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A GENERALIZATION OF w-LINKED EXTENSIONS

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Abstract. In this paper, the concepts of w-linked homomorphisms, the w_ϕ-operation, and DW_ϕ rings are introduced. Also the relationships between w_ϕ-ideals and w-ideals over a w-linked homomorphism $\phi : R \to T$ are discussed. More precisely, it is shown that every w_ϕ-ideal of T is a w-ideal of T. Besides, it is shown that if T is not a DW_ϕ ring, then T must have an infinite number of maximal w_ϕ-ideals. Finally we give an application of Cohen's Theorem over w-factor rings, namely it is shown that an integral domain R is an SM-domain with w-$\dim(R) \le 1$, if and only if for any nonzero w-ideal I of R, $(R/I)_w$ is an Artinian ring, if and only if for any nonzero element $a \in R$, $(R/(a))_w$ is an Artinian ring, if and only if for any nonzero element $a \in R$, R satisfies the descending chain condition on w-ideals of R containing a.

1. Introduction

Throughout this paper, R denotes a commutative ring with identity. Let R be an integral domain with quotient field K.

As is well known, an integral domain R is a Prüfer domain if and only if every overring of R is integrally closed. In order to give a Prüfer-like characterization of PVMDs (Prüfer v-multiplication domains), the concept of t-linked extensions was introduced in [3]. Namely, let $R \subseteq T \subseteq K$ be an extension. If $J^{-1} = R$ for a finitely generated (abbreviated to f.g.) nonzero ideal J of R implies that $(JT)^{-1} = T$, then T is called a t-linked extension of R. By virtue of the concept of t-linked extensions, Dobbs *et al.* proved that R is a PVMD if and only if every t-linked overring of R is integrally closed. More generally, by the concept of t-linked extensions in [2], the authors tried to learn the relationships between the t-operation of R and t-operation of T in an extension $R \subseteq T$ of rings. In [2], the concept of t-linkative domains is introduced. An integral domain R is said to be t-linkative if it satisfies that every extension ring of R is a t-linked extension. In [12], a f.g. nonzero ideal J such that $J^{-1} = R$ is called a GV-ideal (Glaz-Vasconcelos ideal) by Wang *et al.*, denoted by $J \in \mathrm{GV}(R)$, where $\mathrm{GV}(R)$ is the set of all GV-ideals of R. Clearly, $\mathrm{GV}(R)$ is a multiplicative set of ideals

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of R. Let M be an R-module. Define

$$\mathrm{tor}_{\mathrm{GV}}(M) = \{x \in M \mid Jx = 0 \text{ for some } J \in \mathrm{GV}(R)\}.$$

Therefore, $\mathrm{tor}_{\mathrm{GV}}(M)$ is a submodule of M. And an R-module M is called a GV-torsion-free module if whenever $Jx = 0$ for some $J \in \mathrm{GV}(R)$ and $x \in M$, one has $x = 0$. A GV-torsion-free module M is called a w-module if $\mathrm{Ext}^1_R(R/J, M) = 0$ for any $J \in \mathrm{GV}(R)$, and the w-envelope of M is the set given by

$$M_w = \{x \in E(M) \mid Jx \in M \text{ for some } J \in \mathrm{GV}(R)\},$$

where $E(M)$ is the injective hull of M. Therefore, M is a w-module if and only if $M_w = M$. For w-modules, readers are referred to [11]. Besides, in an extension $R \subseteq T$ (T not necessary in the quotient field K) of domains, if T as an R-module is a w-module, then T is called a w-domain over R in [4]. In [10], it is shown that T is a t-linked extension of R if and only if T is a w-domain over R for any extension $R \subseteq T$ (T not necessary in the quotient field K) of domains. In [10], it is pointed out that R is a t-linkative domain if and only if every ideal is a w-ideal, subsequently, Mimouni called it a DW domain in [9]. Also in [7], Kim studied it module-theoretically.

The Krull-Akizuki Theorem states that if R is a Noetherian domain with $\dim(R) = 1$, then every overring T of R is a Noetherian domain with $\dim(R) \leq 1$. In 1976, this theorem was generalized to reduced Noetherian rings by Matijevic. Namely, let R be a reduced Noetherian ring. Then every extension ring T of R contained in the global transform is a Noetherian ring. In 1999, Wang and McCsland in [4] generalized Krull-Akizuki Theorem to strong Mori domains. That is, let R be an SM domain with w-$\dim(R) \leq 1$. Then they showed that every t-linked overring T of R is an SM domain with w-$\dim(T) \leq 1$. Park proved a w-version of Krull-Akizuki Theorem over domain in 2002, that is, if R is an SM domain, then the w-global transform of R is a w-overring, and every w-overring of R contained in the w-global transform is also an SM domain. As a corollary, she obtained the result of Wang and McCsland again. Yin *et al.* observed that the w-operation has good torsion-theoretic properties. They in [15] generalized the w-operation to commutative rings and introduced the concept of w-Noetherian rings. In 2011, in order to gave a w-version of Krull-Akizuki Theorem over commutative rings, Xie *et al.* in [14] unified t-linked extensions over integral domains and w-domains into w-linked extensions. Let $R \subseteq T$ be an extension of rings. If T as an R-module is a w-module, then the ring extension is called a w-linked extension. In [14], it is proved that: If R is a reduced w-Noetherian ring, then every w-linked extension ring of R contained in the w-global transform is a w-Noetherian ring. More properties of w-linked extension, we can refer to [14].

Let R be a commutative ring and I be a w-ideal of R. Although the use of "w-linked" can learn many properties of ring extensions, the experience of this approach is rarely used for the natural ring homomorphism $R \to R/I$. Besides, the discussion of factor rings in the star-operation theory is mostly avoided by researchers. The main reason is that there is not enough connection between a star operation on R and the same star operation on the factor ring R/I. Let R be an integral domain and let $u \in R$ be a nonzero element. The a-operation and the b-operation over a factor ring $R/(u)$ are introduced by Costa *et al.* in [1]. Let $I = A/(u)$ be an ideal of $R/(u)$. Define $I_a := \mathrm{Ann}(\mathrm{Ann}(I))$ and $I_b := \bigcup\{J_a \mid \text{where } J \text{ runs over all the f.g. ideals of } I\}$. So $I_a = A_v/(u)$ and $I_b = A_t/(u)$. Although the a-operation and the b-operation over $R/(u)$ correlate well with the v-operation and the t-operation over R respectively, they are different from the v-operation and the t-operation of commutative rings with zero divisor, which Kang *et al.* discussed in [5, 6].

As is well known, the w-linked extension can well describe the relationship between the w-operators on R and T. In order for the "w-linked" idea to play a role in the discussion of the factor ring R/I, we introduce the concept of the w-linked homomorphism. Let $\phi : R \to T$ be a ring homomorphism. If T as an R-module is a w-module, then ϕ is called a w-linked homomorphism. Many classical theorems can have natural w-version representations with the help of the w-linked homomorphism. For example, let R be an integral domain, in 1950, Cohen proved that R is a Noetherian ring with $\dim(R) \leq 1$, if and only if R/I is an Artinian ring for every nonzero proper ideal I of R, if and only if $R/(a)$ is an Artinian ring for every nonzero and non-unit element a of R. In 1999, Wang *et al.* in [4] gave a w-version of Cohen's Theorem: An integral domain R is an SM domain with w-$\dim(R) \leq 1$, if and only if for any nonzero w-ideal I of R, every descending chain on w-ideals of R containing I stabilizes. In this paper, by virtue of the concept of w-linked homomorphisms, the "w-linked" idea plays an important role in the discussion of the factor ring R/I. As is well known, a ring R is said to be local if R has only one maximal ideal. If every ideal of R is a w-ideal, then R is said to be a DW ring. However, the w-operation does not play a role over DW rings, so the naturally arising question if R isn't a DW ring, whether we can introduce a local w-ring, which has only one maximal w-ideal, but in this paper, according to Theorem 3.11 and Corollary 3.12, we get that it can't come true. Namely, let $\phi : R \to T$ be a w-linked homomorphism. Let T be not a DW_ϕ ring. Then T must have an infinite number of maximal w_ϕ-ideals. And let R be not a DW ring. Then R must have an infinite number of maximal w-ideals. Moreover, let R be a ring, let I be a proper w-ideal of R, and let $\overline{R} = R/I$, $\phi : R \to \overline{R}_w$ is a natural w-linked homomorphism, where $\overline{R}_w$ is a w-factor ring of R. By virtue of the concept of w-factor rings, we give an application of Cohen's Theorem over w-factor rings, namely, we give a new characterization of an SM domain with w-$\dim(R) \leq 1$: Let R be an integral domain. Then R is an SM-domain with w-$\dim(R) \leq 1$, if and only if for any nonzero w-ideal I of R, $\overline{R}_w$ is an Artinian ring, if and only if for any nonzero element $a \in R$, $(R/(a))_w$ is an Artinian ring, if and only if for any nonzero element $a \in R$, R has the descending chain condition on w-ideals of R containing a.

