{B_2})(x_0)} \right| \\ \lesssim & r^{-Q} M^{2Q} \chi_{B(x_0,r)}(x) \left| \int_{B_1 \times B_2} |K(x_0, y_1, y_2) - K(x, y_1, y_2)| d\mu(y_1) d\mu(y_2) \right| \\ \lesssim & r^{-Q} M^{-\beta} \chi_{B(x_0,r)}(x). \end{aligned} \tag{3.18}$$

Similarly, if $M \geq 10A_0^3$, then $\rho(y_1, x_0) < r \leq \frac{1}{2}\rho(y_2, x_0)$. By (2.9), we have

$$|K(x_0, x, y_2) - K(y_1, x, y_2)| \lesssim \frac{r^\beta}{(Mr)^{2Q+\beta}}.$$

Applying the above inequality, the properties of atom a and (3.16), we have

$$
\begin{aligned}
&|W_2(x)| \\
&= \chi_{B_1}(x)\left|\int_{G\times G} K(y_1,x,y_2)\frac{a(y_1)\chi_{B_2}(y_2)}{T(\chi_{B_1},\chi_{B_2})(x_0)}d\mu(y_1)d\mu(y_2)\right. \\
&\qquad\left. -\int_{G\times G} K(x_0,x,y_2)\frac{a(y_1)\chi_{B_2}(y_2)}{T(\chi_{B_1},\chi_{B_2})(x_0)}d\mu(y_1)d\mu(y_2)\right| \\
&\lesssim \chi_{B_1}(x)r^{-Q}M^{2Q}\int_{B_0\times B_2}|K(x_0,x,y_2)-K(y_1,x,y_2)|d\mu(y_1)d\mu(y_2) \\
&\lesssim \chi_{B_1}(x)r^{-Q}M^{-\beta}.
\end{aligned}
\tag{3.19}
$$

Combining (3.17), (3.18) and (3.19), we obtain

$$|a(x)-\Pi_1(f,g_1,g_2)(x)|\lesssim r^{-Q}M^{-\beta}(\chi_{B(x_0,r)}(x)+\chi_{B(x_1,r)}(x)).$$

By (3.13) and the fact that the atom a has mean value zero, we have

$$\int_G [a(x)-\Pi_j(f,g_1,g_2)(x)]d\mu(x)=0.$$

By Lemma 3.4 and choosing M large enough such that $\frac{\log M}{M^\beta}<C\varepsilon$, we obtain

$$\|a-\Pi_1(f,g_1,g_2)\|_{H^1(G)}<\varepsilon.$$

Next we will consider the case $j=2$. For x_0, select $x_2\in G$ such that $\rho(x_0,x_2)=Mr$, and then select $x_1\in G$ such that $\rho(x_1,x_2)=2A_0r$. Let $B_1=B(x_1,r)$, $B_2=B(x_2,r)$. Then $B_1\cap B_2=\varnothing$. For any $y_0\in B_0$, $y_1\in B_1$ and $y_2\in B_2$, if $M\geq 5A_0^3$, then for $l=1,2$, $\rho(y_0,y_l)\approx Mr$. Thus (3.16) holds. Set $g_1(x)=\chi_{B_1}(x)$, $g_2(x)=\chi_{B_2}(x)$ and $f(x)=\frac{a(x)}{T(\chi_{B_1},\chi_{B_2})(x_0)}$. Then

$$
\begin{aligned}
&a(x)-\Pi_2(f,g_1,g_2)(x) \\
&=\left(a(x)-\frac{a(x)T(\chi_{B_1},\chi_{B_2})(x)}{T(\chi_{B_1},\chi_{B_2})(x_0)}\right) \\
&\quad+\left(\chi_{B_2}(x)T^{*2}(\chi_{B_1},\frac{a}{T(\chi_{B_1},\chi_{B_2})(x_0)})(x)\right) \\
&=: W_1(x)+W_2(x).
\end{aligned}
\tag{3.20}
$$

