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ON TORIC HAMILTONIAN T-SPACES WITH ANTI-SYMPLECTIC INVOLUTIONS

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Abstract. The aim of this paper is to deal with the realization problem of a given Lagrangian submanifold of a symplectic manifold as the fixed point set of an anti-symplectic involution. To be more precise, let (X,ω,μ) be a toric Hamiltonian T-space, and let $\Delta=\mu(X)$ denote the moment polytope. Let τ be an anti-symplectic involution of X such that τ maps the fibers of μ to (possibly different) fibers of μ, and let p_0 be a point in the interior of Δ. If the toric fiber $\mu^{-1}(p_0)$ is real Lagrangian with respect to τ, then we show that p_0 should be the origin and, furthermore, Δ should be centrally symmetric.

1. Introduction and main results

Our main concern of this paper is an *anti-symplectic involution* τ of a compact connected symplectic manifold X with a symplectic 2-form ω satisfying

$$\tau^*\omega=-\omega.$$

For the rest of this paper, we assume that the fixed point set of τ is non-empty. The non-empty fixed point set of τ is a Lagrangian submanifold of X, called a *real Lagrangian* of X (see [10, Theorem 1] and [11, Lemma 1.2]).

Let T be a torus, and let $\mathfrak{t}$ denote the Lie algebra of T. Let T act on X in a Hamiltonian way, with Hamiltonian functions μ_ξ, $\xi\in\mathfrak{t}$, and moment map $\mu: X\to\mathfrak{t}^*$. As for Hamiltonians, adding a constant vector to the moment map does not change the group action it generates. So in this paper we always assume the normalization $\int_X \mu\omega^{\frac{1}{2}\dim X}=0\in\mathfrak{t}^*$, unless stated otherwise. The triple (X,ω,μ) is usually called a *Hamiltonian T-space*, and it is called *toric* if, in addition, $\dim T=\frac{1}{2}\dim X$ (refer to [1]). Furthermore, we often require that τ and μ satisfy the condition

$$\mu\circ\tau=\mu. \tag{1.1}$$

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Contrary to the first glance, the above condition (1.1) is not so restrictive in that all toric manifolds always admit anti-symplectic involutions satisfying the condition (1.1) (see [9]).

Note that the condition (1.1) implies that τ maps the fibers of the moment map μ to the same fibers of μ. Hence, in order to obtain meaningful results in more general settings it seems to be natural to impose the condition that anti-symplectic involutions with which we are concerned map the fibers of the moment map to (possibly other) fibers of the moment map. Moreover, if an anti-symplectic involution τ maps the fibers of the moment map μ to (possibly other) fibers of μ, then (1.1) can always be arranged by passing to a suitable subtorus T_0 of T and taking the moment map μ composed by the natural projection $\mathfrak{t}^* \to \mathfrak{t}_0^*$. This will be discussed in Proposition 2.3 of Section 2, in more detail.

We remark that there exists an anti-symplectic involution which does not necessarily map the fibers of the moment map μ to those of μ ([2]). For some concrete example, let S^2 be the unit-sphere in the Euclidean space $\mathbb{R}^3$, equipped with the standard symplectic structure ω given by

$$\omega_p(u, v) = \langle p, u \times v \rangle, \quad p \in S^2,\ u, v \in T_pS^2,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^3$. Then (S^2, ω) has the Hamiltonian circle action given by rotating around the z-axis in $\mathbb{R}^3$. Clearly in this case the moment map μ is given by the height function along the z-axis, and the fibers of μ are the circles on S^2 parallel to the xy-plane. Now, let W be a plane in $\mathbb{R}^3$ passing through the origin, and let τ_W be the reflection with respect to the plane W. Then τ_W becomes an anti-symplectic involution for the Hamiltonian S^1-space (S^2, ω, μ). If the plane W is taken to be the xy-plane, then τ_W maps the fibers of μ to fibers of μ. However, for some general plane W (e.g., the plane W given by the equation $z = y$) it is easy to see that τ_W does not map the fibers of μ to other fibers of μ.

Now, let P^n denote a simple convex polytope of dimension n, and let T^n be an n-dimensional torus acting on a compact connected smooth manifold M^{2n} of dimension $2n$ which has P as the orbit space. If, in addition, the action of T^n is locally isomorphic to the standard representation of T^n on the n-dimensional complex vector space $\mathbb{C}^n$, then M^{2n} is called a *quasitoric manifold* over P^n. Similarly, let $\mathbb{Z}_2^n$ act on a compact connected smooth manifold N^n of dimension n which has P^n as the orbit space. If, in addition, the action of $\mathbb{Z}_2^n$ is locally isomorphic to the standard representation of $\mathbb{Z}_2^n$ on the n-dimensional real vector space $\mathbb{R}^n$, then N^n is called a *small cover* over P^n (see [5] for more details).