Undefined terms and terminology are standard as in [11, 14].

2. The ring of finite fractions

Let R be a ring and let x be an indeterminate. For $f(x)=\sum_{i=0}^{n} d_i x^i \in R[x]$, we denote $c(f) := (d_0, d_1, \ldots, d_n)$. Recall that an ideal A of R is called a semiregular ideal if $\mathrm{Ann}(I)=0$ for some f.g. subideal I of A. It is easy to see that the set of f.g. semiregular ideals of R is a multiplicative system of ideals of R.

Lemma 2.1. *Let $f(x)=\sum_{i=0}^{n} d_i x^i \in R[x]$.*

(1) *Let M be an R-module. If $f(x)$ is a zero-divisor of $M[x]$, then $f(x)u=0$ for some $u \in M$ with $u \neq 0$.*
(2) *$f(x)$ is a non-zero-divisor of $R[x]$ if and only if $c(f)$ is a semiregular ideal.*

Proof. (1) Suppose $f(x)$ is a zero-divisor of $M[x]$. Then we may choose $g(x) \in M[x]$ with $g(x) \neq 0$ such that $f(x)g(x)=0$ and the degree of $g(x)$ is minimal. Write $g(x)=\sum_{j=0}^{m} b_j x^j \in M[x]$, where $b_j \in M$, $b_m \neq 0$. Then

$$f(x)g(x)=b_m d_n x^{m+n}+(b_m d_{n-1}+b_{m-1}d_n)x^{m+n-1}+\cdots=0,$$

and thus $b_m d_n=0$. Hence $b_m f(x)=0$. If not and let d_k be the first coefficient of $f(x)$ such that $b_m d_k \neq 0$, then $b_m d_n=0, b_m d_{n-1}=0, \ldots, b_m d_{k+1}=0$. Since $(d_i g(x))f(x)=0$, $\deg(d_i g(x)) < \deg(g(x))$, and the degree of $g(x)$ is minimal, we have $d_i g(x)=0$, $i=n, n-1, \ldots, k+1$. Write

$$f(x)=(d_n x^n+\cdots+d_{k+1}x^{k+1})+(d_k x^k+\cdots+d_0)=f_1(x)+f_2(x).$$

Since $g(x)f(x)=g(x)f_1(x)+g(x)f_2(x)=0$ and $g(x)f_1(x)=0$, we have $b_m d_k=0$, which is a contradiction. Therefore $b_m f(x)=0$. So let $u := b_m \in M$. Then $f(x)u=0$ with $u \neq 0$.

(2) Suppose $f(x)$ is a zero-divisor of $R[x]$. If $a \in R$ with $ac(f)=0$, then $af(x)=0$. Hence $a=0$. Therefore $c(f)$ is a semiregular ideal of R.

Conversely, suppose $c(f)$ is a semiregular ideal of R and $g(x) \in R[x]$ such that $g(x)f(x)=0$. If $g(x) \neq 0$, then according to [11, Theorem 1.7.7], there exists $a \in R$ with $a \neq 0$ such that $af(x)=0$. Then $ac(f)=0$, and so $a=0$, a contradiction. Therefore $f(x)$ is a non-zero-divisor of $R[x]$. $\square$

Set

$$Q_0(R) := \{\alpha \in T(R[x]) \mid I\alpha \subseteq R \text{ for some f.g. semiregular ideal } I \text{ of } R\}.$$

Then $Q_0(R)$ is an extension ring of R contained in $T(R[x])$. Hence $Q_0(R)$ is called a ring of finite fractions of R. By [8], the element α of $Q_0(R)$ can be written as $\alpha=\frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{n} b_i x^i}$, where $a_i, b_i \in R$, $(b_0, b_1, \ldots, b_n)$ is an semiregular ideal, and $a_i b_j = a_j b_i$ for any i, j. Clearly $T(R) \subseteq Q_0(R)$ and $Q_0(R)$ is the quotient field of R when R is an integral domain.

Let $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$. If $c(f) \in \mathrm{GV}(R)$, then $f(x)$ is called a GV-polynomial. When R is a GCD domain, a GV-polynomial is a primitive polynomial. Now let

$$S_w = \{f \in R[x] \mid f \text{ is a GV-polynomial}\}.$$

According to [11], S_w is a multiplicative closed set, that is, the product of two GV-polynomials is a GV-polynomial. Write $R\{x\} = R[x]_{S_w}$.

Let B be a nonempty subset of $Q_0(R)$. We define

$$B^{-1} = \{y \in Q_0(R) \mid yB \subseteq R\}.$$

Hence B^{-1} is an R-submodule of $Q_0(R)$. If (B) represents the submodule generated by B, then clearly $B^{-1} = (B)^{-1}$.

Lemma 2.2. (1) *Let $\alpha = \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{n} b_i x^i} \in Q_0(R)$. If some $b_k = 0$, then we can get $a_k = 0$.*

(2) *Let T' be an extension ring of R contained in $Q_0(R)$. Then $Q_0(T') = Q_0(R)$. Specially, $Q_0(Q_0(R)) = Q_0(R)$.*

(3) *Let J be a f.g. semiregular ideal of R. Then $J \in \mathrm{GV}(R)$ if and only if $J^{-1} = R$.*

(4) $Q_0(R) \cap R\{x\} = R$.

Proof. (1) If $b_k = 0$, then $b_i a_k = b_k a_i = 0$ for any $i = 0, 1, \ldots, n$. Since $J := (b_0, b_1, \ldots, b_n)$ is a semiregular ideal, we have $a_k = 0$.

(2) Let A be a subring of $T(R[x])$ generated by T' and x. If $\sum_{i=0}^{n} \alpha_i x^i = 0$ in $T(R[x])$, where $\alpha_i \in T'$, by [11, Theorem 6.6.7], we have $\alpha_i = 0$ for any $i = 0, 1, \ldots, n$. Therefore x is an indeterminate over T' and $A \cong T'[x]$. Thus we can suppose $T(T'[x]) = T(R[x])$.

Let I be a f.g. semiregular ideal of R. By Lemma 2.1(2), IT' is also a f.g. semiregular ideal of T', and thus $Q_0(R) \subseteq Q_0(T')$.

Let $\alpha \in Q_0(T')$. Then there exists a f.g. semiregular ideal A of T' such that $A\alpha \subseteq T'$. Denoted by $\{\beta_1, \ldots, \beta_n\}$ a generating set of A. Thus $\beta_i \alpha \in T'$. Hence there exists a f.g. semiregular ideal I of R such that $I\beta_i \subseteq R$ and $I\beta_i\alpha \subseteq R$. Set $B = R\beta_1 + \cdots + R\beta_n$. Then IB is a semiregular ideal of R and $IB\alpha \subseteq R$. Hence $\alpha \in Q_0(R)$. Therefore $Q_0(T') \subseteq Q_0(R)$.

(3) This follows from [11, Proposition 6.6.8].