Similarly we can obtain

$$
\begin{aligned}
&|W_1(x)| \\
&\lesssim r^{-Q}M^{2Q}\chi_{B_0}(x)\int_{B_1\times B_2}|K(x_0,y_1,y_2)-K(x,y_1,y_2)|d\mu(y_1)d\mu(y_2) \\
&\lesssim r^{-Q}M^{-\beta}\chi_{B_0}(x).
\end{aligned}
\tag{3.21}
$$

and

$$\begin{aligned}
&|W_2(x)| \\
&\lesssim \chi_{B_2}(x) r^{-Q} M^{2Q} \int_{B_1\times B_0} |K(y_2,y_1,x)-K(x_0,y_1,x)| d\mu(y_1) d\mu(y_2) \\
&\lesssim \chi_{B_2}(x) r^{-Q} M^{-\beta}.
\end{aligned} \tag{3.22}$$

Therefore, combining (3.20), (3.21) and (3.22), we obtain

$$|a(x) - \Pi_2(f,g_1,g_2)(x)| \lesssim r^{-Q} M^{-\beta} (\chi_{B_0}(x) + \chi_{B_2}(x)).$$

Similarly, if M is large enough, then we have

$$\|a - \Pi_2(f,g_1,g_2)\|_{H^1(G)} < \varepsilon.$$

Therefore we obtain (3.14).

Moreover, by Lemma 3.3, we have

$$\|g_1\|_{L^{p_1(\cdot)}} = \|\chi_{B_1}\|_{L^{p_1(\cdot)}} \sim |B_1|^{\frac{1}{p_{1,\infty}}} \sim (r^Q)^{\frac{1}{p_{1,\infty}}},$$

$$\|g_2\|_{L^{p_2(\cdot)}} = \|\chi_{B_2}\|_{L^{p_2(\cdot)}} \sim |B_2|^{\frac{1}{p_{2,\infty}}} \sim (r^Q)^{\frac{1}{p_{2,\infty}}},$$

and

$$\|f\|_{L^{p'(\cdot)}} \lesssim r^{-Q} M^{2Q} \|\chi_{B_0}\|_{L^{p'(\cdot)}} \sim r^{-Q} M^{2Q} (r^Q)^{\frac{1}{p'_\infty}}.$$

Thus we obtain (3.15). □

Now we will give our main results.

Theorem 3.7. *Let $1 \le j \le 2$. Suppose T is a bilinear Calderón-Zygmund operator which is $2Q$-homogeneous. Let $p_1(\cdot)$, $p_2(\cdot) \in \mathscr{P}_1(G)$ and be log-Hölder continuous. Let $p(\cdot)$ satisfy*

$$\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)} < 1, \qquad \forall\, x \in G.$$

Then for each $f \in H^1(G)$, there exist a sequence $\{\lambda_s^k\} \in l^1$ and functions $g_s^k \in L_c^\infty \cap L^{p'(\cdot)}$, $h_{s,1}^k \in L_c^\infty \cap L^{p_1(\cdot)}$ and $h_{s,2}^k \in L_c^\infty \cap L^{p_2(\cdot)}$ such that

$$f = \sum_{k=1}^\infty \sum_{s=1}^\infty \lambda_s^k \Pi_j(g_s^k, h_{s,1}^k, h_{s,2}^k) \tag{3.23}$$

in the sense of $H^1(G)$. Moreover, we have

$$\inf\left\{ \sum_{k=1}^\infty \sum_{s=1}^\infty |\lambda_s^k| \|g_s^k\|_{L^{p'(\cdot)}} \|h_{s,1}^k\|_{L^{p_1(\cdot)}} \|h_{s,2}^k\|_{L^{p_2(\cdot)}} \right\} \lesssim \|f\|_{H^1(G)}, \tag{3.24}$$

where the infimum is taken over all decomposition of f such that (3.23) *holds.*

Proof. For any $f \in H^1(G)$, according to the atomic decomposition, there exist a sequence $\{\lambda_s^1\} \in l^1$ and atoms $\{a_s^1\}$ such that

$$f = \sum_{s=1}^{\infty} \lambda_s^1 a_s^1 \quad \text{and} \quad \sum_{s=1}^{\infty} |\lambda_s^1| \le C\|f\|_{H^1}.$$