Let $\pi : M^{2n} \to P^n$ be a quasitoric manifold. Then there is an involution ϕ on M^{2n}, with the fixed point set of ϕ being N^n, such that $\pi|_{N^n} : N^n \to P^n$ is a small cover. This involution ϕ is called a *conjugation* of M^{2n} (see [6, Corollary 1.9] for more details).

Let K be a finite simplicial complex of dimension $n-1$. For each $0 \le i \le n-1$, let f_i denote the number of i-simplices in K. Then $(f_0, f_1, \dots, f_{n-1})$ is called the *f-vector* of K. Let h_i be the coefficient of t^{n-i} in the polynomial $\Psi_K(t)$ of degree n given by

$$\Psi_K(t) = (t-1)^n + \sum_{i=0}^{n-1} f_i (t-1)^{n-1-i}.$$

That is, h_i's satisfy the identity

$$\Psi_K(t) = \sum_{i=0}^{n} h_i t^{n-i}.$$

Then $(h_0, h_1, \dots, h_n)$ is called the *h-vector* of the simple polytope P^n dual to K. Clearly we have $h_0 = 1$,

$$h_n = \Psi_K(0) = (-1)^n (1 - \chi(K)), \text{ and } \sum_{i=0}^{n} h_i = \Psi_K(1) = f_{n-1},$$

where $\chi(K)$ denotes the Euler characteristic of K.

As a special case, let K be the boundary of a simplicial polytope of dimension n, and let P^n be the dual simple polytope of dimension n. Then f_i is the number of faces of P^n of codimension $i+1$, $h_n = 1$, and $\sum_{i=0}^{n} h_i$ is the number of vertices of P^n.

With these understood, the topology of toric Hamiltonian T-spaces (X, ω, μ), equipped with an anti-symplectic involution τ satisfying the condition (1.1) can be summarized, as follows.

Theorem 1.1. *Let (X, ω, μ) be a toric Hamiltonian T-space, and let $\Delta = \mu(X)$ denote the moment polytope. Let τ be an anti-symplectic involution of X satisfying $\mu \circ \tau = \mu$, and let $Q = \mathrm{Fix}(\tau)$ denote the non-empty fixed point set of τ. Then the followings are true*:

(1) *X is a quasitoric manifold of dimension $2 \dim T$ with the quotient map $\mu : X \to \Delta$.*
(2) *τ is a conjugation of the quasitoric manifold X such that $\mu|_Q : Q \to \Delta$ is a small cover.*
(3) *For each $\xi \in \mathfrak{t}$, let C_ξ denote the critical set of the map $\mu_\xi|_Q$. Then we have*

$$\dim H^{2i}(X; \mathbb{Z}_2) = h_i = \dim H^i(Q; \mathbb{Z}_2), \quad 0 \le i \le \dim T,$$

and

$$\dim H^{2*}(X; \mathbb{Z}_2) = \sum_{i=0}^{\dim T} h_i = \dim H^*(Q; \mathbb{Z}_2) = \dim H^*(C_\xi; \mathbb{Z}_2),$$

where h_i's denote the h-numbers of Δ.

Remark 1.2. (1) Theorem 1.1(1) seems to be well-known to the experts in the field, while Theorem 1.1(2)-(3) is essentially an easy consequence of [6] (see, e.g., [6, Corollary 1.9]).

(2) Furthermore, in [9, Proposition 3.4] Haug proved a much stronger result asserting that there is a canonical ring isomorphism between $H^{2*}(X;\mathbb{Z}_2)$ and $H^*(Q;\mathbb{Z}_2)$ for a toric Hamiltonian T-space (X,ω,μ), equipped with the anti-symplectic involution τ induced from the complex conjugation on the complex vector space. According to [9], the real Lagrangian induced by the anti-symplectic involution τ induced from the complex conjugation on the complex vector space is the only example of this kind up to composing with fiber preserving symplectic diffeomorphisms. Thus the result of Haug in [9] also implies Theorem 1.1(3). For the sake of reader's convenience, we give a relatively detailed proof of Theorem 1.1 in Section 3. It can be regarded as a by-product of some observations made in Section 2.

One simple example for Theorem 1.1 is given by the unit two-sphere S^2 with its area form ω as a symplectic form. Here the toric structure for (S^2,ω) is given by the rotation around the axis passing through the north and south poles, and its corresponding moment map μ is given by the height function. Hence the moment polytope Δ for a toric Hamiltonian S^1-space (S^2,ω,μ) is equal to the closed interval $[-1,1]$. If we consider the two-sphere S^2 as $\mathbb{C}\cup\{\infty\}$, then we can find an anti-symplectic involution τ given by the complex conjugation $z\mapsto\bar{z}$. Thus the fixed point set $Q=\mathrm{Fix}(\tau)$ of τ is given by the union of the real line and ∞ which is homeomorphic to the circle. Note that the image of Q under μ is the same as $\mu(S^2)=[-1,1]$. Furthermore, $\mu|_Q : Q\to[-1,1]$ is a small cover, and, for any $\xi\in\mathfrak{t}\cong\mathbb{R}$, C_ξ just consists of two points, the north and south poles. Hence we have

$$2=\dim H^{2*}(S^2;\mathbb{Z}_2)=\dim H^*(Q;\mathbb{Z}_2)=\dim H^*(C_\xi;\mathbb{Z}_2).$$

By taking the product of these examples, we can obtain the product of circles, called the *Clifford torus*, as the fixed point set of an anti-symplectic involution on the product manifold $S^2\times\cdots\times S^2$.