(4) Clearly $R \subseteq Q_0(R) \cap R\{x\}$. Let $\alpha = \frac{a(x)}{b(x)} = \frac{c(x)}{d(x)}$, where $a(x) = \sum_{i=0}^{n} a_i x^i$, $b(x) = \sum_{i=0}^{n} b_i x^i$, $c(x) = \sum_{k=0}^{m} c_k x^k$, $d(x) = \sum_{l=0}^{s} d_l x^l$ are polynomials over R, and for any i, i', we have $a_i b_{i'} = a_{i'} b_i$, $(b_0, b_1, \ldots, b_n)$ is a semiregular ideal, and $(d_0, d_1, \ldots, d_s) \in \mathrm{GV}(R)$. For $i = 0, 1, \ldots, n$, we have $b_i \alpha = a_i = b_i \frac{c(x)}{d(x)}$, and hence $d(x) a_i = b_i c(x)$. So we can suppose $s = m$ and $b_i c_j = d_j a_i$ for any j. Therefore we also have $d_k b_i c_j = d_k d_j a_i = d_j b_i c_k$ for any k. Hence we have $b_i(d_k c_j - d_j c_k) = 0$ for any $i = 0, 1, \ldots, n$. So $d_k c_j = d_j c_k$ for any j, k, and thus $d_k \alpha = c_k \in R$ for any $k = 0, 1, \ldots, m$. Therefore $\alpha \in R$. $\square$

Lemma 2.3. *Let $\phi : R \to T$ be a ring homomorphism, $f(x) = \sum_{i=0}^{n} d_i x^i \in R[x]$ be a GV-polynomial and M be a T-module such that M as an R-module is a GV-torsion-free module. Then $\phi(f)$ is not a zero-divisor of $M[x]$, $\phi(f)$ is satisfied that $\phi(d_i) = d_i, i = 1, \dots, n$.*

Proof. If there exists $\alpha \in M[x]$ with $\alpha \neq 0$ such that $\phi(f)\alpha = 0$, then by Lemma 2.1, we can assume that $\alpha \in M$. Thus $d_i\alpha = \phi(d_i)\alpha = 0$ for any $i = 0, 1, \dots, n$. Since M is a GV-torsion-free R-module, we have $\alpha = 0$, which is a contradiction. Therefore $\phi(f)$ is not a zero-divisor of $M[x]$. □

Let $\phi : R \to T$ be a ring homomorphism. Let

$$S_\phi = \{\phi(f) \in T[x] \mid f \in R[x] \text{ is a GV-polynomial}\}.$$

Obviously the induced map $S_w \to S_\phi$ by ϕ is a surjection.

Lemma 2.4. *Let $\phi : R \to T$ be a ring homomorphism. Then S_ϕ is a multiplicatively closed set of $T[x]$.*

Proof. This follows from the facts that S_w is a multiplicatively closed set of $R[x]$ and $\phi : S_w \to S_\phi$ is a surjection. □

In [16], Zhou, Kim and Hu provided an element-wise characterization of w-modules [16, Lemma 3.1 and Theorem 3.3] and proved that $(R/I)_w$ as the natural w-version of the factor ring R/I is also a ring, where I is a w-ideal of R [16, Remark 3.4]. Next we will obtain more general results and properties than theirs by considering ring homomorphisms. Although the proof is essentially the same as in [16], we give a proof for completeness.

Proposition 2.5. *Let $\phi : R \to T$ be a ring homomorphism, where T as an R-module is a GV-torsion-free module. Let M be a T-module and let M as an R-module be a GV-torsion-free module. Then the following statements hold.*

(1) $M_w = \left\{ \frac{\sum_{i=0}^{n} u_i x^i}{\sum_{i=0}^{n} \phi(d_i) x^i} \in M[x]_{S_\phi} \mid \sum_{i=0}^{n} d_i x^i \text{ is a GV-polynomial and } \phi(d_i)u_j = \phi(d_j)u_i \text{ for any } i, j \right\}.$

(2) *$T[x]_{S_\phi} \subseteq Q_0(T)$ and T_w is a subring of $T[x]_{S_\phi}$ containing T.*

(3) *M_w is a T_w-module. Therefore M is a T_w-module when M is a w-module.*

(4) *Let A be a T-submodule of M. Then A_w is a T_w-submodule of M_w. Especially, if A is an ideal of T, then A_w is an ideal of T_w.*

(5) *$Q_0(T_w) = Q_0(T)$ and $Q_0(T)$ as an R-module is a w-module.*

(6) *Let T be an integral domain. Then $T_w \subseteq qf(T)$, and*

$$\begin{aligned} T_w &= \{z \in qf(T) \mid Jz \subseteq T \text{ for some } J \in \mathrm{GV}(R)\} \\ &= \bigcap \{T_{\mathfrak{m}} \mid \mathfrak{m} \in w\text{-}\mathrm{Max}(R) \text{ and } \ker(\phi) \subseteq \mathfrak{m}\}. \end{aligned}$$

Proof. (1) Let H be the righthand side of (1). Let $y \in H$ with $y \neq 0$. Write $y = \frac{\sum_{i=0}^{n} u_i x^i}{\sum_{i=0}^{n} \phi(d_i) x^i}$, where $u_i \in M$, $d_i \in R$, $i = 0, 1, \dots, n$, $f(x) = \sum_{i=0}^{n} d_i x^i$

is a GV-polynomial. Then $d_k y = \frac{\sum_{i=0}^n \phi(d_k)u_i x^i}{\sum_{i=0}^n \phi(d_i)x^i} = u_k \in M$. Since M is a GV-torsion-free R-module, we have $(d_0, d_1, \dots, d_n)y \neq 0$. Therefore H is an essential extension of M, and so $H \subseteq E(M)$. By the same process as above, $H \subseteq M_w$ is also obtained.

On the other hand, when $y \in M_w$, there exists $J = (d_0, d_1, \dots, d_n) \in \mathrm{GV}(R)$ such that $Jy \subseteq M$. Write $d_k y = u_k$ and let $f(x) = \sum_{i=0}^n d_i x^i$. Then $f(x)y = \sum_{i=0}^n u_i x^i \in M[X]$. Therefore $y = \frac{\sum_{i=0}^n u_i x^i}{\sum_{i=0}^n \phi(d_i)x^i} \in M[X]_{S_\phi}$. Since $d_k y = u_k$, we have $d_k u_i = d_k d_i y = d_i d_k y = d_i u_k$ for any i and k, that is $\phi(d_k)u_i = \phi(d_i)u_k$. Therefore $M_w \subseteq H$. So we get $H = M_w$.

(2) Let $y, z \in T_w$. Then there exist $J_1, J_2 \in \mathrm{GV}(R)$ such that $J_1 y$, $J_2 z \subseteq T$. Thus $J_1 J_2 yz \in T$, and so $yz \in T_w$. Thus T_w is a multiplicatively closed set of $T[x]_{S_\phi}$. Therefore T_w is a subring of $T[x]_{S_\phi}$.

(3) Let $h = \frac{\sum_{i=0}^n b_i x^i}{\sum_{i=0}^n \phi(d_i)x^i} \in T_w$, $y = \frac{\sum_{j=0}^m u_j x^j}{\sum_{j=0}^m \phi(c_j)x^j} \in M_w$, where $b_i \in T$, $u_j \in M$, $J_1 := (d_0, d_1, \dots, d_n)$ and $J_2 := (c_0, c_1, \dots, c_m)$ are GV-ideals of R. Since $d_i b_j = d_j b_i$ for any i, j, and $c_s u_t = c_t u_s$ for any s, t, it is easy to see that

$$hy = \frac{\sum\limits_{k=0}^{n+m} (\sum\limits_{i+j=k} b_i u_j) x^k}{\sum\limits_{k=0}^{n+m} (\sum\limits_{i+j=k} \phi(d_i c_j)) x^k} \in M_w.$$

Therefore M_w is a T_w-module.

(4) This is obtained directly from (3).

(5) By Lemma 2.2(2), we can get $Q_0(T_w) = Q_0(T)$. To prove that $Q_0(T)$ is a w-R-module, let T as an R-module be a w-module. According to [11, Theorem 6.6.6](3), $Q_0(T)$ is a w-T-module. By Theorem 3.3, $Q_0(T)$ is a w-R-module.

(6) Let $y = \frac{\sum_{i=0}^n a_i x^i}{\sum_{i=0}^n \phi(d_i)x^i} \in T_w$, where $a_i \in T$. Then $\phi(d_k) \neq 0$ for some k, and so $\lambda_k := \frac{a_k}{\phi(d_k)} \in qf(T)$. Since $a_i = d_i \frac{a_k}{\phi(d_k)}$ for $i = 0, 1, \dots, n$, it follows that $y = \lambda_k \in qf(T)$.

Write $H_1 = \{z \in qf(T) \mid Jz \subseteq T$ for some $J \in \mathrm{GV}(R)\}$. Since T is an integral domain, we have $Q_0(T) = qf(T)$. By Proposition 2.5(5), $qf(T)$ is a w-R-module. Thus $T_w = H_1$.