By Lemma 3.6, for each a_s^1, there exist g_s^1, $h_{s,1}^1$, $h_{s,2}^1$ such that

$$\|a_s^1 - \Pi_j(g_s^1, h_{s,1}^1, h_{s,2}^1)\|_{H^1} < \varepsilon \quad \text{and} \quad \|g_s^1\|_{L^{p'(\cdot)}} \|h_{s,1}^1\|_{L^{p_1(\cdot)}} \|h_{s,2}^1\|_{L^{p_2(\cdot)}} \lesssim M^{2Q}.$$

So

$$f = \sum_{s=1}^{\infty} \lambda_s^1 \left[a_s^1 - \Pi_j(g_s^1, h_{s,1}^1, h_{s,2}^1)\right] + \sum_{s=1}^{\infty} \lambda_s^1 \Pi_j(g_s^1, h_{s,1}^1, h_{s,2}^1) =: A_1 + B_1,$$

and

$$\|A_1\|_{H^1} \le \sum_{s=1}^{\infty} |\lambda_s^1| \left\|a_s^1 - \Pi_j(g_s^1, h_{s,1}^1, h_{s,2}^1)\right\|_{H^1} < C\varepsilon\|f\|_{H^1}.$$

Since $A_1 \in H^1(G)$, we can find a sequence $\{\lambda_s^2\} \in l^1$ and atoms $\{a_s^2\}$ such that

$$A_1 = \sum_{s=1}^{\infty} \lambda_s^2 a_s^2 \quad \text{and} \quad \sum_{s=1}^{\infty} |\lambda_s^2| \le C\|A_1\|_{H^1} < C^2\varepsilon\|f\|_{H^1}.$$

For each a_s^2 and the same ε, by Lemma 3.6, there exist g_s^2, $h_{s,1}^2$, $h_{s,2}^2$ such that

$$\|a_s^2 - \Pi_j(g_s^2, h_{s,1}^2, h_{s,2}^2)\|_{H^1} < \varepsilon \quad \text{and} \quad \|g_s^2\|_{L^{p'(\cdot)}} \|h_{s,1}^2\|_{L^{p_1(\cdot)}} \|h_{s,2}^2\|_{L^{p_2(\cdot)}} \lesssim M^{2Q}.$$

Then

$$A_1 = \sum_{s=1}^{\infty} \lambda_s^2 \left[a_s^2 - \Pi_j(g_s^2, h_{s,1}^2, h_{s,2}^2)\right] + \sum_{s=1}^{\infty} \lambda_s^2 \Pi_j(g_s^2, h_{s,1}^2, h_{s,2}^2) =: A_2 + B_2,$$

and

$$\|A_2\|_{H^1} \le \sum_{s=1}^{\infty} |\lambda_s^2| \left\|a_s^2 - \Pi_j(g_s^2, h_{s,1}^2, h_{s,2}^2)\right\|_{H^1} < (C\varepsilon)^2\|f\|_{H^1}.$$

Thus we have

$$f = A_2 + \sum_{k=1}^{2} \sum_{s=1}^{\infty} \lambda_s^k \Pi_j(g_s^k, h_{s,1}^k, h_{s,2}^k).$$

Repeating this step, then for $1 \le k \le K$, we obtain

$$f = A_K + \sum_{k=1}^{K} \sum_{s=1}^{\infty} \lambda_s^k \Pi_j(g_s^k, h_{s,1}^k, h_{s,2}^k).$$

Moreover, we have

$$\|g_s^k\|_{L^{p'(\cdot)}} \|h_{s,1}^k\|_{L^{p_1(\cdot)}} \|h_{s,2}^k\|_{L^{p_2(\cdot)}} \lesssim M^{2Q} \quad \text{and} \quad \|A_K\|_{H^1} < (C\varepsilon)^K \|f\|_{H^1}.$$