Another typical example is given by the complex projective space $\mathbb{CP}^n$ with the complex conjugation as an anti-symplectic involution. In this case, the fixed point set of τ is the real projective space $\mathbb{RP}^n$. In fact, any symplectic manifold (M,ω) can be realized as a real Lagrangian in $(M\times M,\omega\oplus-\omega)$. Here the embedding is given by the diagonal map and the anti-symplectic involution τ on $M\times M$ is given by the map $(p,q)\mapsto(q,p)$. Further, the cotangent bundle T^*M of any smooth manifold M, equipped with the canonical symplectic form $-d\lambda$ admits a natural anti-symplectic involution τ_0 given by $(q,p)\mapsto(q,-p)$. In this case, the fixed point set of τ_0 is just the zero section of the bundle $T^*M\to M$. On the other hand, in [10, Theorem 2] Meyer proved that any anti-symplectic involution with non-empty fixed point set is of this form (see also [8, Proposition 2.2]).

Recall that by assumption the origin $0 \in \mathfrak{t}^*$ lies in Δ°. We say that Δ is *centrally symmetric* if there is an involution $\sigma \in \mathrm{Aut}_{\mathbb{Z}}(\mathfrak{t}^*)$ such that $\sigma(\Delta) = \Delta$ and the origin is the only fixed point of σ. Here $\mathrm{Aut}_{\mathbb{Z}}(\mathfrak{t}^*)$ denotes the group of automorphisms of $\mathfrak{t}^*$ which preserves the lattice $\mathfrak{t}^*_{\mathbb{Z}}$.

In [3], Brendel asked if a given Lagrangian submanifold of a symplectic manifold can be realized as the fixed point set of an anti-symplectic involution. That is, he asked if a given Lagrangian submanifold of a symplectic manifold can always be real and, furthermore, the moment polytope is centrally symmetric. In case of a toric Hamiltonian T-space (X, ω, μ) with moment polytope $\Delta = \mu(X)$, for any point p in the interior Δ° of Δ the toric fiber $\mu^{-1}(p)$ is always Lagrangian. Thus it is natural to ask if the toric fiber $\mu^{-1}(p)$ is always real Lagrangian and Δ is centrally symmetric.

Our main result of this paper is:

Theorem 1.3. *Let (X, ω, μ) be a toric Hamiltonian T-space, and let $\Delta = \mu(X)$ denote the moment polytope. Let τ be an anti-symplectic involution of X such that τ maps the fibers of μ to (possibly different) fibers of μ, and let $p_0 \in \Delta^\circ$. If the toric fiber $\mu^{-1}(p_0)$ is real Lagrangian with respect to τ, then p_0 should be the origin and, furthermore, Δ should be centrally symmetric.*

Theorem 1.3 affirmatively gives a partial answer to [3, Question 1.5] that is stated only for toric monotone symplectic manifolds. Recall that a symplectic manifold is *monotone* if its first Chern class is positively proportional to the cohomology class of the symplectic form. As also noticed in [3], it is easy to see that by Theorem 1.3 the toric fiber $\mu^{-1}(0)$ for $S^2 \times S^2$ as well as that for the two-sphere S^2 is real Lagrangian, while the toric fiber $\mu^{-1}(0)$ for $\mathbb{CP}^2$ is not.

We organize this paper, as follows. In Section 2, we first show some basic properties of anti-symplectic involution on a Hamiltonian T-spaces. When an anti-symplectic involution maps the fibers of the moment map to (possibly other) fibers of the moment map, in the same section we also show that the property that the anti-symplectic involution leaves the moment map invariant can always be achieved by passing to a suitable subtorus T_0 of T and taking the moment map μ composed by the natural projection $\mathfrak{t}^* \to \mathfrak{t}_0^*$. In Section 3, we provide a proof of Theorem 1.1 through a series of lemmas (refer to Theorem 3.3). In Section 4, we give a proof of Theorem 1.3 as a consequence of Proposition 2.3 given in Section 2 (refer to Theorem 4.1).

2. Properties of anti-symplectic involutions

The aim of this section is to collect some basic facts about anti-symplectic involutions on a Hamiltonian T-spaces. Refer to [8] and [4] for more details.