Write $H = \bigcap\{T_\mathfrak{m} \mid \mathfrak{m} \in w\text{-Max}(R)$ and $\ker(\phi) \subseteq \mathfrak{m}\}$. Since T is an integral domain, it follows that $T \subseteq T_\mathfrak{m} \subseteq qf(T)$ for a maximal ideal $\mathfrak{m}$ of R. Therefore $T \subseteq H$. Since every $T_\mathfrak{m}$ is a w-R-module, H is a w-R-module. Therefore $T_w \subseteq H$.

Conversely, suppose $z \in H$. Let $I = \{r \in R \mid rz \in T_w\}$. Then I is a w-ideal of R containing $\ker(\phi)$. Since $z \in T_\mathfrak{m}$ for any maximal w-ideal $\mathfrak{m}$ of R with $\ker(\phi) \subseteq \mathfrak{m}$, there exists $s \in R \setminus \mathfrak{m}$ such that $sz \in T$. Thus $s \in I$. Hence $I \not\subseteq \mathfrak{m}$. Thus $I = R$. So we get $z \in T_w$. □

Proposition 2.6. *Let $\phi : R \to T$ be a ring homomorphism, where T is a GV-torsion-free R-module. Let P be a prime ideal of T. Then the following statements hold.*

(1) *If $\phi^{-1}(P)$ is a w-ideal of T, then $P_w \neq T_w$.*
(2) *If $P_w \neq T_w$, then P_w is a prime ideal of T_w and $P_w \cap T = P$.*
(3) *If $P_w \neq T_w$ and P_1 is a prime ideal of T_w such that $P_1 \subseteq P_w$ and $P_1 \cap T = P$, then $P_1 = P_w$.*

Proof. (1) If $P_w \neq T_w$, then $J \subseteq P$ for some $J \in \mathrm{GV}(R)$. Thus $J \subseteq P \cap R$, a contradiction.

(2) Suppose $x \in P_w \cap T$. Then $Jx \subseteq P$ for some $J \in \mathrm{GV}(R)$. Since $J \nsubseteq P$, we have that $P_w \cap T = P$.

(3) Suppose $x, y \in T_w$, $xy \in P_w$. Then $J_1 x \subseteq T$, $J_2 y \subseteq T$ for $J_1, J_2 \in \mathrm{GV}(R)$. Hence $Jxy \subseteq P$ for some $J = J_1 J_2 \in \mathrm{GV}(R)$, and $Jx \subseteq P$ or $Jy \subseteq P$, therefore $x \in P_w$ or $y \in P_w$. □

3. w-linked homomorphisms and the w_ϕ-operation

We begin this section by introducing the concept of w-linked homomorphisms.

Definition 3.1. Let $\phi : R \to T$ be a ring homomorphism. If T as an R-module is a w-module, then ϕ is called a *w-linked homomorphism*.

Clearly the identity homomorphism $\mathbf{1} : R \to R$ is a w-linked homomorphism. Recall that a ring extension $R \subseteq T$ is said to be w-linked if T as an R-module is a w-module. In this case, the inclusion map $\lambda : R \to T$ is a w-linked homomorphism.

For a ring homomorphism $\phi : R \to T$, there are w-operations on R and T, respectively. For a T-module N, we denote by N_w the w-envelope of N as an R-module and by N_W the w-envelope of N as a T-module.

Lemma 3.2. *Let $\phi : R \to T$ be a ring homomorphism, $J \in \mathrm{GV}(R)$, L be a T-module, and let L as an R-module be a GV-torsion-free module. Then the following statements hold.*

(1) $\mathrm{Hom}_T(JT, L) \cong \mathrm{Hom}_T(T \otimes_R J, L)$.
(2) $\mathrm{Ext}^1_T(T/JT, L) \cong \mathrm{Ext}^1_R(R/J, L)$.

Proof. (1) Let $0 \to A \to J \otimes_R T \xrightarrow{f} JT$ be an exact sequence of R-modules, where $A = \ker(f)$. Then we have the following exact sequence:

$$0 \to A_{\mathfrak{m}} \to (J \otimes_R T)_{\mathfrak{m}} \xrightarrow{f_{\mathfrak{m}}} (JT)_{\mathfrak{m}},$$

where $\mathfrak{m}$ is a maximal w-ideal of R. Since $(J \otimes_R T)_{\mathfrak{m}} = J_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} T_{\mathfrak{m}} = R_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} T_{\mathfrak{m}} = T_{\mathfrak{m}}$, we have $(JT)_{\mathfrak{m}} = J_{\mathfrak{m}} T_{\mathfrak{m}} = T_{\mathfrak{m}}$. Then $f_{\mathfrak{m}}$ is an isomorphism, and thus $A_{\mathfrak{m}} = 0$. Therefore A is a GV-torsion module. Since L is a GV-torsion-free R-module and $\mathrm{Hom}_T(A, L) = 0$, we have the following exact sequence:

$$0 \to \mathrm{Hom}_T(JT, L) \to \mathrm{Hom}_T(J \otimes_R T, L) \to \mathrm{Hom}_T(A, L) = 0.$$

Therefore $\mathrm{Hom}_T(JT, L) \cong \mathrm{Hom}_T(J \otimes_R T, L)$.

(2) Let $0 \to J \to R \to R/J \to 0$ and $0 \to JT \to T \to T/JT \to 0$ be short exact sequences. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} \mathrm{Hom}_R(T, L) & \longrightarrow & \mathrm{Hom}_R(JT, L) & \longrightarrow & \mathrm{Ext}_R^1(T/JT, L) & \longrightarrow & 0 \\ \downarrow g & & \downarrow h & & \downarrow & & \\ \mathrm{Hom}_T(R, L) & \longrightarrow & \mathrm{Hom}_T(J, L) & \longrightarrow & \mathrm{Ext}_T^1(R/J, L) & \longrightarrow & 0 \end{array}$$

By Lemma 3.2(1), we can get:

$$\mathrm{Hom}_T(JT, L) \cong \mathrm{Hom}_T(J \otimes_R T, L) \cong \mathrm{Hom}_R(J, \mathrm{Hom}_T(T, L)) = \mathrm{Hom}_R(J, L),$$

i.e., h is an isomorphism. It is easy to see that g is also an isomorphism. So in the above commutative diagram, by Five Lemma we can get $\mathrm{Ext}_T^1(T/JT, L) \cong \mathrm{Ext}_R^1(R/J, L)$. □

Theorem 3.3. *Let $\phi : R \to T$ be a ring homomorphism, where T as an R-module is a GV-torsion-free module. Then the following statements are equivalent.*

(1) *$\phi(I)_w \subseteq (IT)_W$ for any ideal I of R.*
(2) *$(I_wT)_W = (IT)_W$ for any ideal I of R.*
(3) *$\phi^{-1}((IT)_W)$ is a w-ideal of R for any ideal I of R.*
(4) *$\phi^{-1}(A)$ is a w-ideal of R for any w-ideal A of T.*
(5) *$\phi^{-1}(P)$ is a w-ideal of R for any prime w-ideal P of T.*
(6) *If $J \in \mathrm{GV}(R)$, then $JT = \phi(J)T \in \mathrm{GV}(T)$.*
(7) *ϕ is a w-linked homomorphism.*
(8) *Let L be a T-module. If L as a T-module is a w-module, then L as an R-module is a w-module.*
(9) *Let L be a T-module. If L as a T-module is a GV-torsion-free module, then L as an R-module is a GV-torsion-free module.*
(10) *Let L be a T-module. If L as an R-module is a GV-torsion-free module, then L is a GV-torsion T-module.*

Proof. (1)⇒(2) Since $\phi(I)_w \subseteq (IT)_W$, it follows that

$$(I_wT)_W \subseteq (\phi(I)_wT)_W \subseteq ((IT)_W T_W)_W = (IT)_W.$$

(2)⇒(6) Let $J \in \mathrm{GV}(R)$. Then $J_w = R$, and so $T = (J_wT)_W = (JT)_W$. Therefore $JT \in \mathrm{GV}(T)$.

(6)⇒(1) Let $z \in T$ and $z \in \phi(I)_w$. Then there exists $J \in \mathrm{GV}(R)$ such that $Jz \subseteq \phi(I)$. Since $JTz \subseteq IT$, by the hypothesis $JT \in \mathrm{GV}(T)$, and so $z \in (IT)_W$. Hence $\phi(I)_w \subseteq (IT)_W$.