Let $K \to \infty$. Then

$$f = \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \lambda_s^k \Pi_j(g_s^k, h_{s,1}^k, h_{s,2}^k)$$

in the sense of $H^1(G)$. And

$$\sum_{k=1}^{\infty} \sum_{s=1}^{\infty} |\lambda_s^k| \le \sum_{k=1}^{\infty} \varepsilon^{-1}(C\varepsilon)^k \|f\|_{H^1} = \frac{C}{1 - C\varepsilon} \|f\|_{H^1}.$$

Thus

$$\sum_{k=1}^{\infty} \sum_{s=1}^{\infty} |\lambda_s^k| \|g_s^k\|_{L^{p'(\cdot)}} \|h_{s,1}^k\|_{L^{p_1(\cdot)}} \|h_{s,2}^k\|_{L^{p_2(\cdot)}} \lesssim \frac{M^{2Q}C}{1 - C\varepsilon} \|f\|_{H^1} \lesssim \|f\|_{H^1}. \tag{3.25}$$

Then we complete the proof. □

Theorem 3.8. *Let $1 \le j \le 2$. Suppose T is a bilinear Calderón-Zygmund operator and $T : L^1 \times L^1 \to L^{1/2,\infty}$. Let $p_1(\cdot)$, $p_2(\cdot) \in \mathscr{P}_1(G)$ and be log-Hölder continuous. Let $p(\cdot)$ satisfy*

$$\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)} < 1, \qquad \forall\, x \in G.$$

If $b \in BMO$, then $[b, T]_j$ is bounded from $L^{p_1(\cdot)}(G) \times L^{p_2(\cdot)}(G) \to L^{p(\cdot)}(G)$. Moreover,

$$\left\| [b, T]_j : L^{p_1(\cdot)} \times L^{p_2(\cdot)} \to L^{p(\cdot)} \right\| \lesssim \|b\|_{BMO}. \tag{3.26}$$

Conversely, for $b \in \cup_{q>1} L^q_{loc}(G)$, if T is $2Q$-homogeneous and $[b, T]_j$ is bounded from $L^{p_1(\cdot)}(G) \times L^{p_2(\cdot)}(G) \to L^{p(\cdot)}(G)$, then $b \in BMO(G)$. Moreover,

$$\|b\|_{BMO} \lesssim \left\| [b, T]_j : L^{p_1(\cdot)} \times L^{p_2(\cdot)} \to L^{p(\cdot)} \right\|. \tag{3.27}$$

Proof. We have proved the inequality (3.26) in [26]. It suffices to prove (3.27). Since $b \in \cup_{q>1} L^q_{loc}(G)$, we can assume $b \in L^q_{loc}(G)$ for some $q > 1$. For any $f \in H^1 \cap L^\infty_c$, according to Theorem 3.7, there exist a sequence $\{\lambda_s^k\} \in l^1$ and functions $g_s^k \in L^\infty_c \cap L^{p'(\cdot)}$, $h_{s,1}^k \in L^\infty_c \cap L^{p_1(\cdot)}$ and $h_{s,2}^k \in L^\infty_c \cap L^{p_2(\cdot)}$ such that

$$f = \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \lambda_s^k \Pi_j(g_s^k, h_{s,1}^k, h_{s,2}^k)$$

in the sense of $H^1(G)$ and

$$\sum_{k=1}^{\infty} \sum_{s=1}^{\infty} |\lambda_s^k| \|g_s^k\|_{L^{p'(\cdot)}} \|h_{s,1}^k\|_{L^{p_1(\cdot)}} \|h_{s,2}^k\|_{L^{p_2(\cdot)}} \lesssim \|f\|_{H^1}.$$

Since $f \in L^\infty_c(G)$, we can assume $\operatorname{supp} f \subset E$. So $f \in L^{q'}(E)$. Thus

$$\langle b, f \rangle := \int_G b(x) f(x) d\mu(x)$$

is well defined. Set $b_i(x) = b(x)\chi_{\{x\in G:\ |b(x)|\le i\}}(x)$. Then we have

$$\int_G b(x)f(x)d\mu(x) = \lim_{i\to\infty}\int_G b_i(x)f(x)d\mu(x). \tag{3.28}$$