To do so, we begin with the definition of a Hamiltonian T-space (X, ω, μ). In fact, we have a linear map $\xi \mapsto \mu_\xi = \langle \mu, \xi \rangle$ from the Lie algebra $\mathfrak{t}$ to the space of smooth functions on X such that:

(1) For each $\xi \in \mathfrak{t}$, the infinitesimal action $\tilde{\xi}$ of ξ on X is given by the Hamiltonian vector field of the function μ_ξ. That is, we have

$$\widetilde{\xi} \rfloor \omega = -d\mu_\xi. \tag{2.1}$$

(2) For each $\xi \in \mathfrak{t}$, the function μ_ξ is involutive. That is, for any $\xi_1, \xi_2 \in \mathfrak{t}$, we have

$$\widetilde{\xi_1} \rfloor d\mu_{\xi_2} = 0, \tag{2.2}$$

which means that μ_{ξ_2} is constant along the T-orbits on X.

As before, let τ be an anti-symplectic involution of X satisfying the identity

$$\tau^*\mu_\xi = \mu_\xi \circ \tau = \mu_\xi, \quad \xi \in \mathfrak{t}. \tag{2.3}$$

Lemma 2.1. *The followings are true*:

(1) *For any* $\xi \in \mathfrak{t}$, *we have*

$$\tau_*(\widetilde{\xi}) = -\widetilde{\xi}. \tag{2.4}$$

(2) *For any* $g \in T$, *let* $\widetilde{g}$ *denote the action of* g *on* X. *Then we have*

$$\tau \circ \widetilde{g} = \widetilde{g}^{-1} \circ \tau. \tag{2.5}$$

Proof. (1) It follows from (2.3) that we have

$$\tau^*(d\mu_\xi) = d\mu_\xi, \quad \xi \in \mathfrak{t}. \tag{2.6}$$

By combining (2.6) with (2.1), we can obtain

$$-\tilde{\xi} \rfloor \omega = d\mu_\xi = \tau^*(d\mu_\xi) = \tau^*(-\tilde{\xi} \rfloor \omega).$$

Since τ is anti-symplectic, we have

$$-\omega(\tau_*(\widetilde{\xi}), \cdot) = \omega(\widetilde{\xi}, \tau_*(\cdot)) = \omega(\widetilde{\xi}, \cdot), \quad \xi \in \mathfrak{t}.$$

This implies that $\tau_*(\widetilde{\xi}) = -\widetilde{\xi}$ for any $\xi \in \mathfrak{t}$, since ω is non-degenerate. This completes the proof of (1).

For the proof of (2), note first that (2.4) is equivalent to

$$\frac{d}{dt}\Big|_{t=0} \tau(\exp(t\xi) \cdot m) = \frac{d}{dt}\Big|_{t=0} \exp(-t\xi) \cdot \tau(m), \quad m \in X.$$

This in turn is equivalent to

$$\tau \circ \widetilde{g} = \widetilde{g}^{-1} \circ \tau, \quad g \in T,$$

which completes the proof of (2). □

Next, we show that if an anti-symplectic involution τ maps the fibers of the moment map μ to (possibly other) fibers of μ, then the condition (1.1) can always be achieved by passing to a suitable subtorus T_0 of T and taking the moment map μ composed by the natural projection $\mathfrak{t}^* \to \mathfrak{t}_0^*$. It seems that this fact has already been known to the experts in the field (refer to [8, p. 417]), but we cannot find a proof of the fact in any literature. So in this paper we provide its proof, for the sake of reader's convenience.

To do so, we first need to recall some general facts necessary for the proof. Let $\mathfrak{t}_{\mathbb{Z}}$ denote the standard lattice defined by the kernel of the exponential map $\exp : \mathfrak{t} \to T$. Then clearly T is isomorphic to $\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}}$. Let $\mathrm{Aut}_{\mathbb{Z}}(\mathfrak{t})$ denote the group of automorphisms of $\mathfrak{t}$ which preserves the lattice $\mathfrak{t}_{\mathbb{Z}}$. Any element $\alpha \in \mathrm{Aut}_{\mathbb{Z}}(\mathfrak{t})$ induces a group automorphism A_α of T, and, conversely, any element $A \in \mathrm{Aut}(T)$ induces an automorphism $\alpha_A \in \mathrm{Aut}_{\mathbb{Z}}(\mathfrak{t})$ which is given by the differential of A. The dual $\mathfrak{t}^*_{\mathbb{Z}}$ of the standard lattice $\mathfrak{t}_{\mathbb{Z}}$ is defined by

$$\mathfrak{t}^*_{\mathbb{Z}} = \{\eta \in \mathfrak{t}^* \mid \langle \eta, \xi\rangle \in \mathbb{Z},\ \xi \in \mathfrak{t}_{\mathbb{Z}}\}\,.$$

It is clear that the automorphism group $\mathrm{Aut}_{\mathbb{Z}}(\mathfrak{t}^*)$ can be defined in a similar way.

The following lemma holds (refer to [4, Proposition 2.5]).