(6)⇒(8) By the hypothesis, L is a GV-torsion-free R-module. Let $J \in \mathrm{GV}(R)$. Then by Lemma 3.2, we can get $\mathrm{Ext}_R^1(R/J, L) \cong \mathrm{Ext}_T^1(T/JT, L) = 0$. Therefore L as an R-module is a w-module.

(8)⇒(7) Take $L := T$. Then T as an R-module is a w-module, i.e., ϕ is a w-linked homomorphism.

(7)⇒(6) Let $J \in \mathrm{GV}(R)$. By Lemma 3.2, there exists an isomorphism

$$\mathrm{Ext}_T^1(T/JT, T) \cong \mathrm{Ext}_R^1(R/J, T) = 0,$$

and hence $JT \in \mathrm{GV}(T)$.

(8)⇒(4) Write $I := \phi^{-1}(A)$. Since $\phi(I_w) \subseteq \phi(I)_w \subseteq A_w = A$, we have $I_w \subseteq \phi^{-1}(A) = I$, and hence I is a w-ideal of R.

(4)⇒(3) By letting $A := (IT)_W$, we can get the conclusion.

(3)⇒(6) Let $J \in \mathrm{GV}(R)$. Then $J_w = R$. Hence $R \subseteq \phi^{-1}((JT)_W)$ by assumption. Since $1 = \phi(1) \in (JT)_W$, we have $(JT)_W = T$, namely $JT \in \mathrm{GV}(T)$.

(4)⇒(5) This is clear.

(5)⇒(6) Let $J \in \mathrm{GV}(R)$ and suppose that $JT \notin \mathrm{GV}(T)$. Then $(JT)_W \neq T$, and so there exists a w-prime ideal P of T such that $(JT)_W \subseteq P$. Hence $J \subseteq \phi^{-1}(P)$, since $\phi^{-1}(P)$ is a w-prime ideal of R, a contradiction.

(8)⇒(9) By the hypothesis, L_W is a w-module over R. Therefore L is a GV-torsion-free R-module.

(9)⇒(10) Set $A = \{z \in L \mid Jz = 0 \text{ for some } J \in \mathrm{GV}(T)\}$. Then L/A is a GV-torsion-free T-module. By the hypothesis, L/A is a GV-torsion R-module. Then $L/A = 0$, namely, $L = A$. Therefore L is a GV-torsion T-module.

(10)⇒(6) Let $J \in \mathrm{GV}(R)$. Then R/J is a GV-torsion R-module. From the natural isomorphism $T \otimes_R (R/J) \cong T/JT$, it follows that T/JT is a GV-torsion R-module. By the hypothesis, T/JT is a GV-torsion T-module. Therefore $JT \in \mathrm{GV}(T)$. □

Let $\phi : R \to T$ be a w-linked homomorphism. Let A be a T-module. It is easy to see that $\mathrm{tor}_{\mathrm{GV}}(A)$ is a T-submodule of A. When A is an ideal of T, the mapping $w_\phi : A \mapsto A_w$ gives a w-liked operation over T, which is called the *w_ϕ-operation*. If an ideal A of T satisfies $A_w = A$, then we call A a *w_ϕ-ideal*. By Theorem 3.3, $\mathrm{GV}(\phi) := \{JT \mid J \in \mathrm{GV}(R)\} \subseteq \mathrm{GV}(T)$. Thus there exists the relationship of operations $w_\phi \leqslant w$ over T.

Accordingly let N be a T-module and let N as an R-module be a w-module. Then we also call N a *w_ϕ-T-module*.

Proposition 3.4. *Let $\phi : R \to T$ be a w-linked homomorphism. Then the following statements hold.*

1. *Let P be a prime ideal of T. Then P is a w_ϕ-ideal of T if and only if $P_w \neq T$.*
2. *Let A be a w_ϕ-ideal of T. Then $A = \cup B_w$, where B runs over all the f.g. subideals of A.*
3. *Let A be a w_ϕ-ideal of T. Then there exists a maximal w_ϕ-ideal M of T such that $A \subseteq M$.*
4. *Every maximal w_ϕ-ideal of T is prime.*
5. $\mathrm{Ann}_T(y)$ *is a w_ϕ-ideal of T for any $y \in T$.*
6. *Let M be a T-module. Then M as an R-module is a GV-torsion module if and only if $M_{\mathfrak{m}} = 0$ for any maximal w_ϕ-ideal $\mathfrak{m}$ of T.*

Proof. The proof is similar to the w-module case in [11]. □

Theorem 3.5. *Let $\phi : R \to T$ be a w-linked homomorphism. Then the following statements are equivalent.*

(1) *Every w_ϕ-ideal of T is also a w-ideal of T, in other words, $w_\phi = w$ over T.*
(2) *Every maximal w_ϕ-ideal of T is also a maximal w-ideal of T.*
(3) *Let $J \in \mathrm{GV}(T)$. Then there exists $I \in \mathrm{GV}(R)$ such that $\phi(I) \subseteq J$.*
(4) *Let M be a T-module. If M as an R-module is a GV-torsion-free module, then M is a GV-torsion-free T-module.*
(5) *Let M be a T-module. If M as an R-module is a w-module, then M is a w-T-module.*
(6) *Let M be a T-module. If M as a T-module is a GV-torsion module, then M is also a GV-torsion R-module.*
(7) *Let N be a T-module that is a GV-torsion-free R-module. Then $\mathrm{Hom}_R(T, N)$ is a GV-torsion-free T-module.*
(8) *Let N be a T-module that is a w-module over R. Then $\mathrm{Hom}_R(T, N)$ is a w-module over T.*

Proof. (1)⇒(2) This is clear.

(2)⇒(3) Suppose that $J_w \neq T$. Then there exists a maximal w_ϕ-ideal P of T such that $J \subseteq P$. By the hypothesis, P is also a maximal w-ideal of T. Thus $J_W \neq T$, a contradiction to the fact that $J \in \mathrm{GV}(T)$.

Now since $J_w = T$, we have $1 \in J_w$. Hence there is $I \in \mathrm{GV}(R)$ such that $\phi(I) = I1 \subseteq J$.

(3)⇒(4) Let $J \in \mathrm{GV}(T)$, $z \in M$, $Jz = 0$. Let $I \in \mathrm{GV}(R)$ such that $\phi(I) \subseteq J$. Then $Iz = 0$. Since M is a GV-torsion-free R-module, we have that $z = 0$. Therefore M is a GV-torsion-free T-module.

(4)⇒(5) Let E be the injective hull of M, where M is a T-module. Since M is a w-R-module, by the hypothesis, M is a GV-torsion-free T-module. Hence E is a w-T-module. By Theorem 3.3, we can get E is also a w-R-module.

Consider the exact sequence $0 \to M \to E \to E/M \to 0$. Since M is a w-R-module, according to [11, Theorem 6.1.17], E/M is a GV-torsion-free R-module. By the hypothesis, E/M is a GV-torsion-free T-module. By [11, Theorem 6.1.17], M is a w-T-module.

(5)⇒(1) This is easy.

(3)⇒(6) Let $z \in M$. Since M is a GV-torsion T-module, there exits $J \in \mathrm{GV}(T)$ such that $Jz = 0$. By the hypothesis, there exists $I \in \mathrm{GV}(R)$ such that $I \subseteq J$. Thus we can get $Iz = 0$. Therefore M is also a GV-torsion R-module.

(6)⇒(2) Let P be a maximal w_ϕ-ideal of T. Then T/P as an R-module is a GV-torsion-free module. If P is not a maximal w-ideal of T, then T/P is a GV-torsion T-module, which is a contradiction.

(4)⇒(7) By [11, Proposition 6.1.10], $\mathrm{Hom}_R(T, N)$ is a GV-torsion-free R-module. By the hypothesis, $\mathrm{Hom}_R(T, N)$ is a GV-torsion-free T-module.

(7)⇒(2) Let P be a maximal w_ϕ-ideal of T. Then T/P is a GV-torsion-free R-module. By the hypothesis, $\mathrm{Hom}_R(T, T/P)$ is a GV-torsion-free T-module. Consider the exact sequence $0 \to \mathrm{Hom}_R(T/P, T/P) \to \mathrm{Hom}_R(T, T/P)$. Then $\mathrm{Hom}_R(T/P, T/P)$ is also a GV-torsion-free T-module. If P is not a maximal w-ideal of T, then there exists $J \in \mathrm{GV}(T)$ such that $J \subseteq P$. Use $\mathbf{1}$ to denote the identity mapping over T/P. Then in $\mathrm{Hom}_R(T/P, T/P)$, we have that $J\mathbf{1} = 0$. Hence T/P is not a GV-torsion-free T-module, which is a contradiction. Therefore P is a maximal w-ideal of T.