Since $g_s^k, h_{s,1}^k, h_{s,2}^k \in L_c^\infty$, then by the boundedness of Calderón-Zygmund operators, we know that $\Pi_j(g_s^k, h_{s,1}^k, h_{s,2}^k) \in L^{q'}(G)$. Thus $\langle b_i, \Pi_j(g_s^k, h_{s,1}^k, h_{s,2}^k)\rangle$ is well defined. According to the definition of b_i, we have $b_i \in L^\infty$, and hence $b_i \in BMO(G)$. Therefore, for each i, we obtain

$$\begin{aligned}\int_G b_i(x)f(x)d\mu(x) &= \int_G b_i(x)\sum_{k=1}^\infty\sum_{s=1}^\infty \lambda_s^k \Pi_j(g_s^k, h_{s,1}^k, h_{s,2}^k)d\mu(x)\\ &= \sum_{k=1}^\infty\sum_{s=1}^\infty \lambda_s^k\langle b_i, \Pi_j(g_s^k, h_{s,1}^k, h_{s,2}^k)\rangle.\end{aligned} \tag{3.29}$$

By (3.28) and (3.29), we have

$$\langle b, f\rangle = \sum_{k=1}^\infty\sum_{s=1}^\infty \lambda_s^k\langle b, \Pi_j(g_s^k, h_{s,1}^k, h_{s,2}^k)\rangle.$$

Then by Hölder's inequality and the boundedness of $[b,T]_j$, we have

$$\begin{aligned}|\langle b, f\rangle| &= \left|\sum_{k=1}^\infty\sum_{s=1}^\infty \lambda_s^k\langle g_s^k, [b,T]_j(h_{s,1}^k, h_{s,2}^k)\right|\\ &\le \sum_{k=1}^\infty\sum_{s=1}^\infty |\lambda_s^k|\|g_s^k\|_{L^{p'(\cdot)}}\left\|[b,T]_j(h_{s,1}^k, h_{s,2}^k)\right\|_{L^{p(\cdot)}}\\ &\lesssim \sum_{k=1}^\infty\sum_{s=1}^\infty |\lambda_s^k|\|g_s^k\|_{L^{p'(\cdot)}}\|h_{s,1}^k\|_{L^{p_1(\cdot)}}\|h_{s,2}^k\|_{L^{p_2(\cdot)}}\\ &\quad\times\left\|[b,T]_j : L^{p_1(\cdot)}\times L^{p_2(\cdot)}\to L^{p(\cdot)}\right\|\\ &\lesssim \|f\|_{H^1}\left\|[b,T]_j : L^{p_1(\cdot)}\times L^{p_2(\cdot)}\to L^{p(\cdot)}\right\|.\end{aligned}$$

Therefore

$$\|b\|_{BMO(G)} \approx \sup_{f\in H^1(G):\ \|f\|_{H^1}\le 1}|\langle b, f\rangle| \lesssim \left\|[b,T]_j : L^{p_1(\cdot)}\times L^{p_2(\cdot)}\to L^{p(\cdot)}\right\|.$$

Applying the fact $H^1\cap L_c^\infty$ is dense in H^1, then we obtain (3.27). □

Acknowledgements. The authors would like to express great gratitude to the referees for their careful reading of the manuscript and their valuable comments which improve the presentation of this article.

References

[1] M. Bramanti, *Commutators of integral operators with positive kernels*, Matematiche (Catania) **49** (1994), no. 1, 149–168.

[2] M. Bramanti and M. C. Cerutti, *Commutators of singular integrals and fractional integrals on homogeneous spaces*, in Harmonic analysis and operator theory (Caracas, 1994), 81–94, Contemp. Math., 189, Amer. Math. Soc., Providence, RI, 1995. https://doi.org/10.1090/conm/189/02257

[3] L. Chaffee, *Commutators of multilinear singular integral operators with pointwise multiplication*, Thesis (Ph.D.), University of Kansas, 2015.