Lemma 2.2. *Let (X, ω, μ) be a Hamiltonian T-space, and let τ be an anti-symplectic involution of X such that τ maps the fibers of μ to (possibly different) fibers of μ. Then there is an involution $\sigma \in \mathrm{Aut}_{\mathbb{Z}}(\mathfrak{t}^*)$ such that*

$$\mu \circ \tau = \sigma \circ \mu. \tag{2.7}$$

Proof. For the proof, assume first that there is such an automorphism $\sigma \in \mathrm{Aut}_{\mathbb{Z}}(\mathfrak{t}^*)$ satisfying the identity $\mu\circ\tau = \sigma\circ\mu$. Then, for any $\xi \in \mathfrak{t}$ we can obtain

$$d\langle \mu\circ\tau, \xi\rangle = d\langle \sigma\circ\mu, \xi\rangle. \tag{2.8}$$

Thus, for any $\eta \in TX$ we have

$$\begin{aligned} \tau_*(\widetilde{\xi\circ\tau}) \rfloor \omega(\eta) &= \omega(\tau_*(\widetilde{\xi\circ\tau}), \eta) = -\tau^*\omega(\tau_*(\widetilde{\xi\circ\tau}), \eta) \\ &= -\omega(\widetilde{\xi\circ\tau}, \tau_*(\eta)) = -\widetilde{\xi\circ\tau} \rfloor \omega(\tau_*(\eta)) \\ &= \langle \mu_*(\tau_*(\eta)), \xi\rangle = d\langle \mu\circ\tau, \xi\rangle(\eta). \end{aligned} \tag{2.9}$$

On the other hand, we also have

$$d\langle \mu\circ\tau, \xi\rangle = d\langle \sigma\circ\mu, \xi\rangle = d\langle \mu, \sigma^*(\xi)\rangle = -\widetilde{\sigma^*(\xi)} \rfloor \omega. \tag{2.10}$$

By (2.9) and (2.10), we thus have

$$\tau_*(\widetilde{\xi\circ\tau}) = -\widetilde{\sigma^*(\xi)}. \tag{2.11}$$

In view of (2.11), we now define a map $\alpha : \mathfrak{t} \to \mathfrak{t}$ such that

$$\widetilde{\alpha(\xi)} = -\tau_*(\widetilde{\xi\circ\tau}), \quad \xi \in \mathfrak{t}.$$

We then claim that α is an automorphism in $\mathrm{Aut}_{\mathbb{Z}}(\mathfrak{t})$. Indeed, since the infinitesimal action $\tilde{\xi}$ of ξ on X is given by the Hamiltonian vector field of the function μ_ξ, α is well-defined, and injective as well. Furthermore, the map α is linear, so that α is an isomorphism from $\mathfrak{t}$ to itself.

Now it remains to show that α preserves the lattice $\mathfrak{t}_{\mathbb{Z}}$. To see it, note first that if $\xi \in \mathfrak{t}_{\mathbb{Z}}$, then its corresponding element in $T = \mathfrak{t}/\mathfrak{t}_{\mathbb{Z}}$ is zero. Hence $\widetilde{\xi}$ is the zero vector field on X. By definition of α, $\widetilde{\alpha(\xi)}$ is also the zero vector field on X. Since the infinitesimal action $\widetilde{\alpha(\xi)}$ of $\alpha(\xi)$ on X is given by the

Hamiltonian vector field of the function $\mu_{\alpha(\xi)}$, this implies that $\alpha(\xi)$ lies in the lattice $\mathfrak{t}_{\mathbb{Z}}$. This completes the proof of the claim.

Finally, let $\sigma = \alpha^*$ denote the adjoint of α. Then clearly σ lies in $\mathrm{Aut}_{\mathbb{Z}}(\mathfrak{t}^*)$, and it follows from (2.8) that

$$\mu \circ \tau - \sigma \circ \mu$$

is a constant function of X. Since in this paper we always assume the normalization $\int_X \mu\omega^{\frac{1}{2}\dim X} = 0 \in \mathfrak{t}^*$, this implies that we have

$$\mu \circ \tau = \sigma \circ \mu,$$

as desired.

Since τ is an involution of X, we have

$$\begin{aligned} \sigma^2 \circ \mu = \sigma \circ (\sigma \circ \mu) &= \sigma \circ (\mu \circ \tau) = (\sigma \circ \mu) \circ \tau = (\mu \circ \tau) \circ \tau \\ &= \mu \circ \tau^2 = \mu. \end{aligned} \tag{2.12}$$

Since $\sigma \in \mathrm{Aut}_{\mathbb{Z}}(\mathfrak{t}^*)$ and $\mu : X \to \Delta$ is surjective at least around $0 \in \Delta^\circ \subset \mathfrak{t}^*$, it follows from (2.12) that σ is actually an involution. This completes the proof of Lemma 2.2. □

Proposition 2.3. *Let (X, ω, μ) be a Hamiltonian T-space, and let τ be an anti-symplectic involution of X such that τ maps the fibers of μ to (possibly different) fibers of μ. Then there is a subtorus T_0 of T such that*

1. *there is a natural inclusion $j : \mathfrak{t}_0 \to \mathfrak{t}$ and its adjoint $j^* : \mathfrak{t}^* \to \mathfrak{t}_0^*$, and*
2. *the moment map $\tilde{\mu} := j^* \circ \mu : X \to \mathfrak{t}_0^*$ satisfies $\tilde{\mu} \circ \tau = \tilde{\mu}$.*

Here $\mathfrak{t}_0$ denotes the Lie algebra of T_0, as usual.