(7)⇒(8) Let E be the injective hull of N. Let $C = E/N$. Then E is a w-module. By [11, Theorem 6.1.17], C is a GV-torsion-free module. Consider the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(T, N) \longrightarrow \mathrm{Hom}_R(T, E) \longrightarrow \mathrm{Hom}_R(T, C).$$

By the hypothesis, $\mathrm{Hom}_R(T, E)$ and $\mathrm{Hom}_R(T, C)$ are GV-torsion-free T-modules. Notice that $\mathrm{Hom}_R(T, E)$ is also an injective T-module. Hence $\mathrm{Hom}_R(T, N)$ is a w-module over T.

(8)⇒(7) Let E be the injective hull of N. Then E is a w-module. By the hypothesis, $\mathrm{Hom}_R(T, E)$ is a w-module. Since $\mathrm{Hom}_R(T, N)$ is a submodule of $\mathrm{Hom}_R(T, E)$, it follows that $\mathrm{Hom}_R(T, N)$ is a GV-torsion-free T-module. □

Recall that a ring R is said to be a *DW ring* if every ideal of R is a w-ideal. Clearly if $\dim(R) = 0$, then R is a DW ring. Accordingly we can define DW_ϕ rings.

Definition 3.6. Let $\phi : R \to T$ be a w-linked homomorphism. If every ideal of T is a w_ϕ-ideal, namely it as an R-module is a w-module, then T is called a *DW_ϕ ring*.

Lemma 3.7. *Let M be a GV-torsion module. Then there exists a continuous ascending chain of submodules of M*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\alpha \subseteq M_{\alpha+1} \subseteq \cdots \subseteq M_\tau = M$$

such that $M_{\alpha+1}/M_\alpha$ is a cyclic GV-torsion module for each ordinal α.

Proof. Set $M_0 := 0$. Considering an element $x \in M$ with $x \neq 0$, $M_1 := Rx$ is a cyclic GV-torsion module. For a given ordinal α, by induction hypothesis, we may assume that M_β meets the conditions for all $\beta < \alpha$. If $M_\beta = M$, then the chain terminates. Otherwise, when α is not a limit ordinal number, consider an element $y \in M \setminus M_{\alpha-1}$ and set $M_\alpha := M_{\alpha-1} + Ry$. Then $M_\alpha/M_{\alpha-1}$ is a cyclic GV-torsion module. And when α is a limit ordinal number, set $M_\alpha := \bigcup_{\beta<\alpha} M_\beta$. By transfinite induction, the assertion follows. □

Recall that an R-module N is said to be a *strong w-module* if $\mathrm{Ext}_R^k(R/J, N) = 0$ for any $J \in \mathrm{GV}(R)$ and for any $k \geqslant 1$. For the discussion about strong w-modules, we can refer to [13].

Theorem 3.8. *Let $\phi : R \to T$ be a w-linked homomorphism. Then the following statements are equivalent.*

(1) *T is a DW_ϕ ring.*
(2) *Every prime ideal of T as an R-module is a w-module.*
(3) *Every maximal ideal of T as an R-module is a w-module.*
(4) $\mathrm{GV}(\phi) = \{T\}$*, in other words, if $J \in \mathrm{GV}(R)$, then $JT = T$.*
(5) *Every f.g. ideal of T as an R-module is a w-module.*
(6) *Every T-module as an R-module is a GV-torsion-free module.*
(7) *Every cyclic T-module as an R-module is a GV-torsion-free module.*
(8) *Every T-module as an R-module is a w-module.*
(9) *Every cyclic T-module as an R-module is a w-module.*
(10) *Every T-module as an R-module is a strong w-module.*
(11) *$T \otimes_R R_1 = 0$ for any cyclic GV-torsion R-module R_1.*
(12) *$T \otimes_R R_1 = 0$ for any GV-torsion R-module R_1.*
(13) *Let $\xi : 0 \to A \to B \to C \to 0$ be a sequence of T-modules. If ξ is a w-exact sequence of R-modules, then ξ is already an exact sequence.*

Proof. (1)⇒(2)⇒(3)⇒(4) Trivial.

(4)⇒(6) Let N be a T-module, $J \in \mathrm{GV}(R)$, $z \in M$, $Jz = 0$. Then $Tz = JTz = 0$. Thus $z = 0$. Therefore N is a GV-torsion-free R-module.

(6)⇒(10) Let $J \in \mathrm{GV}(R)$ and $k \geqslant 1$ an integer. Then $\mathrm{Ext}_R^k(R/J, N)$ is a GV-torsion R-module. By the condition that $\mathrm{Ext}_R^k(R/J, N)$ is also a GV-torsion-free R-module, we have $\mathrm{Ext}_R^k(R/J, N) = 0$. Therefore N is a strong w-R-module.

(10)⇒(9)⇒(8)⇒(7) Trivial.

(7)⇒(5) Let $I = (a_1, \ldots, a_n)$ be an ideal of T. Use the method of induction on n. When $n = 1$, this is the hypothesis. When $n > 1$, let $I_1 = (a_1, \ldots, a_{n-1})$. Then according to the exact sequence $0 \to I_1 \to I \to I/I_1 \to 0$ and the fact that I_1 and I/I_1 are w-modules, we can get I is a w-module.

(5)⇒(1) Let I be an ideal of T. Then $I = \bigcup I_0$, where I_0 runs over all f.g. ideals of R. By the hypothesis, we can get I is also a w-module.

(4)⇒(11) Let $R_1 = Rx$ be a cyclic GV-torsion module. Then $Rx \cong R/I$ for some ideal I of R. Since Rx is a GV-torsion module, we have $I_w = R$. So there exists $J \in \mathrm{GV}(R)$ such that $J \subseteq I$. Thus $R/J \to T \to 0$ is an exact sequence. Therefore we can get $T \otimes_R R_1 = 0$ by tensoring with T.

(11)⇒(12) According to transfinite induction and Lemma 3.7, we can finish the proof.

(12)⇒(4) Let $R_1 = R/J$ for any $J \in \mathrm{GV}(R)$. Then applying the known condition, we can get the conclusion.

(8)⇒(13) By [11, Theorem 6.3.5], $0 \to A \to B \to C$ is an exact sequence. Let $g : B \to C$ be a given homomorphism. Since g is a w-epimorphism, we have $\mathrm{Im}(g) = \mathrm{Im}(g)_w = C$. Hence g is also an epimorphism, and so ξ is also an exact sequence.

(13)⇒(6) Let N be a T-module and let $L = \mathrm{tor}_{\mathrm{GV}}(N)$. Then T as an R-module is a GV-torsion module. So $0 \to L \to 0$ is a w-exact sequence. It follows the assumption that $L = 0$. Hence N is a GV-torsion-free R-module. □

Corollary 3.9. *Let $\phi : R \to T$ be a w-linked homomorphism.*

(1) *Let T be a DW_ϕ ring. Then* $\dim(T) = w_\phi\text{-}\dim(T)$.
(2) *Let T be a DW ring. Then T is a DW_ϕ ring.*

Proof. (1) This follows from Theorem 3.8(2).

(2) This follows from the fact that $\mathrm{GV}(\phi) \subseteq \mathrm{GV}(T) = \{T\}$. □

Example 3.10. (1) Let R be a DW ring. Then any ring homomorphism $\phi : R \to T$ is a w-linked homomorphism and T is a DW_ϕ ring.

(2) Let $\mathfrak{m}$ be a maximal w-ideal of R and let $\phi : R \to R_\mathfrak{m}$ be a natural homomorphism. Then by [11, Proposition 6.2.18], $R_\mathfrak{m}$ is a DW_ϕ ring.

(3) Let $R\{x\}$ be the Nagata ring of R and let $\phi : R \to R\{x\}$ be a natural homomorphism. By [11, Theorem 6.6.17], $R\{x\}$ is a DW ring, and so $R\{x\}$ is a DW_ϕ ring.

(4) Let R be an integral domain but not a field. Let K be the quotient field of R and let $\phi : R \to K$ be an including homomorphism. Then K is a DW_ϕ ring. So we can notice that even if T is a DW_ϕ ring, R is not necessary a DW ring.