[4] S. Chanillo, *A note on commutators*, Indiana Univ. Math. J. **31** (1982), no. 1, 7–16. https://doi.org/10.1512/iumj.1982.31.31002

[5] X. Chen and Q. Xue, *Weighted estimates for a class of multilinear fractional type operators*, J. Math. Anal. Appl. **362** (2010), no. 2, 355–373. https://doi.org/10.1016/j.jmaa.2009.08.022

[6] R. R. Coifman, R. Rochberg, and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. (2) **103** (1976), no. 3, 611–635. https://doi.org/10.2307/1970954

[7] D. V. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces*, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, Heidelberg, 2013. https://doi.org/10.1007/978-3-0348-0548-3

[8] D. Cruz-Uribe, A. Fiorenza, J. M. Martell, and C. Pérez, *The boundedness of classical operators on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **31** (2006), no. 1, 239–264.

[9] D. Cruz-Uribe and L.-A. D. Wang, *Variable Hardy spaces*, Indiana Univ. Math. J. **63** (2014), no. 2, 447–493. https://doi.org/10.1512/iumj.2014.63.5232

[10] X. T. Duong, H. Li, J. Li, and B. D. Wick, *Lower bound of Riesz transform kernels and commutator theorems on stratified nilpotent Lie groups*, J. Math. Pures Appl. (9) **124** (2019), 273–299. https://doi.org/10.1016/j.matpur.2018.06.012

[11] J. Fang and J. Zhao, *Variable Hardy spaces on the Heisenberg group*, Anal. Theory Appl. **32** (2016), no. 3, 242–271. https://doi.org/10.4208/ata.2016.v32.n3.4

[12] G. B. Folland and E. M. Stein, *Hardy spaces on homogeneous groups*, Mathematical Notes, 28, Princeton University Press, Princeton, NJ, 1982.

[13] X. Fu, D. Yang, and W. Yuan, *Generalized fractional integrals and their commutators over non-homogeneous metric measure spaces*, Taiwanese J. Math. **18** (2014), no. 2, 509–557. https://doi.org/10.11650/tjm.18.2014.3651

[14] L. Grafakos and R. H. Torres, *Multilinear Calderón-Zygmund theory*, Adv. Math. **165** (2002), no. 1, 124–164. https://doi.org/10.1006/aima.2001.2028

[15] Z. Guo, P. Li, and L. Peng, *L^p boundedness of commutators of Riesz transforms associated to Schrödinger operator*, J. Math. Anal. Appl. **341** (2008), no. 1, 421–432. https://doi.org/10.1016/j.jmaa.2007.05.024

[16] P. A. Hästö, *Local-to-global results in variable exponent spaces*, Math. Res. Lett. **16** (2009), no. 2, 263–278. https://doi.org/10.4310/MRL.2009.v16.n2.a5

[17] G. Hu, Y. Meng, and D. Yang, *Multilinear commutators of singular integrals with non doubling measures*, Integral Equations Operator Theory **51** (2005), no. 2, 235–255. https://doi.org/10.1007/s00020-003-1251-y

[18] S. Janson, *Mean oscillation and commutators of singular integral operators*, Ark. Mat. **16** (1978), no. 2, 263–270. https://doi.org/10.1007/BF02386000

[19] O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41(116)** (1991), no. 4, 592–618.

[20] P. Li and L. Peng, *L^p boundedness of commutator operator associated with Schrödinger operators on Heisenberg group*, Acta Math. Sci. Ser. B (Engl. Ed.) **32** (2012), no. 2, 568–578. https://doi.org/10.1016/S0252-9602(12)60039-3

[21] J. Li and B. D. Wick, *Weak factorizations of the Hardy space $H^1(\mathbb{R}^n)$ in terms of multilinear Riesz transforms*, Canad. Math. Bull. **60** (2017), no. 3, 571–585. https://doi.org/10.4153/CMB-2017-033-9

[22] Y. Liang, L. D. Ky, and D. Yang, *Weighted endpoint estimates for commutators of Calderón-Zygmund operators*, Proc. Amer. Math. Soc. **144** (2016), no. 12, 5171–5181. https://doi.org/10.1090/proc/13130