Proof. It follows from Lemma 2.2 that there is an automorphism $\sigma \in \mathrm{Aut}_{\mathbb{Z}}(\mathfrak{t}^*)$ such that

$$\mu \circ \tau = \sigma \circ \mu.$$

Let

$$\mathfrak{t}_0 = \{\xi \in \mathfrak{t} \,|\, \sigma^*(\xi) = \xi\},$$

and let T_0 be the subtorus of T whose Lie algebra is equal to $\mathfrak{t}_0$.

As above, let

$$\tilde{\mu} = j^* \circ \mu : X \to \mathfrak{t}_0^*.$$

Then, for any $\xi \in \mathfrak{t}_0$ we have

$$\begin{aligned} \langle \tilde{\mu} \circ \tau, \xi \rangle = \langle j^* \circ \mu \circ \tau, \xi \rangle &= \langle \mu \circ \tau, \xi \rangle \\ &= \langle \sigma \circ \mu, \xi \rangle = \langle \mu, \sigma^*(\xi) \rangle = \langle j^* \circ \mu, \xi \rangle \\ &= \langle \tilde{\mu}, \xi \rangle. \end{aligned}$$

This completes the proof of Proposition 2.3. □

3. Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. To do so, we show a series of lemmas, as follows.

Lemma 3.1. *Let (X, ω, μ) be a toric Hamiltonian T^n-space, and let $\Delta = \mu(X)$ denote the moment polytope. Then X is a quasitoric manifold of dimension $2 \dim T^n$ with the quotient map $\mu : X \to \Delta$.*

Proof. Since (X, ω, μ) is a toric Hamiltonian T^n-space, we should have

$$\dim X = 2 \dim T^n = 2n,$$

and the moment map $\mu : X \to (\mathfrak{t}^n)^*$ is a quotient map for the torus action of T^n. Here $(\mathfrak{t}^n)^*$ denotes the dual of the Lie algebra $\mathfrak{t}^n$ of T^n. Moreover, our choice of the normalization implies that the barycenter of the moment polytope $\Delta = \mu(X)$ is located at $0 \in (\mathfrak{t}^n)^*$.

Recall also that in [7] Delzant gave an explicit construction of the toric Hamiltonian T^n-space (X, ω, μ) from a given Delzant polytope

$$\Delta = \{\eta \in (\mathfrak{t}^n)^* \mid \langle \eta, v_i \rangle \leq \kappa_i,\ 1 \leq i \leq k\}$$

such that $\mu(X) = \Delta$. Here v_i's denote the outward primitive normal vectors to the facets of Δ and κ_i's are positive real numbers. To be more precise, let $\mathfrak{t}^k$ denote the Lie algebra of the k-dimensional torus T^k, and let

$$\begin{aligned} &\nu_0 : \mathbb{C}^k \to (\mathfrak{t}^k)^* \cong \mathbb{R}^k, \\ &(z_1, z_2, \ldots, z_k) \mapsto \frac{1}{2}(|z_1|^2, |z_2|^2, \ldots, |z_k|^2) - (\kappa_1, \kappa_2, \ldots, \kappa_k). \end{aligned}$$

Then ν_0 is a moment map for the standard action of T^k on $\mathbb{C}^k$. Let

$$\pi : \mathfrak{t}^k \to \mathfrak{t}^n, \quad e_i \mapsto v_i,$$

where e_i's denote the standard coordinate vectors of $\mathfrak{t}^k \cong \mathbb{R}^k$. This map π is called a *characteristic map* for the polytope Δ, and it is a linear map of full rank equal to n. Moreover, π preserves the integral lattices $\mathfrak{t}^k_{\mathbb{Z}}$ and $\mathfrak{t}^n_{\mathbb{Z}}$. Hence π induces a map from T^k to T^n, denoted by the same letter π.

Let $K = \ker \pi$ with the inclusion $j : K \to T^k$, and let $\mathfrak{k}$ denote the Lie algebra of K. Then we have the following short exact sequences

$$0 \longrightarrow \mathfrak{k} \xrightarrow{j_*} \mathfrak{t}^k \xrightarrow{\pi} \mathfrak{t}^n \longrightarrow 0,$$

$$0 \longrightarrow (\mathfrak{t}^n)^* \xrightarrow{\pi^*} (\mathfrak{t}^k)^* \xrightarrow{(j_*)^*} \mathfrak{k}^* \longrightarrow 0.$$