(5) The converse of Corollary 3.9 is not necessarily true. For example, let R be a DW domain but not a field. Let $\phi : R \to R[x]$ be an inclusion homomorphism. By Corollary 3.9, the polynomial ring $R[x]$ is a DW_ϕ ring. Let $a \in R$ be a nonzero and nonunit. Then $J = (a, x) \in \mathrm{GV}(R[x])$. Therefore $R[x]$ is not a DW ring.

(6) Let $\phi : R \to R[x]$ be an inclusion homomorphism and let R be not a DW ring. Then there exists a maximal ideal A of R such that A isn't a w-ideal. So $A[x]$ as an R-module is not a w-module. Therefore a polynomial ring extension is not a DW_ϕ ring in general.

Recall that a ring R is said to be local if R has only one maximal ideal. However, the w-operation does not play a role over DW rings. So if R isn't a DW ring, we can introduce a local w-ring, which has the only one maximal w-ideal, but by the next theorem and corollary, we can see that it can't come true.

Theorem 3.11. *Let $\phi : R \to T$ be a w-linked homomorphism. Let T be a non-DW_ϕ ring. Then T must have an infinite number of maximal w_ϕ-ideals.*

Proof. Since T is not a DW_ϕ ring, by Theorem 3.8, there exists a maximal w_ϕ-ideal M_1 of T such that M_1 is not a maximal ideal of T. Suppose on the contrary that T has only a finite number of maximal w_ϕ-ideals, say $M_1, M_2, \ldots, M_n$. Let P be a maximal ideal containing M_1. Then P is not a w_ϕ-ideal. According to Prime Avoidance Theorem, $P \nsubseteq \bigcup_{i=1}^n M_i$. Let $y \in P \setminus \bigcup_{i=1}^n M_i$.

If y is a non-zero-divisor of T, then Ty is a proper w_ϕ-ideal of T. If y is a zero-divisor of R, then $\mathrm{Ann}_T(y) \neq 0$, and so $Ty \subseteq \mathrm{Ann}_T(\mathrm{Ann}(y)) \neq T$. By Proposition 3.4, $\mathrm{Ann}_T(\mathrm{Ann}(y))$ is a w_ϕ-ideal of T. By Proposition 3.4 again, there exists a maximal w_ϕ-ideal M of T such that $Tx \subseteq M$. Clearly $M \neq M_i$, $i = 1, 2, \ldots, n$, which is a contradiction. □

Corollary 3.12. *Let R be a non-DW ring. Then R must have an infinite number of maximal w-ideals.*

Proof. The assertion follows immediately by letting $T := R$ and ϕ be the identity homomorphism in Theorem 3.11. □

4. Properties of a w-factor ring $\overline{R}_w$

Let R be a ring, let I be a proper w-ideal of R, and let $\overline{R} = R/I$. Let $\pi : R \to \overline{R}$ be a natural homomorphism and let $\lambda : \overline{R} \to \overline{R}_w$ be the inclusion homomorphism. Then $\phi : R \to \overline{R}_w$ is a natural w-linked homomorphism. We also call $\overline{R}_w$ a *w-factor ring* of R.

Let I be an ideal of R. Write $\mathcal{Q}$ as the multiplicative system of f.g. semiregular ideals of R. Recall that I is said to be a q-ideal, if $z \in R$ and $J \in \mathcal{Q}$ with $Jz \subseteq I$ imply $z \in I$.

Remark 4.1. (1) Let $\alpha \in \overline{R}_w$. By Proposition 2.5, write $\alpha = \frac{\sum_{i=0}^n \overline{a_i} x^i}{\sum_{i=0}^n \overline{d_i} x^i}$, where $a_i, d_i \in R$, $\sum_{i=0}^n d_i x^i$ is a GV-polynomial and $\overline{d_i}\,\overline{a_j} = \overline{d_j}\,\overline{a_i}$ for any i, j. So we have $\frac{\sum_{i=0}^n a_i x^i}{\sum_{i=0}^n d_i x^i} \in R\{x\}$. In other words, there exists a $u \in R\{x\}$ such that $\phi(u) = \alpha$.

(2) Let I be a q-ideal of R. For $\alpha = \frac{\sum_{i=0}^n a_i x^i}{\sum_{i=0}^n b_i x^i} \in Q_0(R)$, let $\pi(\alpha) = \frac{\sum_{i=0}^n \overline{a_i} x^i}{\sum_{i=0}^n \overline{b_i} x^i}$. Then the ring homomorphism $\pi : R \to \overline{R}$ can be extended to the map from $Q_0(R)$ to $Q_0(\overline{R})$.

(3) Let M be a GV-torsion-free R-module. Considering the identity homomorphism $\mathbf{1} : R \to R$, by Proposition 2.5, $M_w = \left\{ \frac{\sum_{i=0}^n u_i x^i}{\sum_{i=0}^n d_i x^i} \in M[x]_{S_w} \mid \sum_{i=0}^n d_i x^i \text{ is a GV-ploynomial and } d_i u_j = d_j u_i \text{ for any } i, j \right\}$.

(4) Let M be an $\overline{R}$-module and let M as an R-module be a GV-torsion-free module. Denote $a \in R$ over $\overline{R}$ by $\overline{a}$. Then $M_w = \left\{ \frac{\sum_{i=0}^n u_i x^i}{\sum_{i=0}^n \overline{d_i} x^i} \in M[x]_{S_w} \mid \sum_{i=0}^n d_i x^i \text{ is a GV-ploynomial and } \overline{d_i} u_j = \overline{d_j} u_i \text{ for any } i, j \right\}$.

Proposition 4.2. *Let B be an ideal of R containing I. Denote $\overline{B} = B/I$. Then*:

(1) $\overline{B}_w = (\overline{B_w})_w = (B\overline{R}_w)_w$.

(2) $\overline{B}_w = \overline{R}_w$ *if and only if* $B_w = R$.

(3) *Let B be a prime w-ideal of R. Then $\overline{B}_w$ is a prime w_ϕ-ideal of $\overline{R}_w$ and $\overline{B}_w \cap \overline{R} = \overline{B}$.*

Proof. (1) By [11, Exercise 6.20(1)], $\overline{B}_w = (\overline{B_w})_w$. Notice that $B\overline{R} = \overline{B}$. Hence $(B\overline{R}_w)_w = (B\overline{R})_w = \overline{B}_w$.

(2) Let $B_w = R$. Then by (1), $\overline{B}_w = (\overline{B_w})_w = \overline{R}_w$.

Conversely, let $\overline{B}_w = \overline{R}_w$. Then $(B/I)_\mathfrak{m} = B_\mathfrak{m}/I_\mathfrak{m} = R_\mathfrak{m}/I_\mathfrak{m}$ for any $\mathfrak{m} \in w$-Max(R), and so $B_\mathfrak{m} = R_\mathfrak{m}$. Thus $B_w = R$.

(3) Let $y, z \in \overline{R}_w$, $yz \in \overline{B}_w$. Then there exists $J \in \mathrm{GV}(R)$ such that $Jy, Jz \subseteq \overline{R}$ and $J^2yz \subseteq \overline{B}$. Since $\overline{B}$ is a prime ideal of $\overline{R}$, it follows that $Jy \subseteq \overline{B}$ or $Jz \subseteq \overline{B}$. Hence $y \in \overline{B}_w$ or $z \in \overline{B}_w$. Therefore $\overline{B}_w$ is a prime w_ϕ-ideal of $\overline{R}_w$.