[23] H. Lin, S. Wu, and D. Yang, *Boundedness of certain commutators over non-homogeneous metric measure spaces*, Anal. Math. Phys. **7** (2017), no. 2, 187–218. https://doi.org/10.1007/s13324-016-0136-6

[24] L. Liu, D. Chang, X. Fu, and D. Yang, *Endpoint boundedness of commutators on spaces of homogeneous type*, Appl. Anal. **96** (2017), no. 14, 2408–2433. https://doi.org/10.1080/00036811.2017.1341628

[25] Z. Liu and S. Lu, *Two-weight weak-type norm inequalities for the commutators of fractional integrals*, Integral Equations Operator Theory **48** (2004), no. 3, 397–409. https://doi.org/10.1007/s00020-002-1237-1

[26] D. Liu, J. Tan, and J. Zhao, *Multilinear commutators in variable Lebesgue spaces on stratified groups*, in Analysis of pseudo-differential operators, 97–120, Trends Math, Birkhäuser/Springer, Cham, 2019. https://doi.org/10.1007/978-3-030-05168-6_5

[27] H. Liu and L. Tang, *Compactness for higher order commutators of oscillatory singular integral operators*, Internat. J. Math. **20** (2009), no. 9, 1137–1146. https://doi.org/10.1142/S0129167X09005698

[28] G. Lu, S. Lu, and D. Yang, *Singular integrals and commutators on homogeneous groups*, Anal. Math. **28** (2002), no. 2, 103–134. https://doi.org/10.1023/A:1016568918973

[29] E. Nakai and Y. Sawano, *Hardy spaces with variable exponents and generalized Campanato spaces*, J. Funct. Anal. **262** (2012), no. 9, 3665–3748. https://doi.org/10.1016/j.jfa.2012.01.004

[30] W. Orlicz, *Über konjugierte Exponentenfolgen*, Stud. Math. **3** (1931), 200–211.

[31] C. Pérez and R. H. Torres, *Sharp maximal function estimates for multilinear singular integrals*, in Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001), 323–331, Contemp. Math., 320, Amer. Math. Soc., Providence, RI, 2003. https://doi.org/10.1090/conm/320/05615

[32] J. Tan, Z. Liu, and J. Zhao, *On some multilinear commutators in variable Lebesgue spaces*, J. Math. Inequal. **11** (2017), no. 3, 715–734. https://doi.org/10.7153/jmi-2017-11-57

[33] J. Tan and J. Zhao, *Rough fractional integrals and its commutators on variable Morrey spaces*, C. R. Math. Acad. Sci. Paris **353** (2015), no. 12, 1117–1122. https://doi.org/10.1016/j.crma.2015.09.024

[34] A. Uchiyama, *On the compactness of operators of Hankel type*, Tohoku Math. J. (2) **30** (1978), no. 1, 163–171. https://doi.org/10.2748/tmj/1178230105

[35] H. Wang and Z. Liu, *The wavelet characterization of Herz-type Hardy spaces with variable exponent*, Ann. Funct. Anal. **3** (2012), no. 1, 128–141. https://doi.org/10.15352/afa/1399900030

[36] D. Yang, C. Zhuo, and W. Yuan, *Triebel-Lizorkin type spaces with variable exponents*, Banach J. Math. Anal. **9** (2015), no. 4, 146–202. https://doi.org/10.15352/bjma/09-4-9

[37] Y. Zhu and L. Liu, *Weighted sharp boundedness for multilinear commutators of singular integral on spaces of homogeneous type*, Vietnam J. Math. **39** (2011), no. 4, 381–390.

Dongli Liu
Department of Mathematics and Physics
Shijiazhuang Tiedao University
Shijiazhuang 050043, P. R. China
and
School of Mathematical Sciences
Beijing Normal University
Beijing 100875, P. R. China
Email address: `dongliliu03@163.com`

Jian Tan
School of Science
Nanjing University of Posts and Telecommunications
Nanjing 210023, P. R. China
Email address: `tanjian89@126.com`

Jiman Zhao
School of Mathematical Sciences
Beijing Normal University
Beijing 100875, P. R. China
Email address: `jzhao@bnu.edu.cn`