Let

$$\nu = (j_*)^* \circ \nu_0 : \mathbb{C}^k \to \mathfrak{k}^*.$$

Then ν is the moment map for the action of K, and $0 \in \mathfrak{k}^*$ is a regular value of ν. Since K acts freely on the pre-image $\nu^{-1}(0) \subset \mathbb{C}^k$, we can take the symplectic reduction $\nu^{-1}(0)/K$, equipped with the induced symplectic form

ω_{red} and the residual action of T^n. Furthermore, there is a moment map $\nu_{\mathrm{red}} : \nu^{-1}(0)/K \to (\mathfrak{t}^n)^*$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} & & \nu^{-1}(0)/K & \longleftarrow & \nu^{-1}(0) & \hookrightarrow & \mathbb{C}^k & & \\ & & & \overset{\nu_{\mathrm{red}}}{\searrow} & \downarrow & & \downarrow \nu_0 & \overset{\nu}{\searrow} & \\ 0 & \longrightarrow & & & (\mathfrak{t}^n)^* & \xrightarrow{\pi^*} & (\mathfrak{t}^k)^* & \xrightarrow{(j_*)^*} & \mathfrak{k}^* \longrightarrow 0 \end{array}$$

It is well-known and also can be shown that the Hamiltonian T^n-space

$$(\nu^{-1}(0)/K, \omega_{\mathrm{red}}, \nu_{\mathrm{red}})$$

is isomorphic to the Hamiltonian T^n-space (M, ω, μ). It follows from the above construction of (X, ω, μ) that the action of T^n on X is locally isomorphic to the standard action of T^n on the complex vector space $\mathbb{C}^n$. This completes the proof of Lemma 3.1 (or, equivalently Theorem 1.1(1)). □

The following lemma also holds.

Lemma 3.2. *Let (X, ω, μ) be a toric Hamiltonian T^n-space of dimension $2n$, and let $\Delta = \mu(X)$ denote the moment polytope. Let τ be an anti-symplectic involution of X satisfying $\mu \circ \tau = \mu$, and let $Q = \mathrm{Fix}(\tau)$ denote the non-empty fixed point set of τ. Then τ is a conjugation of the quasitoric manifold X such that $\mu|_Q : Q \to \Delta$ is a small cover.*

Proof. As before, for any $g \in T^n$ let $\widetilde{g}$ denote the action of g on X. Then it follows from Lemma 2.1 that we have

$$\tau \circ \widetilde{g} = \widetilde{g}^{-1} \circ \tau. \tag{3.1}$$

In other words, τ maps T^n-orbits to T^n-orbits, but at the same time reverses the time on the orbits of the 1-parameter subgroups of T^n. This means that for any $g \in T^n$ and any $m \in X$ we have

$$\tau(g \cdot m) = g^{-1} \cdot \tau(m).$$

Furthermore, since we have the identity $\mu \circ \tau = \mu$, for each $p \in \Delta$, τ maps the fiber $\mu^{-1}(p)$ to itself.

On the other hand, it follows from Lemma 3.1 that the toric Hamiltonian T^n-space (X, ω, μ) is a quasitoric manifold over the moment polytope Δ. For each $p \in \Delta$, let $F(p)$ denote the unique face of Δ which contains p in its relative interior. Then X can be obtained as the quotient construction $T \times \Delta/\sim$, where $(g, p) \sim (h, q)$ if and only if $p = q$ and $g^{-1}h \in (T^n)_{F(p)}$ for the stabilizer $(T^n)_{F(p)}$ corresponding to $F(p)$. Hence, if we consider the involution $\bar{\tau}$ on $T^n \times \Delta$ defined by $(g, p) \mapsto (g^{-1}, p)$, then we may assume without loss of generality that $\bar{\tau}$ descends to the involution τ on X.

Moreover, it is easy to show that the fixed point set of $\bar{\tau}$ is equal to $\mathbb{Z}_2^n \times \Delta$. Thus the fixed point set Q of τ is given by the quotient space $\mathbb{Z}_2^n \times \Delta/\sim$, where $(g, p) \sim (h, q)$ if and only if $p = q$ and $g^{-1}h \in (\mathbb{Z}_2^n)_{F(p)}$ for the stabilizer

$(\mathbb{Z}_2^n)_{F(p)}$ corresponding to $F(p)$. This implies that τ is a conjugation of the quasitoric manifold X such that $\mu|_Q : Q \to \Delta$ is a small cover. This completes the proof of Lemma 3.2 (or, equivalently Theorem 1.1(2)). □

Now, we are ready to complete the proof of Theorem 1.1, as follows.

Theorem 3.3. *Let (X, ω, μ) be a toric Hamiltonian T^n-space, and let $\Delta = \mu(X)$ denote the moment polytope. Let τ be an anti-symplectic involution of X satisfying $\mu \circ \tau = \mu$, and let $Q = \mathrm{Fix}(\tau)$ denote the non-empty fixed point set of τ. For each $\xi \in \mathfrak{t}^n$, let C_ξ denote the critical set of the map $\mu_\xi|_Q$. Then we have*

$$\dim H^{2i}(X; \mathbb{Z}_2) = h_i = \dim H^i(Q; \mathbb{Z}_2), \quad 0 \le i \le n,$$

and

$$\dim H^{2*}(X; \mathbb{Z}_2) = \sum_{i=0}^{n} h_i = \dim H^*(Q; \mathbb{Z}_2) = \dim H^*(C_\xi; \mathbb{Z}_2).$$

Here h_i's denote the h-numbers of Δ, as before.