Let $r \in R$, $\overline{r} = \frac{\sum_{i=0}^n \overline{b_i} x^i}{\sum_{i=0}^n \overline{d_i} x^i}$, where $b_i \in B$, $\overline{d_j b_i} = \overline{d_i b_j}$, $J := (d_0, d_1, \ldots, d_n) \in \mathrm{GV}(R)$. Then $\overline{d_k}\overline{r} = \overline{b_k}$, and so $Jr \subseteq B$. Since B is a prime w-ideal, we have $r \in B$. Therefore $\overline{B}_w \cap \overline{R} = \overline{B}$. □

Lemma 4.3. *Let M, N be w-modules over R, $f : M \to N$ be a homomorphism, and A be a w-submodule of N. Then $B := f^{-1}(A)$ is a w-submodule of M.*

Proof. Let $J \in \mathrm{GV}(R)$, $x \in M$, $Jx \subseteq B$. Then $Jf(x) = f(Jx) \subseteq f(B) \subseteq A$. Since A is a w-submodule of N, we have $f(x) \in A$. Thus $x \in B$. Therefore $B := f^{-1}(A)$ is a w-submodule of M. □

Proposition 4.4. *Let $\phi : R \to \overline{R}_w$ be a natural w-linked homomorphism. Then*:

(1) *Let A be a w_ϕ-ideal of $\overline{R}_w$. Write $B = \phi^{-1}(A)$. Then $I \subseteq B$ and $A = (B/I)_w$.*

(2) *Let A_i be a w_ϕ-ideal of $\overline{R}_w$ for $i = 1, 2$. Write $B_i = \phi^{-1}(A_i)$. Then $A_1 = A_2$ if and only if $B_1 = B_2$.*

(3) *There is a one-to-one correspondence between the set of w-ideals* (*resp., prime w-ideals, maximal w-ideals*) *of R containing I and the set of w_ϕ-ideals* (*resp., prime w_ϕ-ideals, maximal w_ϕ-ideals*) *of $\overline{R}_w$.*

(4) $(\sqrt{I}/I)_w = nil(\overline{R}_w)$.

Proof. (1) By Lemma 4.3, B is a w-ideal of R. Clearly $I \subseteq B$. Since $\phi(x) = \pi(x) = \overline{x} \in A$ for $x \in B$, we have $B/I \subseteq A$. Thus $(B/I)_w \subseteq A$.

Conversely, let $\alpha = \frac{\sum_{i=0}^n \overline{r_i} x^i}{\sum_{i=0}^n \overline{d_i} x^i} \in A$. Then $\bar{d}_i \alpha = \overline{r_i}$. So $r_i \in B$, and thus $\overline{r_i} \in B/I$. Hence $\alpha \in (B/I)_w$. So we can get $A = (B/I)_w$.

(2) Let $A_1 = A_2$. Then it is easy to get $B_1 = B_2$.

Conversely, let $B_1 = B_2$. Then $A_1 = (B_1/I)_w = (B_2/I)_w = A_2$.

(3) This follows from (2).

(4) Let $\alpha = \frac{\sum_{i=0}^n \overline{r_i} x^i}{\sum_{i=0}^n \overline{d_i} x^i} \in nil(\overline{R}_w)$. Then there exists a positive integer m such that $\alpha^m = 0$. So $\sum_{i=0}^n \overline{r_i} x^i$ is a nilpotent element. Hence every $\overline{r_i}$ is

a nilpotent element. Thus $\overline{r_i} \in nil(\overline{R}) = \sqrt{I}/I$. Hence $\alpha \in (\sqrt{I}/I)_w$. So $nil(\overline{R}_w) \subseteq (\sqrt{I}/I)_w$.

Conversely, since $\sqrt{I}/I \subseteq nil(\overline{R}_w)$, we have that $(\sqrt{I}/I)_w \subseteq nil(\overline{R}_w)$. □

Let I_1, I_2 be w-ideals of R such that $I_1 \subseteq I_2$. Then there exists the natural homomorphism $\sigma : R/I_1 \to R/I_2$ such that $\sigma(\overline{r}) = \overline{r}$. Notice that the bars in the two locations have different meanings. So σ induces a ring homomorphism $\sigma : (R/I_1)_w \to (R/I_2)_w$ such that

$$\sigma\left(\frac{\sum_{i=0}^{n} \overline{a_i} x^i}{\sum_{i=0}^{n} \overline{b_i} x^i}\right) = \frac{\sum_{i=0}^{n} \overline{a_i} x^i}{\sum_{i=0}^{n} \overline{b_i} x^i}.$$

Theorem 4.5. *Let $\phi : R \to \overline{R}_w$ be a natural w-linked homomorphism. Then:*

(1) *I is a prime w-ideal of R if and only if $\overline{R}_w$ is an integral domain.*
(2) *I is a maximal w-ideal of R if and only if $\overline{R}_w$ is a field.*

Proof. (1) Let I be a prime w-ideal of R. Then R/I is an integral domain. By Proposition 2.5(6), $\overline{R}_w$ is an integral domain.

Conversely, let $\overline{R}_w$ be an integral domain. Since $\overline{R} = R/I \subseteq \overline{R}_w$, it follows that $\overline{R}$ is an integral domain. Therefore I is a prime ideal.

(2) Let I be a maximal w-ideal of R. By [11, Proposition 6.5.5], $\overline{R}_w = qf(\overline{R})$ is a field.

Conversely, let P be a maximal w-ideal of R and $I \subseteq P$. Then there exists a natural homomorphism $\sigma : \overline{R}_w \to (R/P)_w$. Since $\overline{R}_w$ is a field, σ is a monomorphism. Thus the natural homomorphism $R/I \to R/P$ is a monomorphism. Therefore $I = P$ is a maximal w-ideal of R. □

Theorem 4.6. *The following statements are equivalent.*

(1) *$\overline{R}_w$ satisfies the descending chain condition on w_ϕ-ideals of $\overline{R}_w$.*
(2) *$\overline{R}_w$ satisfies the minimal condition on w_ϕ-ideals of $\overline{R}_w$.*
(3) *$\overline{R}_w$ is an Artinian ring.*

Proof. (1)⇒(2) It is trivial.

(2)⇒(3) We should prove that $\overline{R}_w$ has only a finite number of maximal w_ϕ-ideals, and then by Theorem 3.11, $\overline{R}_w$ is a DW$_\phi$ ring. Hence every ideal of $\overline{R}_w$ is a w_ϕ-ideal. Therefore $\overline{R}_w$ is an Artinian ring.

Set

$$S = \{M_1 \cap M_2 \cap \cdots \cap M_k \mid k \geqslant 1, M_i \text{ is a maximal } w_\phi\text{-ideal of } \overline{R}_w\}.$$

By the hypothesis, S has a minimal element $M_1 \cap M_2 \cap \cdots \cap M_n$. Now we prove that $M_1, M_2, \ldots, M_n$ are all the maximal w_ϕ-ideals of R.

Let M be a maximal ideal of R. By the minimal property of $M_1 \cap M_2 \cap \cdots \cap M_n$, we have

$$M \cap M_1 \cap M_2 \cap \cdots \cap M_n = M_1 \cap M_2 \cap \cdots \cap M_n.$$

Then $M_1 M_2 \cdots M_n \subseteq M$. So there exists i such that $M_i \subseteq M$. Since M_i is the maximal w_ϕ-ideal, we have $M_i = M$.

(3)⇒(1) This is clear. □

Theorem 4.7. *Let R be an integral domain. Then the following statements are equivalent.*

(1) *R is an SM-domain with w-$\dim(R) \leq 1$.*
(2) *For any nonzero w-ideal I of R, $(R/I)_w$ is an Artinian ring.*
(3) *For any nonzero element $a \in R$, $(R/(a))_w$ is an Artinian ring.*
(4) *For any nonzero element $a \in R$, R has the descending chain condition on w-ideals of R containing a.*

Proof. (1)⇒(2) Let $(\xi) : A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$ be a descending chain of w_ϕ-ideals of $(R/I)_w$. For every n, let $B_n = \phi^{-1}(A_n)$. By Proposition 4.4, $(\eta) : B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n \supseteq \cdots$ is a descending chain of w-ideals of R. By [4, Theorem 3.2], the descending chain (η) is stationary. By Proposition 4.4, the descending chain (ξ) is stationary. By Theorem 4.6, $\overline{R}_w$ is an Artinian ring.

(2)⇒(3) This is trivial.

(3)⇒(4) Let $(\xi) : I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ be a descending chain of w-ideals of R containing a. Then $(\eta) : (I_1/(a))_w \supseteq (I_2/(a))_w \supseteq \cdots \supseteq (I_n/(a))_w \supseteq \cdots$ is a descending chain of w_ϕ-ideals of $(R/(a))_w$. Since $(R/(a))_w$ is an Artinian ring, the descending chain (η) is stationary. By Proposition 4.4, the descending chain (ξ) is stationary.

(4)⇒(1) Let I be a nonzero w-ideal of R and let $(\xi) : I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ be a descending chain of w-ideals of R containing I. For any $a \in I$ with $a \neq 0$, (ξ) is also a descending chain of w-ideals of R containing a. By the hypothesis, (ξ) is stationary. By [4, Theorem 3.2] again, R is an SM domain with w-$\dim(R) \leq 1$. □

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