Proof. Since (X, ω, μ) is a quasitoric manifold over a simple convex polytope Δ such that $\mu|_Q : Q \to \Delta$ is a small cover by Lemmas 3.1 and 3.2, it follows from [6, Theorem 3.1] that we have

$$\dim H^{2i}(X; \mathbb{Z}_2) = h_i = \dim H^i(Q; \mathbb{Z}_2), \quad 0 \le i \le n. \tag{3.2}$$

Hence, by [8, Theorem 3.1] we have

$$\dim H^{2*}(X; \mathbb{Z}_2) \overset{(3.2)}{=} \sum_{i=0}^{n} h_i \overset{(3.2)}{=} \dim H^*(Q; \mathbb{Z}_2) \overset{[8,\ \text{Theorem 3.1}]}{=} \dim H^*(C_\xi; \mathbb{Z}_2).$$

This completes the proof of Theorem 3.3 (or, equivalently Theorem 1.1(3)). □

4. Proof of Theorem 1.3

In this section, we give a proof of Theorem 1.3. Proposition 2.3 as well as [8, Theorem 2.5] plays an important role in the proof.

In order to prove Theorem 1.3, it suffices to prove the following theorem.

Theorem 4.1. *Let (X, ω, μ) be a toric Hamiltonian T-space, and let $\Delta = \mu(X)$ denote the moment polytope. Let τ be an anti-symplectic involution of X such that τ maps the fibers of μ to (possibly different) fibers of μ, and let $p_0 \in \Delta^\circ$. Assume that the toric fiber $\mu^{-1}(p_0)$ is real Lagrangian with respect to an anti-symplectic involution τ. Then the followings are true:*

(1) *There is an involution $\sigma \in \mathrm{Aut}_{\mathbb{Z}}(\mathfrak{t}^*)$ such that*
   (i) *$\mu \circ \tau = \sigma \circ \mu$ over X,*
   (ii) *$\{\xi \in \mathfrak{t} \mid \sigma^*(\xi) = \xi\} = \{0\}$, and*
   (iii) *the point p_0 is the origin of $\mathfrak{t}^*$.*
(2) *Δ is centrally symmetric.*

Proof. (1) Since the toric fiber $\mu^{-1}(p_0)$ is real Lagrangian with respect to an anti-symplectic involution τ, the fixed point set $\mathrm{Fix}(\tau)$ of τ is equal to $\mu^{-1}(p_0)$ by definition. Furthermore, it follows from Lemma 2.2 that over X there is an involution $\sigma \in \mathrm{Aut}_{\mathbb{Z}}(\mathfrak{t}^*)$ such that

$$\mu \circ \tau = \sigma \circ \mu.$$

As in the proof of Proposition 2.3, let

$$\mathfrak{t}_0 = \{\xi \in \mathfrak{t} \,|\, \sigma^*(\xi) = \xi\}.$$

By Proposition 2.3, there is a natural inclusion $j : \mathfrak{t}_0 \to \mathfrak{t}$ and its adjoint $j^* : \mathfrak{t}^* \to \mathfrak{t}_0^*$ such that the moment map $\tilde{\mu} := j^* \circ \mu : X \to \mathfrak{t}_0^*$ satisfies $\tilde{\mu} \circ \tau = \tilde{\mu}$. Thus, if $x \in \mu^{-1}(p_0)$, then we have

$$\tilde{\mu}(x) = j^* \circ \mu(x) = j^*(p_0) = p_0. \tag{4.1}$$

On the other hand, by [8, Theorem 2.5] we have

$$\tilde{\mu}(X) = \tilde{\mu}(\mathrm{Fix}(\tau)) \subset \mathfrak{t}_0^*. \tag{4.2}$$

Since $\mathrm{Fix}(\tau)$ is assumed to be equal to $\mu^{-1}(p_0)$, it follows from (4.1) and (4.2) that we have

$$\tilde{\mu}(X) = \tilde{\mu}(\mathrm{Fix}(\tau)) = \{p_0\}.$$

This implies that p_0 is actually the origin and the origin is the only fixed point of σ^*, which proves (1).

(2) Since $\mu \circ \tau = \sigma \circ \mu$ over X and τ is an automorphism of X, it is easy to show that

$$\Delta = \mu(X) = \mu \circ \tau(X) = \sigma \circ \mu(X) = \sigma(\mu(X)) = \sigma(\Delta).$$

Note also that the origin is the only fixed point of the involution σ by (1) above. Hence, Δ should be centrally symmetric around the origin by definition. This completes the proof of (2). □

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